MATHEMATICS

Coordinate Geometry

Ghanshyam Tewani

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Mathematics:
Coordinate Geometry

Ghanshyam Tewani

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, NITs, IIITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

GHANSHYAM TEWANI

COORDINATE GEOMETRY

The coordination of algebra and geometry is called coordinate geometry. Historically, coordinates were introduced to help geometry. And so well did they do this job that the very identity of geometry was changed. The word 'geometry' today generally means coordinate geometry.

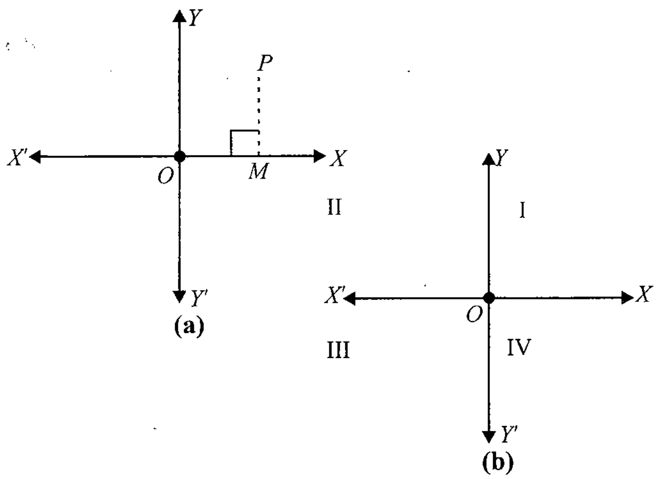
In coordinate geometry, all the properties of geometrical figures are studied with the help of algebraic equations. Students should note that the object of coordinate geometry is to use some known facts about a curve in order to obtain its equation and then deduce other properties of the curve from the equation so obtained. For this purpose, we require a coordinate system. There are various types of coordinate systems present in two dimensions e.g., rectangular, oblique, polar, triangular system, etc. Here, we will only discuss rectangular coordinate system in detail.

Cartesian Coordinates

Let $X'OX$ and $Y'OY$ be two fixed straight lines at right angles. $X'OX$ is called axis of x and $Y'OY$ is called axis of y, and O is named as origin. From any point 'P', a line is drawn parallel to OY. The directed line $OM = x$ and $MP = y$. Here, OM is abscissa and MP is ordinate of the point 'P'. The abscissa OM and the ordinate MP together written as (x, y) are called coordinates of point 'P'. Here, (x, y) is an ordered pair of real numbers x and y, which determine the position of point 'P'.

Since $X'OX$ is perpendicular to $Y'OY$, this system of representation is called rectangular (or orthogonal) coordinate system (Fig. 1.1(a)).

When the axes coordinates $X'OX$ and $Y'OY$ are not at right angles, they are said to be oblique axes.



(a) (b)

Fig. 1.1

The axes of rectangular coordinate system divide the plane into four infinite regions, called quadrants, each bounded by two half-axes. These are numbered from 1st to 4th and denoted by roman numerals (Fig. 1.1 (b)). The signs of the two coordinates are given below:

	x	y
First quadrant	+	+
Second quadrant	–	+
Third quadrant	–	–
Fourth quadrant	+	–

Lattice point (with respect is coordinate geometry): Lattice point is defined as a point whose abscissa and ordinate are integers.

DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

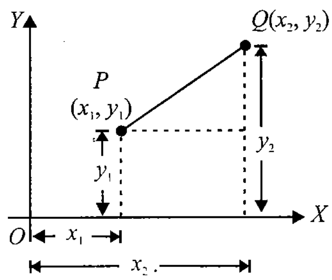


Fig. 1.2

Therefore, distance of (x_1, y_1) from origin $= \sqrt{x_1^2 + y_1^2}$.

Note:

- *If distance between two points is given, then use* $\pm$ *sign.*
- *Distance between* $(x_1, 0)$ *and* $(x_2, 0)$ is $|x_1 - x_2|$.
- *Distance between* $(0, y_1)$ *and* $(0, y_2)$ is $|y_1 - y_2|$.
- *Circumcentre* $P(x, y)$ *is a point which is equidistant from the vertices of triangle* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.
  Hence, $AP = BP = CP$, *which gives two equations in* x *and* y, *solving which we get circumcentre.*

Example 1.1 **In ΔABC, prove that $AB^2 + AC^2 = 2(AO^2 + BO^2)$, where O is the middle point of BC.**

Sol.

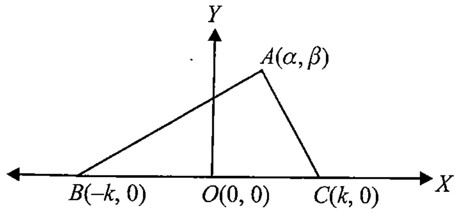


Fig. 1.3

We take O as the origin and OC and OY as the x-and y-axes, respectively.

Let $BC = 2k$, then $B \equiv (-k, 0)$, $C \equiv (k, 0)$.

Let $A \equiv (\alpha, \beta)$

Now L.H.S.

$$= AB^2 + AC^2$$
$$= (\alpha + k)^2 + (\beta - 0)^2 + (\alpha - k)^2 + (\beta - 0)^2$$
$$= \alpha^2 + k^2 + 2\alpha k + \beta^2 + \alpha^2 + k^2 - 2\alpha k + \beta^2$$
$$= 2\alpha^2 + 2\beta^2 + 2k^2$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

and R.H.S

$$= 2(AO^2 + BO^2)$$
$$= 2[(\alpha - 0)^2 + (\beta - 0)^2 + (-k - 0)^2 + (0 - 0)^2]$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

$\therefore$ L.H.S. = R.H.S.

Example 1.2 **Find the coordinates of the circumcentre of the triangle whose vertices are $A(5, -1)$, $B(-1, 5)$, and $C(6, 6)$. Find its radius also.**

Sol.

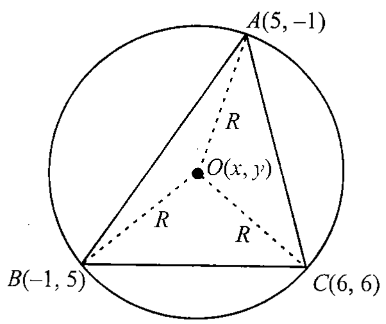


Fig. 1.4

Let circumcentre be $O(x, y)$, then

$$(OA)^2 = (OB)^2 = (OC)^2 = (\text{radius})^2 = R^2 \quad \text{(i)}$$

$$\Rightarrow (x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$
$$= (x - 6)^2 + (y - 6)^2$$

Taking first two relations, we get

$$(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$
$$\Rightarrow \quad x = y \quad \text{(ii)}$$

Taking last two relations, we get

$$(x + 1)^2 + (y - 5)^2 = (x - 6)^2 + (y - 6)^2$$
$$\Rightarrow (x + 1)^2 + (x - 5)^2 = (x - 6)^2 + (x - 6)^2 \quad \text{[from (ii)]}$$
$$\Rightarrow \quad 2x^2 - 8x + 26 = 2x^2 - 24x + 72$$
$$x = 23/8$$
$$\Rightarrow \quad \text{Circumcentre} = (23/8, 23/8)$$
$$\Rightarrow \quad R^2 = (x - 5)^2 + (y + 1)^2 = (OA)^2$$
$$\Rightarrow \quad = [(23/8) - 5]^2 + [(23/8) + 1]^2$$
$$= (-17)^2/(64) + (31)^2/(64)$$
$$= 1250/64$$
$$\Rightarrow \quad \text{Radius} = 25\sqrt{2}/8 \text{ units}$$

Example 1.3 **Two points $O(0, 0)$ and $A(3, \sqrt{3})$ with another point P form an equilateral triangle. Find the coordinates of P.**

Sol. Let the coordinates of P be (h, k).

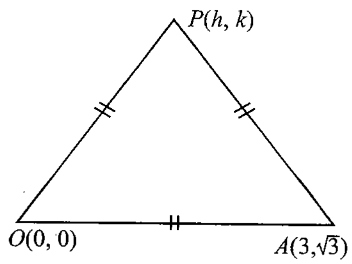


Fig. 1.5

$$\therefore \quad OA = OP = AP \text{ or } OA^2 = OP^2 = AP^2$$
$$\therefore \quad OA^2 = OP^2$$
$$\Rightarrow \quad 12 = h^2 + k^2 \quad \text{(i)}$$
$$\text{and} \quad OP^2 = AP^2$$
$$\Rightarrow \quad h^2 + k^2 = (h - 3)^2 + (k - \sqrt{3})^2$$
$$\Rightarrow \quad 3h + \sqrt{3}k = 6 \text{ or } h = 2 - k/\sqrt{3} \quad \text{(ii)}$$

Using Eq. (ii) in Eq. (i), we get

$$(2 - k/\sqrt{3})^2 + k^2 = 12 \text{ or } k^2 - \sqrt{3}k - 6 = 0$$

or $(k - 2\sqrt{3})(k + \sqrt{3}) = 0$

$$\therefore \quad k = 2\sqrt{3} \text{ or } -\sqrt{3}$$

From Eq. (ii), we get that

when $k = 2\sqrt{3}, h = 0,$

when $k = -\sqrt{3}, h = 3$

Hence, the coordinates of P are

$(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Example 1.4 **If O is the origin and if coordinates of any two points Q_1 and Q_2 are (x_1, y_1) and (x_2, y_2), respectively, prove that $OQ_1 \cdot OQ_2 \cos \angle Q_1 OQ_2 = x_1x_2 + y_1y_2$.**

Sol.

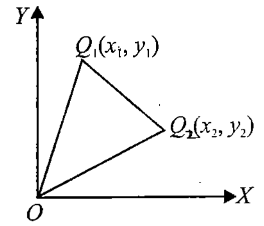


Fig. 1.6

In ΔQ_2OQ_1,

$$Q_1Q_2^2 = OQ_1^2 + OQ_2^2 - 2OQ_1OQ_2 \cos \angle Q_1OQ_2$$

$$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2OQ_1\, OQ_2 \cos\angle Q_1OQ_2$$

(using cosine Rule)

$$\Rightarrow \quad x_1x_2 + y_1y_2 = OQ_1\, OQ_2 \cos Q_1OQ_2$$

Example 1.5 **Let $A = (3, 4)$ and B is a variable point on the lines $|x| = 6$. If $AB \le 4$, then the number of position of B with integral coordinates is**

a. 5 **b.** 4

c. 6 **d.** 10

Sol. $B = (\pm 6, y)$. So, $AB \le 4$

$$\Rightarrow \quad (3 \mp 6)^2 + (y - 4)^2 \le 16$$

$$\therefore \quad 9 + (y - 4)^2 \le 16,$$

$(\because 81 + (y - 4)^2 \le 16$ is absurd)

$$\Rightarrow \quad y^2 - 8y + 9 \le 0$$

$$\Rightarrow \quad 4 - \sqrt{7} \le y \le 4 + \sqrt{7}$$

But y is an integer.

$$\Rightarrow \quad y = 2, 3, 4, 5, 6$$

AREA OF A TRIANGLE

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Proof:

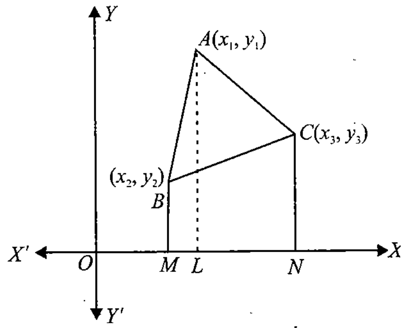


Fig. 1.7

Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Draw AL, BM, and CN as perpendiculars from A, B, and C on the x-axis. Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We have,

Area of Δ ABC = Area of trapezium $ABML$ + Area of trapezium $ALNC$ – Area of trapezium $BMNC$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

Note:

- *Area of a triangle can also be found by easy method, i.e., Stair method.*

$$\Delta = \frac{1}{2}\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{matrix}\right|$$

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}|$$

- *If three points A, B, and C are collinear, then area of triangle ABC is zero.*
- ***Sign of area:** If the points A, B, C are plotted in two-dimensional plane and taken on the anticlockwise sense, then the area calculated of the triangle ABC will be positive, while if the points are taken in clockwise sense, then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in both cases. In case, the area calculated is negative, we will consider it as positive.*

Area of Polygon

The area of polygon whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n) is

$$=\frac{1}{2}|\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\}|$$

Stair Method

Repeat first coordinates one time in last for down arrow use + ve sign and for up arrow use – ve sign.

$$\text{Area of polygon} = \frac{1}{2}\begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{Vmatrix}$$

$$=\frac{1}{2}|\{x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)\}|$$

Note:

Points should be taken in cyclic order in coordinate plane.

Example 1.6 **Find the area of a triangle whose vertices are $A(3, 2)$, $B(11, 8)$, and $C(8, 12)$.**

Sol. Let $A = (x_1, y_1) = (3, 2)$, $B(x_2, y_2) = (11, 8)$, and $C = (x_3, y_3) = (8, 12)$.

Then, area of ΔABC

$$=\frac{1}{2}|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}|$$
$$=\frac{1}{2}|\{3(8 - 12) + 11(12 - 2) + 8(2 - 8)\}|$$
$$=\frac{1}{2}|\{-12 + 110 - 48\}|$$
$$= 25 \text{ sq. units}$$

Example 1.7 **Prove that the area of the triangle whose vertices are $(t, t - 2)$, $(t + 2, t + 2)$, and $(t + 3, t)$ is independent of t.**

Sol. Let $A = (x_1, y_1) = (t, t - 2)$, $B = (x_2, y_2) = (t + 2, t + 2)$, and $C = (x_3, y_3) = (t + 3, t)$ be the vertices of the given triangle.

Then, area of ΔABC

$$=\frac{1}{2}|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}|$$

$$=\frac{1}{2}|\{t(t + 2 - t) + (t + 2)(t - t + 2) + (t + 3)(t - 2 - t - 2)\}|$$

$$=\frac{1}{2}|\{2t + 2t + 4 - 4t - 12\}| = |-4| = 4 \text{ sq. units.}$$

Clearly, area of ΔABC is independent of t.

Example 1.8 **Find the area of the quadrilateral $ABCD$ whose vertices are respectively $A(1, 1)$, $B(7, -3)$, $C(12, 2)$, and $D(7, 21)$.**

Sol. Here, given points are in cyclic order, then

$$\text{Area} = \frac{1}{2}\begin{Vmatrix} 1 & 1 \\ 7 & -3 \\ 12 & 2 \\ 7 & 21 \\ 1 & 1 \end{Vmatrix}$$

$$=\frac{1}{2}|(-3 - 7) + (14 + 36) + (252 - 14) + (7 - 21)|$$
$$= 132 \text{ sq. units}$$

Example 1.9 **For what value of k are the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear?**

Sol. Let three given points be $A = (x_1, y_1) = (k, 2 - 2k)$, $B = (x_2, y_2) = (-k + 1, 2k)$, and $C = (x_3, y_3) = (-4 - k, 6 - 2k)$.

If the given points are collinear, then $\Delta = 0$

$$\Rightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$
$$\Rightarrow k(2k - 6 + 2k) + (-k + 1)(6 - 2k - 2 + 2k) + (-4 - k)(2 - 2k - 2k) = 0$$
$$\Rightarrow k(4k - 6) - 4(k - 1) + (4 + k)(4k - 2) = 0$$
$$\Rightarrow 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$$
$$\Rightarrow 8k^2 + 4k - 4 = 0$$
$$\Rightarrow 2k^2 + k - 1 = 0$$
$$\Rightarrow (2k - 1)(k + 1) = 0$$
$$\Rightarrow k = 1/2 \text{ or } -1$$

Hence, the given points are collinear for $k = 1/2$ or -1.

Example 1.10 **If the vertices of a triangle have rational coordinates, then prove that the triangle cannot be equilateral.**

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle ABC, where x_i, y_i, $i = 1, 2, 3$ are rational. Then, the area of ΔABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$= \text{a rational number} \qquad [\because x_i, y_i \text{ are rational}]$$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad (\because AB = BC = CA)$$

$= \frac{\sqrt{3}}{4}$ (a rational number) [∵ vertices are rational ∴ AB^2 is a rational]

= an irrational number

This is a contradication to the fact that the area is a rational number. Hence, the triangle cannot be equilateral.

Example 1.11 **If the coordinates of two points A and B are (3, 4) and (5, – 2), respectively. Find the coordinates of any point P if $PA = PB$ and area of $\Delta PAB = 10$ sq. units.**

Sol. Let the coordinates of P be (x, y). Then, $PA = PB$.

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$

$\Rightarrow \quad x - 3y - 1 = 0 \qquad$ (i)

Now, area of $\Delta PAB = 10$ sq. units

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$

$\Rightarrow \quad 6x + 2y - 26 = \pm 20$

$\Rightarrow \quad 6x + 2y - 46 = 0$

or $\quad 6x + 2y - 6 = 0$

$\Rightarrow \quad 3x + y - 23 = 0$ or $3x + y - 3 = 0 \qquad$ (ii)

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get $x = 7$, $y = 2$

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1$, $y = 0$

Thus, the coordinates of P are (7, 2) or (1, 0).

Example 1.12 **Given that $P(3, 1)$, $Q(6, 5)$, and $R(x, y)$ are three points such that the angle PRQ is a right angle and the area of $\Delta RQP = 7$, then find the number of such points R.**

Sol. Obviously, R lies on the circle with P and Q as end points of diameter.

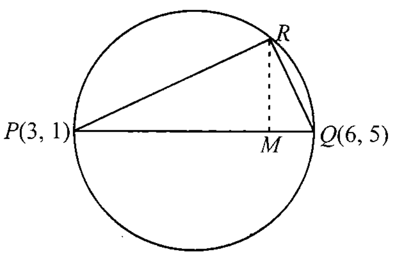


Fig. 1.8

Distance between points $P(3, 1)$ and $Q(6, 5)$ is 5 units. Hence, radius is 2.5 units

Area of triangle $PQR = \frac{1}{2} RM \times PQ = 7$ (given)

$\Rightarrow \quad RM = \frac{14}{5} = 2.8$

Now the value of RM cannot be greater than the radius. Hence. no such triangle is possible.

SECTION FORMULA

Formula for Internal Division

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are given by

$$x = \frac{mx_2 + nx_1}{m+n},$$

$$y = \frac{my_2 + ny_1}{m+n}$$

Proof:

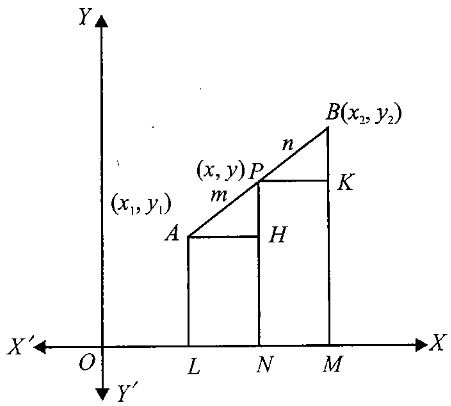


Fig. 1.9

From the figure,

Clearly, ΔAHP and ΔPKB are similar.

$$\Rightarrow \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Now,

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}$$

$$\Rightarrow \quad mx_2 - mx = nx - nx_1$$

$$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m+n}$$

Similarly $\quad \frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$\Rightarrow \quad my_2 - my = ny - ny_1$

$\Rightarrow \quad y = \dfrac{my_2 + ny_1}{m+n}$

Thus, the coordinates of P are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$.

Formula for External Division

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m: n are given by

$$x = \frac{mx_2 - nx_1}{m-n}, y = \frac{my_2 - ny_1}{m-n}$$

Proof:

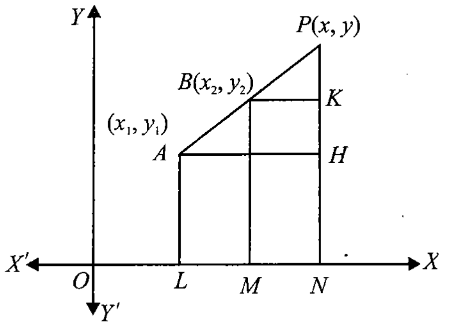


Fig. 1.10

From the figure,

Clearly, triangles PAH and PBK are similar. Therefore,

$$\frac{AP}{PB} = \frac{AH}{BK} = \frac{PH}{PK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2}$$

Now,

$$\frac{m}{n} = \frac{x-x_1}{x-x_2}$$

$\Rightarrow \quad mx - mx_2 = nx - nx_1$

$\Rightarrow \quad x = \dfrac{mx_2 - nx_1}{m-n}$

and

$$\frac{m}{n} = \frac{y-y_1}{y-y_2}$$

$\Rightarrow \quad my - my_2 = ny - ny_1$

$\Rightarrow \quad y = \dfrac{my_2 - ny_1}{m-n}$

Thus, the coordinates of P are $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$.

Note:

- *If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio m:n), we take the ratio* λ: *1 and apply the formula for internal division* $\left(\dfrac{\lambda x_2 + x_1}{\lambda + 1}, \dfrac{\lambda y_2 + y_1}{\lambda + 1}\right)$.

  If the value of λ *turns out to be positive, it is an internal division, otherwise it is an external division.*
- *The midpoint of* (x_1, y_1) *and* (x_2, y_2) is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
- *To prove that A, B, C, D are vertices of*

Parallelogram	*Show that diagonals AC and BD bisect each other*
Rhombus	*Show that diagonals AC and BD bisect each other and adjacent sides AB and BC are equal*
Square	*Show that diagonals AC and BD are equal and bisect each other and adjacent sides AB and BC are equal*
Rectangle	*Show that diagonals AC and BD are equal and bisect each other*

Example 1.13 **Find the coordinates of the point which divides the line segments joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.**

Sol. Let $P(x, y)$ be the required point.

i. For internal division, we have

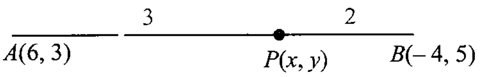


Fig. 1.11

$$x = \frac{3(-4) + 2(6)}{3+2}$$

and $\quad y = \dfrac{3(5) + 2(3)}{3+2}$

$\Rightarrow \quad x = 0$ and $y = 21/5$

So the coordinates of P are $(0, 21/5)$.

ii. For external division, we have

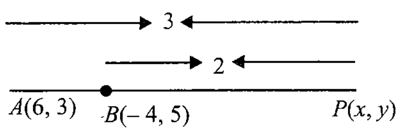


Fig. 1.12

$$x = \frac{3(-4) - 2(6)}{3 - 2}$$

and $$y = \frac{3(5) - 2(3)}{3 - 2}$$

$\Rightarrow$ $x = -24$ and $y = 9$

So the coordinates of P are $(-24, 9)$.

Example 1.14 **In what ratio does the x-axis divide the line segment joining the points (2, – 3) and (5, 6)?**

Sol. Let the required ratio be $\lambda:1$.

Then, the coordinates of the point of division are

$[(5\lambda + 2)/(\lambda - 1), (6\lambda - 3)/(\lambda + 1)]$

But, it is a point on x-axis on which y-coordinates of every point is zero.

$\Rightarrow$ $(6\lambda - 3)/(\lambda + 1) = 0$

$\Rightarrow$ $\lambda = \frac{1}{2}$

Thus, the required ratio is (1/2): 1 or 1 : 2

Example 1.15 **Given that $A(1, 1)$ and $B(2, -3)$ are two points and D is a point on AB produced such that $AD = 3AB$. Find the coordinates of D.**

Sol. We have, $AD = 3AB$. Therefore, $BD = 2AB$.

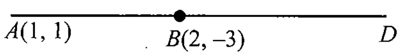


Fig. 1.13

Thus, D divides AB externally in the ratio $AD : BD = 3 : 2$

Hence, the coordinates of D are

$$\left(\frac{3(2) - 2(1)}{3 - 2}, \frac{3(-3) - 2(1)}{3 - 2}\right) = (4, -11)$$

Example 1.16 **Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points (1, 3) and (2, 7).**

Sol. Suppose the line $3x + y - 9 = 0$ divides the line segment joining $A(1, 3)$ and $B(2, 7)$ in the ratio $k : 1$ at point C. Then, the coordinates of C are

$$\left(\frac{2k + 1}{k + 1}, \frac{7k + 3}{k + 1}\right)$$

But, C lies on $3x + y - 9 = 0$. Therefore,

$$3\left(\frac{2k + 1}{k + 1}\right) + \frac{7k + 1}{k + 1} - 9 = 0$$

$\Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0$

$\Rightarrow$ $k = \frac{3}{4}$

So, the required ratio is 3 : 4 internally

Example 1.17 **Prove that the points (–2, – 1), (1, 0), (4, 3), and (1, 2) are the vertices of a parallelogram. Is it a rectangle?**

Sol. Let the given points be A, B, C, and D, respectively.

Then, the coordinates of the midpoint of AC are

$$\left(\frac{-2 + 4}{2}, \frac{-1 + 3}{2}\right) = (1, 1)$$

Coordinates of the midpoint of BD are

$$\left(\frac{1 + 1}{2}, \frac{0 + 2}{2}\right) = (1, 1)$$

Thus, AC and BD have the same midpoint.

Hence, $ABCD$ is a parallelogram.

Now, we shall see whether $ABCD$ is a rectangle or not.

We have

$$AC = \sqrt{(4 - (-2))^2 + (3 - (-1))^2}$$

$$= 2\sqrt{13},$$

and

$$BD = \sqrt{(1 - 1)^2 + (0 - 2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

Example 1.18 **If P divides OA internally in the ratio $\lambda_1 : \lambda_2$ and Q divides OA externally in the ratio $\lambda_1 : \lambda_2$, then prove that OA is the harmonic mean of OP and OQ.**

Sol.

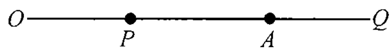


Fig. 1.14

We have, $$\frac{1}{OP} = \frac{\lambda_1 + \lambda_2}{\lambda_1 OA}$$

and $$\frac{1}{OP} = \frac{\lambda_1 - \lambda_2}{\lambda_1 OA}$$

$\Rightarrow$ $$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OA}$$

$\Rightarrow OP$, OA and OQ are in H.P.

Example 1.19 **Given that $A_1, A_2, A_3, \ldots, A_n$ are n points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, respectively. A_1A_2 is bisected at the point P_1, P_1A_3 is divided in the ratio 1 : 2 at P_2, P_2A_4 is divided in the ratio 1 : 3 at P_3, P_3A_5 is divided in the ratio 1 : 4 at P_4, and so on until all n points are exhausted. Find the final point so obtained.**

Sol. The coordinates of P_1(midpoint of A_1A_2) are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

P_2 divides P_1A_3 in 1 : 2; therefore, coordinates of P_2 are

$$\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1}, \frac{2\left(\frac{y_1+y_2}{2}\right)+y_3}{2+1}\right)$$

i.e., $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Now P_3 divides P_1A_4 in 1 : 3, therefore,

$$P_3 = \left(\frac{3\frac{(x_1+x_2+x_3)}{3}+x_4}{3+1}, \frac{3\frac{(y_1+y_2+y_3)}{3}+y_4}{3+1}\right)$$

$$= \left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)\right]$$

Proceeding in this manner, we can show that the coordinates of the final point are

$$[(x_1+x_2+\cdots+x_n)/n, (y_1+y_2+\cdots+y_n)/n]$$

COORDINATES OF THE CENTROID, INCENTRE, AND EX-CENTRES OF A TRIANGLE

Centroid of a Triangle

The point of concurrency of the medians of a triangle is called the centroid of the triangle. The coordinates of the centroid of the triangle with vertices as (x_1, y_1), (x_2, y_2), and (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of ΔABC whose medians are AD, BE, and CF, respectively. So D, E, and F are respectively the midpoint of BC, CA, and AB.

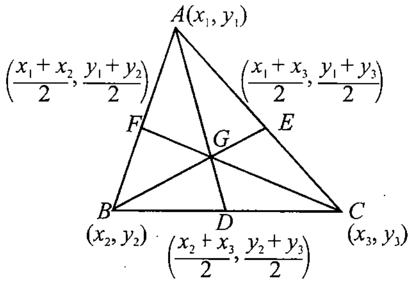


Fig. 1.15

Coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Coordinates of a points G dividing AD in the ratio 2:1 are

$$\left(\frac{1(x_1)+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1(y_1)+2\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Incentre of a Triangle

The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre of a triangle with vertices as $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a = BC, $b = AC$ and $C = AB$

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB, respectively.

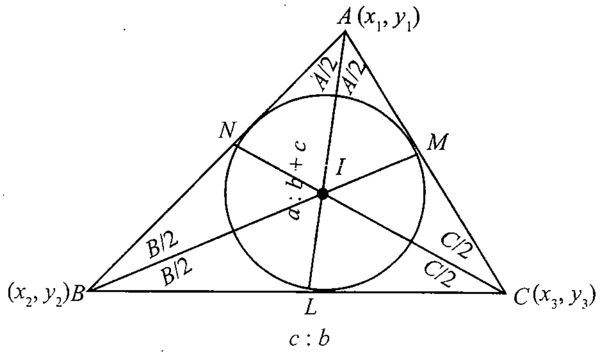


Fig. 1.16

The point at which the bisectors of the angles of a triangle intersect is called the incentre of the triangle.

Let AL, BM, and CN be respectively the internal bisectors of the angles A, B and C.

As AL bisects $\angle BAC$ internally, we have

$$\frac{BL}{LC} = \frac{BA}{AC} = \frac{c}{b} \qquad \text{(i)}$$

$$\Rightarrow \quad \frac{LC}{BL} = \frac{b}{c}$$

$$\Rightarrow \quad \frac{LC}{BL}+1 = \frac{b}{c}+1$$

$$\Rightarrow \quad \frac{LC+BL}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \quad \frac{BC}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \quad \frac{a}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \quad BL = \frac{ac}{b+c} \qquad \text{(ii)}$$

Since BI is the bisector of $\angle B$, so it divides AIL in the ratio $AI : IL$

$$\therefore \quad \frac{AI}{IL} = \frac{AB}{BL} = \frac{c}{(ac)/(b+c)} = \frac{b+c}{a}$$

[Using (ii)]

$$\Rightarrow \quad AI : IL = (b+c) : a \qquad \text{(iii)}$$

From (i), L divides BC in the ratio $c : b$

$\Rightarrow$ Coordinates of L are

$$\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$$

From (iii), I divides AL in the ratio $(b + c) : a$. So the coordinates of I are

$$\left(\frac{ax_1 + (b+c)\left(\frac{bx_2 + cx_3}{b+c}\right)}{a+b+c}, \frac{ay_1 + (b+c)\left(\frac{by_2 + cy_3}{b+c}\right)}{a+b+c}\right)$$

$$\text{or} \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Ex-centre of a Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB, respectively. The circle which touches the side BC and the other two sides AB and AC produced is called the escribed circle opposite to the angle A. The bisectors of the external angle B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A.

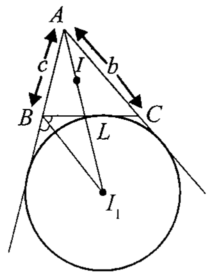


Fig. 1.17

$$\frac{BL}{LC} = \frac{c}{b}, \text{ also } \frac{AI_1}{I_1L} = -\frac{(b+c)}{a}$$

The coordinates of I_1 are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly, coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C, respectively) are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Circumcentre of a Triangle

Circumcentre $P(x, y)$ is a point which is equidistant from the vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Hence, $AP = BP = CP$, which gives two equations in x and y, solving which we get circumcentre.

Also circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Orthocentre

The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

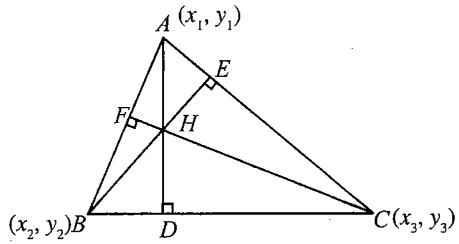


Fig. 1.18

In Fig. 1.18, point H is an orthocentre of ΔABC, and it is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Note:

- *Circumcentre O, Centroid G, and Orthocentre H of an acute ΔABC are collinear. G divides OH in the ratio* 1 : 2, *i.e.,* $OG : GH = 1 : 2$
- *In an isosceles triangle, centroid, orthocentre, incentre, and circumcentre lie on the same line. In an equilateral triangle, all these four points coincide.*

Example 1.20 **If a vertex of a triangle is (1, 1), and the middle points of two sides passing through it are (–2, 3) and (5, 2), then find the centroid and the incentre of the triangle.**

Sol.

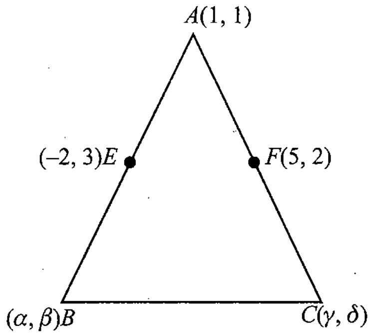


Fig. 1.19

Let E and F be the midpoints of AB and AC.

Let the coordinates of B and C be (α, β) and (γ, δ), respectively, then

$$-2 = \frac{1+\alpha}{2}, 3 = \frac{1+\beta}{2},$$

$$5 = \frac{1+\gamma}{2}, 2 = \frac{1+\delta}{2}$$

$$\therefore \quad \alpha = -5, \beta = 5, \gamma = 9, \delta = 3$$

Therefore, coordinates of B and C are (–5, 5) and (9, 3), respectively.

Then, centroid is

$$\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right), \text{ i.e., } \left(\frac{5}{3}, 3\right)$$

and $\quad a = BC = \sqrt{(-5-9)^2 + (5-3)^2} = 10\sqrt{2}$

$$b = CA = \sqrt{(9-1)^2 + (3-1)^2} = 2\sqrt{17}$$

and $\quad c = AB = \sqrt{(1+5)^2 (1-5)^2} = 2\sqrt{13}$

Then incentre is

$$\left(\frac{10\sqrt{2}(1) + 2\sqrt{17}(-5) + 2\sqrt{13}(9)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}},\right.$$

$$\left.\frac{10\sqrt{2}(1) + 2\sqrt{17}(5) + 2\sqrt{13}(3)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}\right)$$

$$\text{i.e., } \left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}\right)$$

Example 1.21 **Find the orthocentre of the triangle whose vertices are (0, 0), (3, 0), and (0, 4).**

Sol. This is a right-angled (at origin) triangle, therefore orthocentre = (0, 0).

Example 1.22 **If the vertices *P*, *Q*, *R* of a triangle *PQR* are rational points, which of the following points of the triangle *PQR* is (are) not necessarily rational?**

a. **centroid** **b.** **incentre**

c. **circumcentre** **d.** **orthocentre**

(A rational point is a point both of whose coordinates are rational numbers)

Sol. If $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, where x_1, y_1 etc., are rational numbers, then Σx_1, Σy_1 are also rational.

So, the coordinates of the centroid $(\Sigma x_1/3\ \Sigma y_1/3)$ will be rational.

As $AB = c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ may or may not be rational and it may be an irrational number of the form $\sqrt{p}$. Hence, the coordinates of the incentre $(\Sigma ax_1/\Sigma a, \Sigma ay_1/\Sigma a)$ may or may not be rational. If (α, β) is the circumcentre or orthocentre, α and β are found by solving two linear equations in α, β with rational coefficients. So α, β must be rational numbers.

Example 1.23 **If the circumcentre of an acute angled triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$, then find the orthocentre.**

Sol. We know from geometry that the circumcentre (O) centroid (G) and orthocentre (H) of a triangle lie on the line joining the circumcentre (0, 0) and the centroid $((a + 1)^2/2, (a - 1)^2/2)$.

Also $\quad \frac{\text{HG}}{\text{GO}} = \frac{2}{1} \Rightarrow$ H has coordinate

$$\left(\frac{3(a+1)^2}{2}, \frac{3(a-1)^2}{2}\right)$$

Example 1.24 **If a vertex, the circumcentre, and the centroid of a triangle are (0, 0), (3, 4), and (6, 8), respectively, then the triangle must be**

a. **a right-angled triangle**

b. **an equilateral triangle**

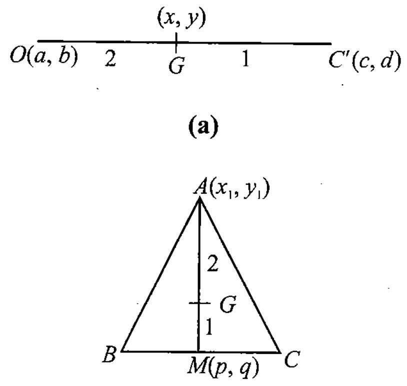
c. **an isosceles triangle**

d. **a right-angled isosceles triangle**

Sol. Clearly, (0, 0), (3, 4), and (6, 8) are collinear. So, the circumcentre M and the centroid G are on the median which is also the perpendicular bisector of the side. So, the Δ must be isosceles.

Example 1.25 **Orthocentre and circumcentre of a Δ*ABC* are (*a*, *b*) and (*c*, *d*), respectively. If the coordinates of the vertex *A* are (x_1, y_1), then find the coordinates of the middle point of *BC*.**

Sol.



(a)

(b)

Fig. 1.20

$$x = \frac{a+2c}{3};\ y = \frac{b+2d}{3}$$

Now

$$x = \frac{x_1+2p}{3};\ y = \frac{y_1+2q}{3}$$

$\therefore$

$$p = \frac{a+2c-x_1}{2};\ q = \frac{b+2d-y_1}{2}$$

Concept Application Exercise 1.1

1. If the points (0, 0), $(2, 2\sqrt{3})$, and (p, q) are the vertices of an equilateral triangle, then (p, q) is
   a. (0, –4) b. (4, 4) c. (4, 0) d. (5, 0)
2. The distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is
   a. $a\cos\frac{\alpha-\beta}{2}$ b. $2a\cos\frac{\alpha-\beta}{2}$
   c. $2a\sin\frac{\alpha-\beta}{2}$ d. $a\sin\frac{\alpha-\beta}{2}$
3. Find the length of altitude through A of the triangle ABC, where $A \equiv (-3, 0)$, $B \equiv (4, -1)$, $C \equiv (5, 2)$.
4. If the point $(x, -1)$, $(3, y)$, $(-2, 3)$, and $(-3, -2)$ taken in order are the vertices of a parallelogram, then find the values of x and y.
5. If the midpoints of the sides of a triangle are (2, 1), (–1, –3), and (4, 5). Then find the coordinates of its vertices.
6. The three points (–2, 2), (8, –2), and (–4, –3) are the vertices of
   a. an isosceles triangle
   b. an equilateral triangle
   c. a right angled triangle
   d. none of these
7. The points (a, b), (c, d), and $\left(\frac{kc+la}{k+l}, \frac{kd+lb}{k+l}\right)$ are
   a. Vertices of an equilateral triangle
   b. Vertices of an isosceles triangle
   c. Vertices of a right-angled triangle
   d. Collinear
8. The points $(-a, -b)$, (a, b), (a^2, ab) are
   a. Vertices of an equilateral triangle
   b. Vertices of a right-angled triangle
   c. Vertices of an isosceles triangle
   d. Collinear
9. Circumcentre of the triangle formed by the line $y = x$, $y = 2x$, and $y = 3x + 4$ is
   a. (6, 8) b. (6, –8)
   c. (3, 4) d. (–3, –4)
10. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7, -3)$, $D(12, 2)$, and $E(0, -3)$.
11. If the middle points of the sides of a triangle are (–2, 3), (4, –3), and (4, 5), then find the centroid of the triangle.
12. The line joining $A(b\cos\alpha, b\sin\alpha)$ and $B(a\cos\beta, a\sin\beta)$ is produced to the point $M(x, y)$ so that AM and BM are in the ratio $b : a$. Then prove that $x + y\tan(\alpha + \beta/2) = 0$.
13. A point moves such that the area of the triangle formed by it with the points (1, 5) and (3, –7) is 21 sq. units. Then, find the locus of the point.
14. If (1, 4) is the centroid of a triangle and the coordinates of its any two vertices are (4, –8) and (–9, 7), find the area of the triangle.
15. A triangle with vertices (4, 0), (– 1, – 1), (3, 5) is
   a. isosceles and right-angled
   b. isosceles but not right-angled
   c. right-angled but not isosceles
   d. neither right-angled nor isosceles

LOCUS AND EQUATION TO A LOCUS

Locus

The curve described by a point which moves under given condition or conditions is called its locus. For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is, therefore, the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper, and P be a variable point in the plane of the paper which moves in such a way that its distance from A and B is always same. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distance from A and B is always equal.

Equation to Locus of a Point

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

Steps to find locus of a points

Step I: Assume the coordinates of the point say (h, k) whose locus is to be found.

Step II: Write the given condition in mathematical form involving h, k.

Step III: Eliminate the variable(s), if any.

Step IV: Replace h by x and k by y in the result obtained in step III.

The equation so obtained is the locus of the point which moves under some stated condition(s).

Example 1.26 **The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.**

Sol. Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points.

Then, we have

$$PA^2 + PB^2 = 2c^2 \text{ (given)}$$

$$\Rightarrow (h-a)^2 + (k-0)^2 + (h+a)^2 + (k-0)^2 = 2c^2$$

$$2h^2 + 2k^2 + 2a^2 = 2c^2$$

$$\Rightarrow \quad h^2 + k^2 = c^2 - a^2$$

Hence, equation to locus (h, k) is $x^2 + y^2 = c^2 - a^2$.

Example 1.27 **Find the locus of a point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point.**

Sol. Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be given points.

By the given condition, we have

$$\angle APB = 90°$$

$\Rightarrow \Delta APB$ is a right-angled triangle

$$\Rightarrow \quad AB^2 = AP^2 + PB^2$$

$$\Rightarrow (3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$$

$$\Rightarrow \quad 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\Rightarrow \quad h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

Example 1.28 **A rod of length l slides with its ends on two perpendicular lines. Find the locus of its midpoint.**

Sol. Let the two perpendicular lines be the coordinates axes.

Let AB be a rod of length l.

Let the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively.

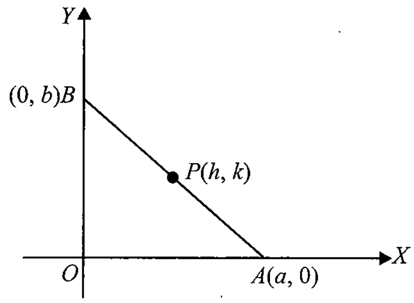


Fig. 1.21

As the rod slides, the values of a and b change.

So a and b are two variables.

Let $P(h, k)$ be the midpoint of the rod AB in one of the infinite positions it attains.

Then, $$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2} \qquad \text{(i)}$$

From ΔOAB, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad a^2 + b^2 = l^2$$

$$\Rightarrow \quad (2h)^2 + (2k)^2 = l^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 4h^2 + 4k^2 = l^2$$

Hence, the locus of (h, k) is $4x^2 + 4y^2 = l^2$

Example 1.29 ***AB* is a variable line sliding between the coordinate axes in such a way that *A* lies on *x*-axis and *B* lies on *y*-axis. If *P* is a variable point on *AB* such that $PA = b$, $PB = a$, and $AB = a + b$, find the equation of the locus of *P*.**

Sol. Let $P(h, k)$ be a variable point on AB such that $\angle OAB = \theta$.

Here θ is a variable.

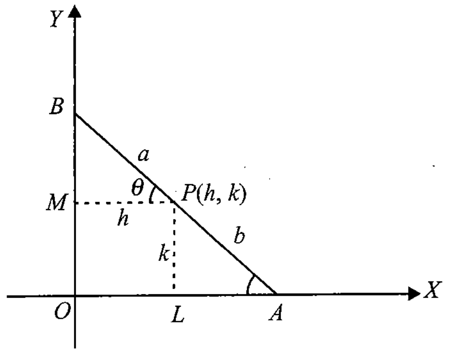


Fig. 1.22

From triangles ALP and PMB, we have

$$\sin\theta = \frac{k}{b} \quad \text{(i)}$$

$$\cos\theta = \frac{h}{a} \quad \text{(ii)}$$

Here θ is a variable. So, we have to eliminate θ.

Squaring (i) and (ii) and adding, we get

$$\frac{k^2}{b^2} + \frac{h^2}{a^2} = 1$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 1.30 **Two points P and Q are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is a positive constant 2α. Find the locus of the point R.**

Sol. Let the x-axis along QP and the middle point of PQ be origin and let $R \equiv (x_1, y_1)$.

Let $OP = OQ = a$ and $\angle RPM = \theta$ and $\angle RQM = \phi$

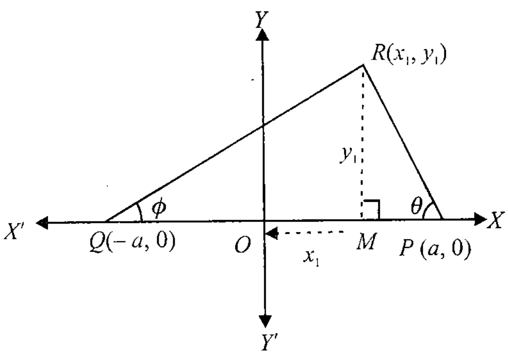


Fig. 1.23

In $\Delta\, RMP$,

$$\tan\theta = \frac{RM}{MP} = \frac{y_1}{a - x_1} \quad \text{(i)}$$

In $\Delta\, RQM$,

$$\tan\phi = \frac{RM}{QM} = \frac{y_1}{a + x_1} \quad \text{(ii)}$$

But given $\angle RPQ - \angle RQP = 2\alpha$ (constant)

$$\Rightarrow \quad \theta - \phi = 2\alpha$$

$$\Rightarrow \quad \tan(\theta - \phi) = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\dfrac{y_1}{a - x_1} - \dfrac{y_1}{a + x_1}}{1 + \dfrac{y_1}{a - x_1}\dfrac{y_1}{a + x_1}} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{2x_1y_1}{a^2 - x_1^2 + y_1^2} = \tan 2\alpha$$

$\Rightarrow a^2 - x_1^2 + y_1^2 = 2x_1y_1 \cot 2\alpha$ or $x_1^2 - y_1^2 + 2x_1y_1 \cot 2\alpha = a^2$

Hence, locus of the point $R(x_1, y_1)$ is $x^2 - y^2 + 2xy \cot 2\alpha = a^2$.

Example 1.31 **If the coordinates of a variable point P is $(a\cos\theta, b\sin\theta)$, where θ is a variable quantity, then find the locus of P.**

Sol. Let $P \equiv (x, y)$. According to the question

$$x = a\cos\theta \quad \text{(i)}$$

$$y = b\sin\theta \quad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 1.32 **Find the locus of a point such that the sum of its distance from the points (0, 2) and (0, – 2) is 6.**

Sol. Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and B(0, – 2) be the given points.

By the given condition, we get

$$PA + PB = 6$$

$\Rightarrow \sqrt{(h-0)^2+(k-2)^2}+\sqrt{(h-0)^2+(k+2)^2}=6$

$\Rightarrow \sqrt{h^2+(k-2)^2}=6-\sqrt{(h-0)^2+(k+2)^2}$

$\Rightarrow h^2+(k-2)^2=36-12\sqrt{h^2+(k+2)^2}+h^2+(k+2)^2$

$\Rightarrow -8k-36=-12\sqrt{h^2+(k+2)^2}$

$\Rightarrow (2k+9)=3\sqrt{h^2+(k+2)^2}$

$\Rightarrow (2k+9)^2=9(h^2+(k+2)^2)$

$\Rightarrow 4k^2+36k+81=9h^2+9k^2+36k+36$

$\Rightarrow 9h^2+5k^2=45$

Hence, locus of (h, k) is $9x^2+5y^2=45$.

Concept Application Exercise 1.2

1. Find the locus of a point whose distance from $(a, 0)$ is equal to its distance from y-axis.
2. The coordinates of the points A and B are $(a, 0)$ and $(-a, 0)$, respectively. If a point P moves so that $PA^2 - PB^2 = 2k^2$, when k is constant, then find the equation to the locus of the point P.
3. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then as α varies, then find the locus of its centroid.
4. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of the triangle moves on the line $2x+3y=1$, then find the locus of the vertex C.
5. Q is a variable point whose locus is $2x+3y+4=0$; corresponding to a particular position of Q, P is the point of section of OQ, O being the origin, such that $OP : PQ = 3 : 1$. Find the locus of P?
6. Find the locus of the middle point of the portion of the line $x\cos\alpha + y\sin\alpha = p$ which is intercepted between the axes, given that p remains constant.
7. Find the locus of the point of intersection of lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ (α is a variable).

SHIFTING OF ORIGIN

Let O be the origin and let $X'OX$ and $Y'OY$ be the axis of x and y, respectively. Let O' and P be two points in the plane having coordinates (h, k) and (x, y), respectively referred to $X'OX$ and $Y'OY$ as the coordinates axes. Let the origin be transferred to O' and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of P referred to new axes as the coordinates axes be (X, Y).

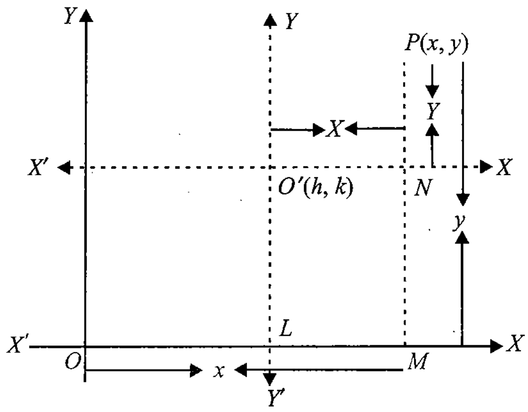


Fig. 1.24

Then,

$O'N = X$, $PN = Y$, $OM = x$, $PM = y$, $OL = h$, and $O'L = k$

Now,

$x = OM = OL + LM = OL + O'N = h + X$

and

$y = PM = PN + NM = PN + O'L = Y + k$

$\Rightarrow x = X + h$ and $y = Y + k$

Thus, if (x, y) are coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then $x = X + h$ and $y = Y + k$. Therefore, the origin is shifted at a point (h, k), we must substitute $X + h$ and $Y + k$ for x and y, respectively.

The transformation formula from new axes to old axes is

$X = x - h$, $Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

Example 1.33 **If the origin is shifted to the point (1, –2) without rotation of axes what do the following equations become?**

i. $2x^2 + y^2 - 4x + 4y = 0$ **and**

ii. $y^2 - 4x + 4y + 8 = 0$.

Sol. i. Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0$

or $2X^2 + Y^2 = 6$

ii. Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0$

or $Y^2 = 4X$

Example 1.34 **At what point the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9)?**

Sol. Let (h, k) be the point to which the origin is shifted. Then,

$$x = 4, y = 5, X = -3, Y = 9$$

$\therefore$ $x = X + h$ and $y = Y + k$

$\Rightarrow$ $4 = -3 + h$ and $5 = 9 + k$

$\Rightarrow$ $h = 7$ and $k = -4$

Hence, the origin must be shifted to $(7, -4)$.

Example 1.35 **Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term.**

Sol. Let the origin be shifted to (h, k). Then,

$$x = X + h \text{ and } y = Y + k$$

Substituting $x = X + h$, $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$\Rightarrow$ $Y^2 + (4 + 2k)\,Y + 8X + (k^2 + 4k + 8h - 2) = 0$

For this equation to be free from the term containing Y and the constant term, we must have

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0$$

$\Rightarrow$ $k = -2$ and $h = 3/4$

Hence, the origin is shifted at the point $(3/4, -2)$.

Example 1.36 **The equation of a curve referred to new axes, axes retaining their directions, and origin (4, 5) is $X^2 + Y^2 = 36$. Find the equation referred to the original axes.**

Sol. With the above notation, we have

$$x = X + 4, y = Y + 5$$

$\Rightarrow$ $X = x - 4$, $Y = y - 5$

$\therefore$ The required equation is

$$(x - 4)^2 + (y - 5)^2 = 36$$

$\Rightarrow$ $x^2 + y^2 - 8x - 10y + 5 = 0$ which is equation referred to the original axes.

Example 1.37 **Find the equation to which the equation**

$x^2 + 7xy - 2y^2 + 17x - 26y - 60 = 0$

is transformed if the origin is shifted to the point (2, – 3), the axes remaining parallel to the original axis.

Sol. Here the new origin is (2, - 3).

Then, $x = X + 2$, $y = Y - 3$.

and the given equation transforms to

$(X + 2)^2 + 7(X + 2)(Y - 3) - 2(Y - 3)^2 + 17(X + 2) - 26(Y - 3) - 60 = 0$

$\Rightarrow$ $X^2 + 7XY - 2Y^2 - 4 = 0$

ROTATION OF AXIS

Rotation of Axes without Changing the Origin

Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$

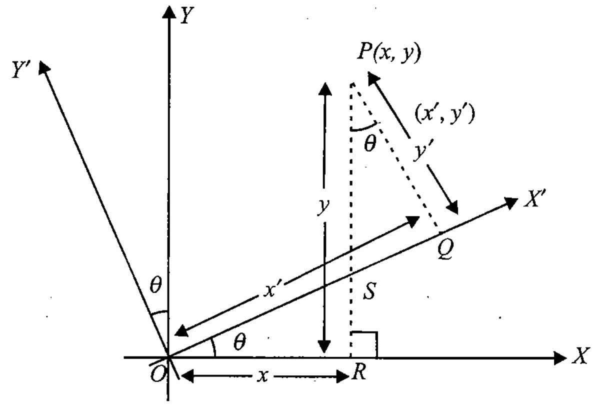


Fig. 1.25

In Fig. 1.25, we have

$$SR = x \tan\theta, \; OS = x \sec\theta,$$
$$PS = y - x\tan\theta$$

Now in triangle PQS,

$$\sin\theta = \frac{SQ}{PS} = \frac{x' - x\sec\theta}{y - x\tan\theta}$$

$\Rightarrow$ $x' = y\sin\theta - x\dfrac{\sin^2\theta}{\cos\theta} + \dfrac{x}{\cos\theta}$

$$= y\sin\theta + x\frac{1 - \sin^2\theta}{\cos\theta}$$

$\Rightarrow$ $x' = x\cos\theta + y\sin\theta$

Also $\cos\theta = \dfrac{PQ}{PS} = \dfrac{y'}{y - x\tan\theta}$

$\Rightarrow$ $y' = -x\sin\theta + y\cos\theta$

$\Rightarrow$ $x = x'\cos\theta - y'\sin\theta$,
$y = x'\sin\theta + y'\cos\theta$

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

Note:

Compare real and imaginary parts of the equation $(x + iy) = (x' + iy')(\cos\theta + i\sin\theta)$ to remember the formula

Example 1.38 **The equation of a curve referred to a given system of axes is $3x^2 + 2xy + 3y^2 = 10$. Find its equation if the axes are rotated through an angle 45°, the origin remaining unchanged.**

Sol. With the above notation, we have

$$x = x' \cos 45° - y' \sin 45° = \frac{x' - y'}{\sqrt{2}}$$

and $y = x' \sin 45° + y' \cos 45° = \dfrac{x' + y'}{\sqrt{2}}$

Thus, $3x^2 + 2xy + 3y^2 = 10$ transforms to

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 3\left(\frac{x'+y'}{\sqrt{2}}\right)^2 = 10$$

$\Rightarrow\ 2x'^2 + y'^2 = 5.$

Removal of the term *xy*, from $f(x, y) = ax^2 + 2hxy + by^2$ without changing the origin

Clearly, $h \neq 0$.

Rotating the axes through an angle θ, we have

$x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta.$

$\therefore f(x, y) = ax^2 + 2hxy + by^2$

$= a(x' \cos\theta - y' \sin\theta)^2 + 2h(x' \cos\theta - \theta y' \sin\theta).(x' \sin\theta + y' \cos\theta) + b(x' \sin\theta + y' \cos\theta)^2$

$= (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)x'^2 + 2[(b-a)\cos\theta \sin\theta + h(\cos^2\theta - \sin^2\theta)]x'y' + (a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta)y'^2$

$= F(x', y')$, (*say*).

In $F(x', y')$, we require that the coefficient of the XY-term to be zero.

$\therefore 2[(b-a)\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta)] = 0.$

$\Rightarrow (a-b)\sin 2\theta = 2h\cos 2\theta.$

$\Rightarrow \tan 2\theta = \dfrac{2h}{a-b}$ or $\cot 2\theta = \dfrac{a-b}{2h}$

We use the former or the later equation according as $a \neq b$ or $a = b$. These yield θ, the angle through which the axes are to be rotated (the origin remaining unchanged) in order to remove the xy-term from $f(x, y)$.

Example 1.39 **Given the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$. Through what angle should the axes be rotated so that the term *xy* is removed from the transformed equation.**

Sol. Comparing the given equation, with

$ax^2 + 2hxy + by^2$, we get $a = 4$, $h = \sqrt{3}$, $b = 2$.

Let θ be the angle through which the axes are to be rotated.

Then $\qquad \tan 2\theta = \dfrac{2h}{a-b}$

$\Rightarrow \qquad \tan 2\theta = \dfrac{2\sqrt{3}}{4-2} = \sqrt{3} = \tan\dfrac{\pi}{3}$

$\Rightarrow \qquad 2\theta = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\Rightarrow \qquad \theta = \dfrac{\pi}{6}, \dfrac{2\pi}{3}$

Change of Origin and Rotation of Axes

If origin is changed to $O'(\alpha, \beta)$ and axes are rotated about the new origin O' by an angle θ in the anticlockwise sense such that the new coordiantes of $P(x, y)$ becomes (x', y'), then the equations of transformation will be

$$x = \alpha + x' \cos\theta - y' \sin\theta$$

and

$$y = \beta + x' \sin\theta + y' \cos\theta$$

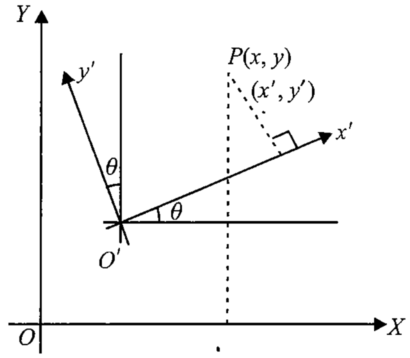


Fig. 1.26

Example 1.40 **What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point (–2, – 3), the new axes being inclined at an angle of 45° with the old axes?**

Sol. Let O' be (–2, –3). Since the axes are rotated about O' by an angle 45° in anticlockwise direction, let (x', y') be the new coordinates with respect to new axes and (x, y) be the coordinates with respect to old axes. Then, we have

$$x = -2 + x' \cos 45° - y' \sin 45° = -2 + \left(\frac{x'-y'}{\sqrt{2}}\right)$$

$$y = -3 + x' \sin 45° + y' \cos 45° = -3 + \left(\frac{x'+y'}{\sqrt{2}}\right)$$

The new equation will be

$$2\left\{-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right\}^2 + 4\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)$$

$$-5\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)^2 + 20\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)$$

$$-22\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right) - 14 = 0$$

$\Rightarrow \quad x'^2 - 14x'y' - 7y'^2 - 2 = 0$

Hence, new equation of curve is

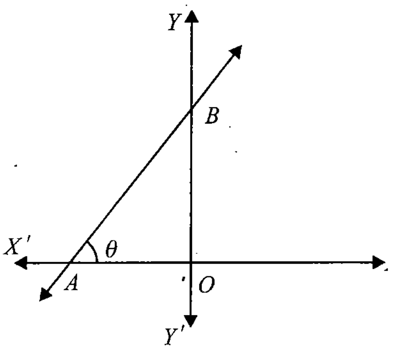
$$x^2 - 14xy - 7y^2 - 2 = 0$$

STRAIGHT LINE

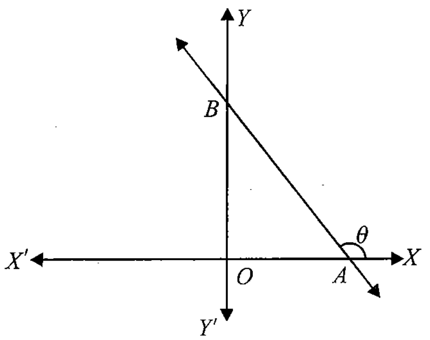
A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Slope (Gradient) of a Line

The trigonometrical tangent of an angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.



(a)



(b)

Fig. 1.27

Note:

- *The slope of a line is generally denoted by m. Thus, $m = \tan\theta$.*
- *Since a line parallel to x-axis makes an angle of* $0°$ *with x-axis; therefore, its slope is tan* $0° = 0$.
- A line parallel to y-axis, i.e., perpendicular to x-axis makes an angle of $90°$ with x-axis, so its slope is $\tan \pi/2 = \infty$.
- *The slope of a line equally inclined with the axis is* 1 *or* -1, *as it makes* $45°$ *or* $135°$ *angle with x-axis. Slope of a line in terms of coordinates of any two points on it is given as shown below:*

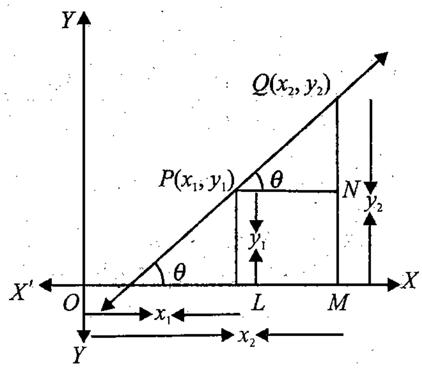


Fig. 1.28

From the figure, slope is

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

Angle between Two Lines

The angle θ between the lines having slope m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}$$

Proof:

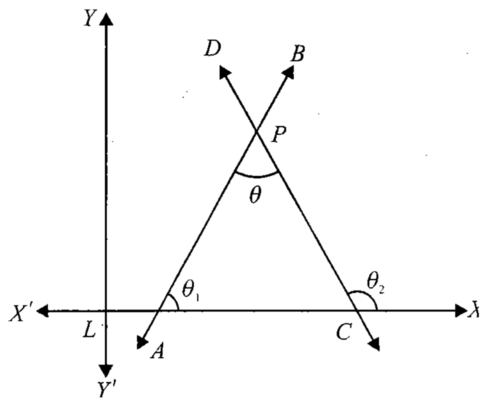


Fig. 1.29

Let m_1 and m_2 be the slopes of two given lines AB and CD which intersect at a point P and make angles θ_1 and θ_2, respectively with the positive direction of x-axis.

Then,

$$m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let $\angle APC = \theta$ be the angle between the given lines.

Then,

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(i)}$$

Since $\angle APD = \pi - \theta$ is also the angle between AB and CD.

Therefore,

$$\tan\angle APD = \tan(\pi - \theta) = -\tan\theta$$

$$= -\frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(ii) [Using (i)]}$$

From (i) and (ii), we find that the angle between two lines of slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\pm\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

The acute angle between the lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Example 1.41 **If $A(-2, 1)$, $B(2, 3)$, and $C(-2, -4)$ are three points, then find the angle between BA and BC.**

Sol. Let m_1 and m_2 be the slopes of BA and BC, respectively. Then,

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$$

and

$$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right|$$

$$= \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

Example 1.42 **Determine x so that the line passing through $(3, 4)$ and $(x, 5)$ makes $135°$ angle with the positive direction of x-axis**

Sol. Since the line passing through $(3, 4)$ and $(x, 5)$ makes an angle of $135°$ with x-axis; therefore, its slope is

$$\tan 135° = -1.$$

But, the slope of the line is also equal to

$$\frac{5-4}{x-3}$$

$$\Rightarrow \quad -1 = \frac{5-4}{x-3}$$

$$\Rightarrow \quad -x + 3 = 1$$

$$\Rightarrow \quad x = 2$$

Condition for Parallelism of Lines

It two lines of slopes m_1 and m_2 are parallel, then the angle θ between is $0°$.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1 m_2} = 0$$

$$\Rightarrow \quad m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

Condition for Perpendicularity of Two Lines

If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is $90°$.

$$\therefore \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{1 + m_1 \cdot m_2}{m_2 - m_1} = 0$$

$$\Rightarrow \quad m_1 m_2 = -1$$

Thus, when lines are perpendicular, the product of their slope is -1.

If m is the slope of a line, then the slope of a line perpendicular to it is $-(1/m)$.

Example 1.43 **Let $A(6, 4)$ and $B(2, 12)$ be two given points. Find the slope of a line perpendicular to AB.**

Sol. Let m be the slope of AB, than

$$m = \frac{12-4}{2-6} = \frac{8}{-4} = -2$$

So, the slope of a line $\perp$ to AB

$$= -\frac{1}{m} = \frac{1}{2}$$

Intercepts of a Line on the Axes

If a straight line cuts x-axis at A and the y – axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

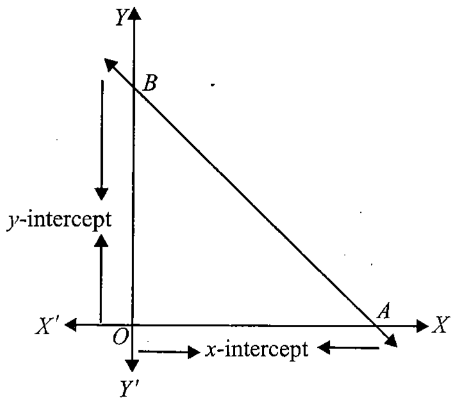


Fig. 1.30

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinates axes.

In figure $OA = x$-intercept, $OB = y$-intercept

OA is positive or negative according as A lies on OX or OX' respectively.

Similarly OB is positive or negative according as B lies on OY or OY' respectively.

Note:

- *If line has equal intercept on axes, then its slope is* -1.

Equation of a Line Parallel to x-Axis

Equation of a line parallel to x-axis at a distance b from it.

Then, clearly the ordinates of each point on AB is b.

Thus, AB can be considered as the locus of a point at a distance b from x-axis.

Thus, if $P(x, y)$ is any point on AB, then $y = b$.

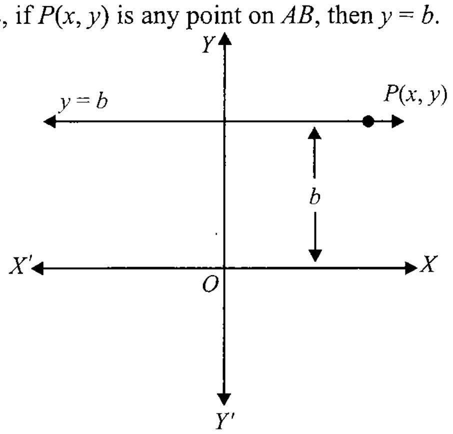


Fig. 1.31

Hence, the equation of a line parallel to x-axis at a distance b from it is $y = b$.

Since x-axis is parallel to itself at a distance 0 from it; therefore, the equation of x-axis is $y = 0$.

Equation of a Line Parallel to y-Axis

Let AB be a line parallel to y-axis and at a distance a from it. Then the abscissa of every point on AB is a. So it can be treated as the locus of a point at a distance a from y-axis.

Thus, if $P(x, y)$ is any point in AB, then $x = a$.

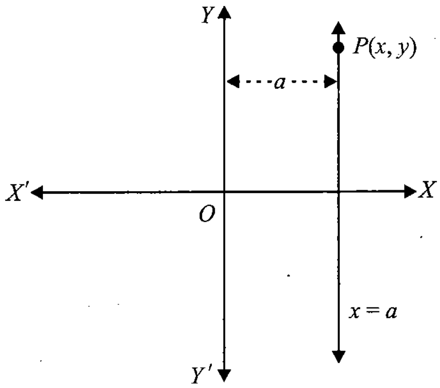


Fig. 1.32

DIFFERENT FORMS OF LINE

Slope Intercept Form of a Line

The equation of a line with slope m that makes an intercept c on y-axis is

$$y = mx + c$$

Proof: Let the given line intersects y-axis at Q and makes an angle θ with x-axis. Then $m = \tan\theta$. Let $P(x, y)$ be any point on the line as shown in the figure.

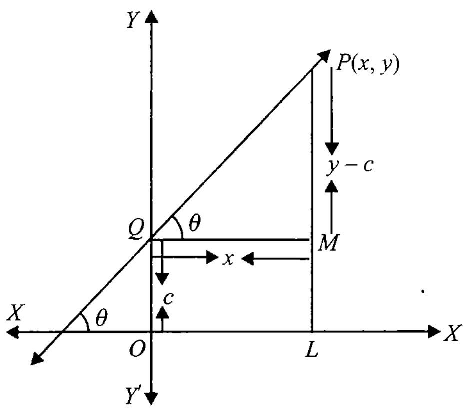


Fig. 1.33

From ΔPMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

$$\Rightarrow \quad m = \frac{y-c}{x}$$

$$\Rightarrow \quad y = mx + c$$

which is the required equation of the line.

Point–Slope Form of a Line

The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof: Let $Q(x_1, y_1)$ be the point through which the line passes and let $P(x, y)$ be any point on the line. Then, the slope of the line is

$$\frac{y-y_1}{x-x_1}$$

But m is the slope of the line. Therefore,

$$m = \frac{y-y_1}{x-x_1} \Rightarrow y - y_1 = m(x - x_1)$$

Thus, $y - y_1 = m(x - x_1)$ is the required equation of the line.

Example 1.44 **Find the equation of a straight line which cuts-off an intercept of 5 units on negative direction of y-axis and makes an angle of 120° with the positive direction of x-axis.**

Sol. Here, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$ and $c = -5$. So, the equation of the line is

$$y = -\sqrt{3}\,x - 5 \Rightarrow \sqrt{3}\,x + y + 5 = 0$$

Example 1.45 **Find the equation of a straight line cutting off an intercept – 1 from y-axis and being equally inclined to the axes.**

Sol. Since the required line is equally inclined with coordinates axes; therefore, it makes an angle of either 45° or 135° with the x-axes.

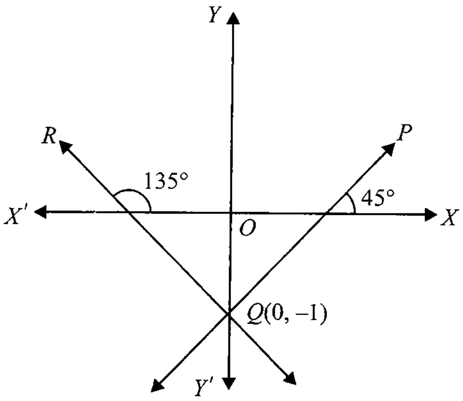


Fig. 1.34

So, its slope is either $m = \tan 45°$ or $m = \tan 135°$, i.e., $m = 1$ or -1. It is given that $c = -1$. Hence, the equations of the lines are

$$y = x - 1 \text{ and } y = -x - 1$$

Example 1.46 **Find the equation of a line that has y- intercept 4 and is perpendicular to the line joining (2, – 3) and (4, 2).**

Sol. Let m be the slope of the required line.

Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$. Therefore,

$$m \times \text{slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{2+3}{4-2} = -1$$

$$\Rightarrow \quad m = -\frac{2}{5}$$

The required line cuts-off an intercept 4 on y-axis, so $c = 4$.

Hence, the equation of the required line is

$$y = -\frac{2}{5}x + 4$$

$$\Rightarrow \quad 2x + 5y - 20 = 0$$

Two-Point Form of a Line

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof:

Let m be the slope of the passing through (x_1, y_1) and (x_2, y_2), then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \text{ (Using point – slope form)}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two-point form.

Example 1.47 **Find the equation of the perpendicular bisector of the line segment joining the points A(2, 3) and B(6,–5).**

Sol. The slope of AB is given by m

$$= \frac{-5-3}{6-2} = -2$$

$\Rightarrow$ The slope of a line $\perp$ to AB

$$= -\frac{1}{m} = \frac{1}{2}$$

Let P be the midpoint of AB, then the coordinates of P are

$\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$, i.e., $(4, -1)$

Thus, the required line passes through $P(4, -1)$ and has slope 1/2.

So its equation is

$$y + 1 = \frac{1}{2}(x-4) \text{ [(Using } y - y_1 = m(x - x_1)]$$

or $\quad x - 2y - 6 = 0$

Example 1.48 **Find the equations of the medians of the triangle ABC whose vertices are $A(2, 5)$, $B(-4, 9)$, and $C(-2, -1)$.**

Sol. Let D, E, F be the midpoints of BC, CA and AB, respectively. Then the coordinates of these points are $D(-3, 4)$, $E(0, 2)$, and $F(-1, 7)$, respectively.

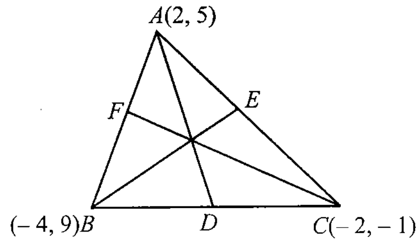


Fig. 1.35

The median AD passes through points $A(2, 5)$ and $D(-3, 4)$.

So, the equation of AD is

$$y - 5 = \frac{4-5}{-3-2}(x-2)$$

$$\Rightarrow \quad y - 5 = \frac{1}{5}(x-2)$$

$$\Rightarrow \quad x - 5y + 23 = 0$$

The median BE passes through points $B(-4, 9)$ and $E(0, 2)$

So, the equation of median BE is

$$(y-9) = \left(\frac{2-9}{0+4}\right)(x+4)$$

$$\Rightarrow \quad 7x + 4y - 8 = 0$$

Similarly, the equation of the median CF is

$$(y+1) = \frac{7+1}{-1+2}(x+2)$$

$$\Rightarrow \quad 8x - y + 15 = 0$$

Example 1.49 **In what ratio does the line joining the points (2, 3) and (4, 1) divide the segment joining the points (1, 2) and (4, 3)?**

Sol. The equation of the line joining the points (2, 3) and (4, 1) is

$$y - 3 = \frac{1-3}{4-2}(x-2)$$

$$\Rightarrow \quad y - 3 = -x + 2$$

$$\Rightarrow \quad x + y - 5 = 0 \qquad \text{(i)}$$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at point P in the ratio $\lambda : 1$.

Then the coordinates of P are

$$\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$$

Clearly, P lies on (i)

$$\Rightarrow \frac{4\lambda+1}{\lambda+1} + \frac{3\lambda+2}{\lambda+1} - 5 = 0$$

$$\Rightarrow \quad \lambda = 1$$

Hence, the required ratio is $\lambda : 1$, i.e., 1 : 1.

Example 1.50 **Find the equations of the altitudes of the triangle whose vertices are $A(7, -1)$, $B(-2, 8)$, and $C(1, 2)$ and hence orthocentre of triangle.**

Sol.

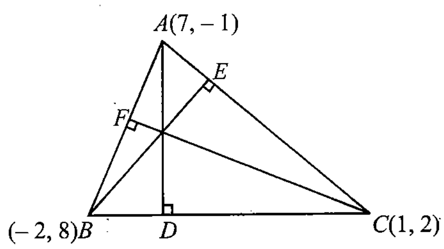


Fig. 1.36

Let AD, BE, and CF be three altitudes of triangle ABC.

Let m_1, m_2, and m_3 be the slopes of AD, BE, and CF, respectively.

Then, $AD \perp BC$

$\Rightarrow$ Slope of AD × Slope of $BC = -1$

$$\Rightarrow \quad m_1 \times \left(\frac{2-8}{1+2}\right) = -1$$

$$\Rightarrow \quad m_1 = \frac{1}{2}$$

$BE \perp AC$

$\Rightarrow$ Slope of BE × Slope of $AC = -1$

$$\Rightarrow \quad m_2 \times \left(\frac{-1-2}{7-1}\right) = -1$$

$$\Rightarrow \quad m_2 = 2$$

And, $CF \perp AB$

$\Rightarrow$ Slope of CF × Slope of $AB = -1$

$$\Rightarrow \quad m_3 \times \frac{-1-8}{7+2} = -1$$

$$\Rightarrow \quad m_3 = 1$$

Since AD passes through $A(7, -1)$ and has slope

$$m_1 = 1/2.$$

So, its equation is

$$y + 1 = \frac{1}{2}(x - 7)$$

$$\Rightarrow \quad x - 2y - 9 = 0$$

Similarly, equation of BE is

$$y - 8 = 2(x + 2)$$

$$\Rightarrow \quad 2x - y + 12 = 0$$

Equation of CF is

$$y - 2 = 1\,(x - 1)$$

$$\Rightarrow \quad x - y + 1 = 0$$

Intercept Form of a Line

The equation of a line which cuts-off intercepts a and b, respectively from the x and y-axes is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof: Let AB be the line which cuts-off intercepts $OA = a$ and $OB = b$ on the x and y-axes respectively. Let $P(x, y)$ be any point on the line.

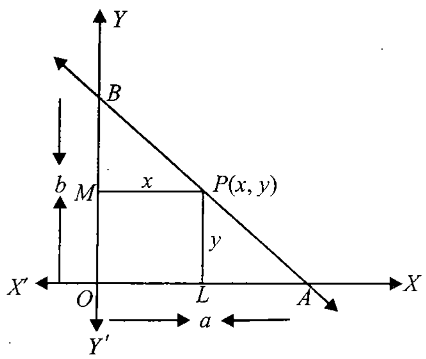


Fig. 1.37

From the diagram, we get that

Area of ΔOAB = Area of ΔOPA + Area of ΔOPB

$$\Rightarrow \quad \frac{1}{2} OA \times OB = \frac{1}{2} OA \times PL + \frac{1}{2} OB \times PM$$

$$\Rightarrow \quad \frac{1}{2} ab = \frac{1}{2} ay + \frac{1}{2} bx$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

Example 1.51 **Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.**

Sol. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(i)}$$

This passes through (3, 4), therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \qquad \text{(ii)}$$

It is given that $a + b = 14$

$$\Rightarrow \quad b = 14 - a$$

Putting $b = 14 - a$ in (ii), we get

$$\frac{3}{a} + \frac{4}{14 - a} = 1$$

$$\Rightarrow \quad a^2 - 13a + 42 = 0$$

$$\Rightarrow \quad (a - 7)(a - 6) = 0$$

$$\Rightarrow \quad a = 7, 6$$

For $a = 7$, $b = 14 - 7 = 7$

and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1$$

or $\quad x + y = 7$ and $4x + 3y = 24$

Example 1.52 **Find the equation of the straight line that**

i. makes equal intercepts on the axes and passes through the point (2, 3),

ii. passes through the point (–5, 4) and is such that the portion intercepted between the axes is divided by the point in the ratio 1 : 2.

Sol. i. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Since it makes equal intercepts on the coordinates axes, therefore $a = b$.

So, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or } x + y = a$$

The line passes through the point (2, 3).

Therefore, $\quad 2 + 3 = a$

$$\Rightarrow \quad a = 5$$

Thus, the equation of the required line is $x + y = 5$.

ii. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Clearly, this line meets the coordinate axes at $A(a, 0)$ and $B(0, b)$, respectively.

The coordinates of the point that divides the line joining $A(a, 0)$ and $B(0, b)$ in the ratio 1 : 2 are

$$\left(\frac{1(0)+2(a)}{1+2}, \frac{1(b)+2(0)}{1+2}\right) = \left(\frac{2a}{3}, \frac{b}{3}\right)$$

It is given that the point (–5, 4) divides AB in the ratio 1 : 2.

Therefore, $2a/3 = -5$ and $b/3 = 4$

$\Rightarrow$ $a = -15/2$ and $b = 12$

Hence, the equation of the required line is

$$-\frac{x}{\frac{15}{2}} + \frac{y}{12} = 1$$

$\Rightarrow$ $8x - 5y + 60 = 0$

Normal Form or Perpendicular Form of a Line

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with +ve direction of x-axis is

$$x \cos \alpha + y \sin \alpha = p$$

Proof: Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ = \alpha$.

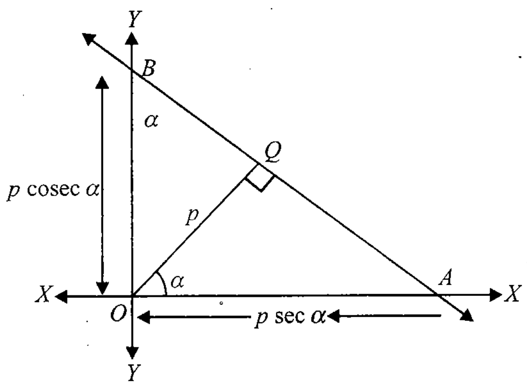


Fig. 1.38

From the diagram, using the intercept form, we get

Equation of line AB is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1$$

or $x \cos \alpha + y \sin \alpha = p$

Example 1.53 **The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.**

Sol. Here $p = 7$ and $\alpha = 30°$

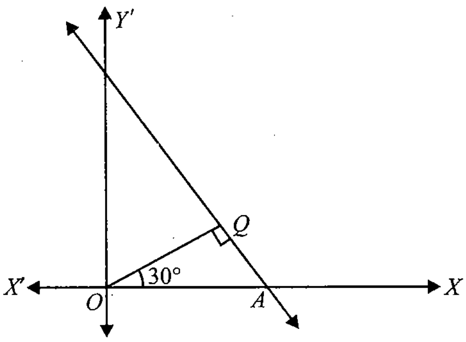


Fig. 1.39

Equation of the required line is

$$x \cos 30° + y \sin 30° = 7$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 7$$

$$\Rightarrow \quad \sqrt{3}x + y = 14$$

ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

Let the angle θ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\tan \theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

Proof: Let m_1 and m_2 be the slopes of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively.

Then,

$$m_1 = -a_1/b_1 \text{ and } m_2 = -a_2/b_2$$

Now,

$$\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\Rightarrow \quad \tan \theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$$\Rightarrow \quad \tan \theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

$$\Rightarrow \quad \theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

Condition for the Lines to be Parallel

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2$$

$$\Rightarrow \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

Condition for the Lines to be Perpendicular

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{b_1}\right) \times \left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$$

It follows from the above discussion that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

i. Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

ii. Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

iii. Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

iv. Perpendicular, if $a_1a_2 + b_1b_2 = 0$

Example 1.54 **Find the angle between the pairs of straight lines**

i. $x - y\sqrt{3} - 5 = 0$ and $\sqrt{3}x + y - 7 = 0$

ii. $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$

Sol. i. The equations of two straight lines are

$$x - y\sqrt{3} - 5 = 0 \quad \text{(i)}$$

and

$$\sqrt{3}x + y - 7 = 0 \quad \text{(ii)}$$

Let m_1 and m_2 be the slopes of these two lines. Then,

$$m_1 = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

and $$m_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

Clearly, $m_1m_2 = -1$. Thus, the two lines are at right angle.

ii. Let m_1 and m_2 be the slopes of the straight lines $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$, respectively. Then,

$$m_1 = 2 - \sqrt{3} \text{ and } m_2 = 2 + \sqrt{3}$$

Let θ be the angle between the lines. Then,

$$\tan\theta = \pm\left(\frac{(m_1 - m_2)}{1 + m_1m_2}\right)$$

$$= \pm\left(\frac{(2-\sqrt{3}) - (2+\sqrt{3})}{1 + (2-\sqrt{3})(2+\sqrt{3})}\right)$$

$$= \pm\left(-\frac{2\sqrt{3}}{1+4-3}\right) = \pm\sqrt{3}$$

So, the acute angle between the lines is given by

$$\tan\theta = |\pm\sqrt{3}| = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Example 1.55 **A straight canal is $4\frac{1}{2}$ miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four east from the place. Does it lie by the nearest edge of the canal?**

Sol. Let the given place be O. Take this as the origin and the east and north directions through O as the x-and y-axes, respectively.

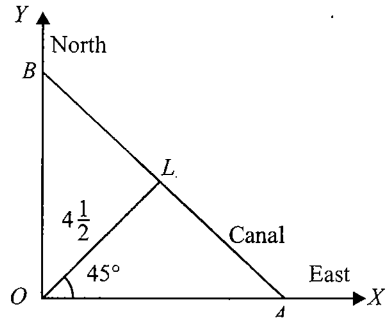


Fig. 1.40

Let AB be the nearest edge of the canal. From the question, OL is perpendicular to AB such that $OL = 4\frac{1}{2}$ miles and $\angle LOA = 45°$

So, the equation of the canal is

$$x\cos 45° + y\sin 45° = 4\frac{1}{2}$$

$$\Rightarrow \quad \sqrt{2}(x + y) = 9 \quad \text{(i)}$$

The position of the village is (4, 3). The village will lie on the edge of the canal if (4, 3) satisfies the Eq. (i).

Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

Example 1.56 **Reduce the line $2x - 3y + 5 = 0$ in slope-intercept, intercept, and normal forms.**

Sol. $y = \frac{2x}{3} + \frac{5}{3}$, $\tan\theta = m = 2/3$, $c = \frac{5}{3}$

Intercept form:

$$\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1, a = -\frac{5}{2}, b = \frac{5}{3}$$

Normal form:

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

$$\sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = -\frac{2}{\sqrt{13}}, p = \frac{5}{\sqrt{13}}$$

Example 1.57 **A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, find the other vertices of the rectangle.**

Sol. Let $A \equiv (1, 2)$, $C \equiv (5, 5)$

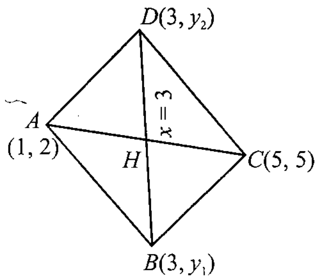


Fig. 1.41

Since vertices B and D lie on line $x = 3$; therefore, let $B \equiv (3, y_1)$ and $D \equiv (3, y_2)$.

Now since AC and BD bisect each other, therefore, middle points of AC and BD will be same

$$\frac{y_1 + y_2}{2} = \frac{2+5}{2} \text{ or } y_1 + y_2 = 7 \quad \text{(i)}$$

Also $BD^2 = AC^2$

$\therefore$ $(y_1 - y_2)^2 = (1-5)^2 + (2-5)^2 = 25$

or $y_1 - y_2 = \pm 5$ (ii)

Solving (i) and (ii), we get

$$y_1 = 6, y_2 = 1 \text{ or } y_1 = 1, y_2 = 6$$

Hence, other vertices of the rectangle are (3, 1) and (3, 6).

Example 1.58 **A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$, find the equation of the other sides of the triangle.**

Sol. Given line is

$$x + y - 2 = 0 \quad \text{(i)}$$

Its slope $m_1 = -1$.

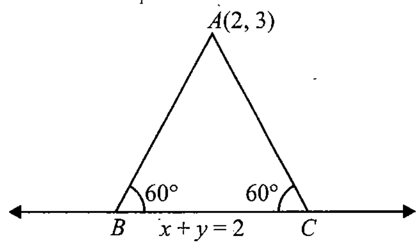


Fig. 1.42

Let the slope of the line be m which makes an angle of 60° with line in Eq. (i), then

$$\tan 60° = \left|\frac{m_1 - m}{1 + m_1 m}\right| \text{ or } \sqrt{3} = \left|\frac{-1-m}{1-m}\right|$$

or $$\sqrt{3} = \left|\frac{1+m}{m-1}\right| \text{ or } \frac{1+m}{m-1} = \pm\sqrt{3}$$

or $$1 + m = \pm\sqrt{3}(m-1)$$

$\Rightarrow$ $$m = \frac{\sqrt{3}+1}{\sqrt{3}-1}, \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= 2+\sqrt{3}, 2-\sqrt{3}$$

Equation of other two sides of the triangle are

$$y - 3 = (2+\sqrt{3})(x-2)$$

and $$y - 3 = (2-\sqrt{3})(x-2)$$

Example 1.59 **Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation of the other diagonal.**

Sol.

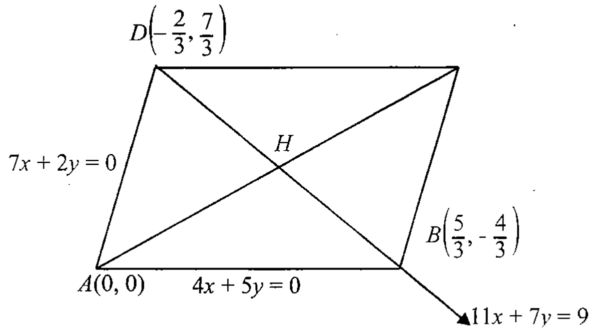


Fig. 1.43

Let the equation of sides AB and AD of the parallelogram $ABCD$ be as given in Eqs. (i) and (ii), respectively, i.e.,

$$4x + 5y = 0 \quad \text{(i)}$$

and $$7x + 2y = 0 \quad \text{(ii)}$$

Solving (i) and (ii), we have

$$x = 0, y = 0$$

$$\therefore \quad A \equiv (0, 0)$$

Equation of one diagonal of the parallelogram is

$$11x + 7y = 9 \quad \text{(iii)}$$

Clearly, $A(0, 0)$ does not lie on diagonal as shown in Eq. (iii), therefore Eq. (iii) is the equation of diagonal BD.

Solving (i) and (iii), we get $B \equiv \left(\frac{5}{3}, -\frac{4}{3}\right)$

Solving (ii) and (iii), we get $D \equiv \left(-\frac{2}{3}, \frac{7}{3}\right)$

Since H is the middle point of BD

$$\therefore \quad H \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$$

Now, equation of diagonal AC which passes through $A(0, 0)$ and $H\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$y - 0 = \frac{0 - \frac{1}{2}}{0 - \frac{1}{2}}(x - 0) \text{ or } y - x = 0$$

Example 1.60 **A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation of the line AC, so that $AB = AC$.**

Sol.

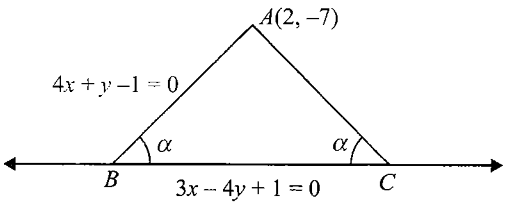


Fig. 1.44

Let the equation of BC be

$$3x - 4y + 1 = 0 \quad \text{(i)}$$

and the equation of AB be

$$4x + y - 1 = 0 \quad \text{(ii)}$$

Since $AB = AC$ $\therefore \angle ABC = \angle ACB = \alpha$ (say)

Slope of line BC = 4/3 and slope of AB = – 4.

Let slope of $AC = m$, equating the two values of tan α, we get

$$\left|\frac{-4 - \frac{3}{4}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\frac{3}{4} - m}{1 + \frac{3}{4}m}\right|$$

$$\Rightarrow \quad \pm\frac{19}{8} = \frac{3 - 4m}{4 + 3m}$$

$$\Rightarrow \quad m = 52/89 \text{ or } m = -4$$

Therefore, equation of AC is

$y + 7 = -(52/89)(x - 2)$ or $52x + 89y + 519 = 0$

Example 1.61 **A variable straight line is drawn through the point of intersection of the straight lines $x/a + y/b = 1$ and $x/b + y/a = 1$ and meets the coordinate axes at A and B. Show that the locus of the midpoint of AB is the curve $2xy(a + b) = ab(x + y)$.**

Sol.

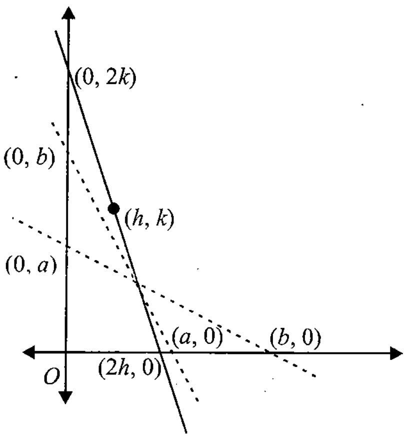


Fig. 1.45

Given lines are

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(i)}$$

and

$$\frac{x}{b} + \frac{y}{a} = 1 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = \frac{ab}{a+b} \text{ and } y = \frac{ab}{a+b}$$

Now a variable line passing through these points meets the axis at points A and B. Let the midpoint of AB be (h, k) whose locus is to be found.

Then coordinates of A and B are $(2h, 0)$ and $(0, 2k)$.

Now points A, B and $[ab/(a + b), ab/(a + b)]$ are collinear.

Then

$$\Delta = \frac{1}{2}\begin{vmatrix} 2h & 0 & 1 \\ 0 & 2k & 1 \\ \frac{ab}{a+b} & \frac{ab}{a+b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4hk - 2h\frac{ab}{a+b} - 2k\frac{ab}{a+b} = 0$$

$$\Rightarrow \quad 2xy(a + b) = ab(x + y)$$

Example 1.62 **If the line $(x/a) + (y/b) = 1$ moves in such a way that $(1/a^2) + (1/b^2) = (1/c^2)$ where c is a constant, prove that the foot of the perpendicular from the origin on the straight line describes the circle $x^2 + y^2 = c^2$.**

Sol. Variable line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(i)}$$

Any line perpendicular to Eq. (i) and passing through the origin will be

$$\frac{x}{b}-\frac{y}{a}=0 \quad \text{(ii)}$$

Now the foot of the perpendicular from the origin to the line Eq. (i) is the point of intersection of Eq. (i) and (ii).

Let be $P(\alpha, \beta)$, then $\frac{\alpha}{a}+\frac{\beta}{b}=1$ (iii)

and

$$\frac{\alpha}{b}-\frac{\beta}{a}=0 \quad \text{(iv)}$$

Squaring and adding Eqs. (iii) and (iv), we get

$$\alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\beta^2\left(\frac{1}{b^2}+\frac{1}{a^2}\right)=1$$

But

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2} \text{ (given)}$$

Hence c is a constant and a, b are parameters (variables). Therefore,

$$(\alpha^2+\beta^2)\frac{1}{c^2}=1.$$

Hence, the locus of $P(\alpha, \beta)$ is

$$x^2+y^2=c^2$$

Equation of a Line Parallel to a Given Line

The equation of a line parallel to a given line $ax + by + c = 0$ is

$$ax+by+\lambda=0$$

where λ is *a* constant. Find λ by using given condition.

Equation of a Line Perpendicular to a Given Line

The equation of a line perpendicular to a given line $ax + by + c = 0$ is

$$bx-ay+\lambda=0$$

where λ is a constant. Find λ by using given condition.

Example 1.63 **Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5, -6)$.**

Sol. The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x-2y+\lambda=0 \quad \text{(i)}$$

This passes through $(5, -6)$, therefore we get

$$3\times(5)-2\times(-6)+\lambda=0$$

$$\Rightarrow \quad \lambda=-27$$

Putting $\lambda = -27$ in (i), we get

$$3x-2y-27=0$$

which is the required equation.

Example 1.64 **Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$.**

Sol. The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x-3y+\lambda=0 \quad \text{(i)}$$

This passes through the point (3, 4), therefore we get

$$2\times(3)-3\times(4)+\lambda=0$$

$$\Rightarrow \quad \lambda=6$$

Putting $\lambda = 6$ in (i), we get

$$2x-3y+6=0$$

which is the required equation.

Example 1.65 **Find the coordinates of the foot of the perpendicular drawn from the point $(1, -2)$ on the line $y = 2x + 1$.**

Sol. Let M be the foot of the perpendicular drawn from $P(1, -2)$ on the line $y = 2x + 1$.

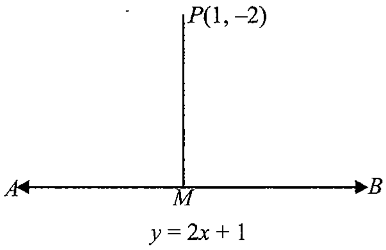


Fig. 1.46

Then M is the point of intersection of $y = 2x + 1$ and a line passing through $P(1, -2)$ and perpendicular to $y = 2x + 1$.

The equation of a line perpendicular to $y = 2x + 1$ or $2x - y + 1 = 0$ is

$$x+2y+\lambda=0 \quad \text{(i)}$$

This passes through $P(1, -2)$, therefore we get

$$\Rightarrow \quad 1-4+\lambda=0$$

$$\Rightarrow \quad \lambda=3$$

Putting $\lambda = 3$ in (i), we get

$$x+2y+3=0$$

Point M is the point of intersection of $2x - y + 1 = 0$ and $x + 2y + 3 = 0$

Solving these equations by cross-multiplication, we get

$$\frac{x}{-5}=\frac{y}{-5}=\frac{1}{5}$$

$$\Rightarrow \quad x=-1 \text{ and } y=-1$$

Hence, the coordinates of the foot of the perpendicular are $(-1, -1)$.

Example 1.66 **Find the image of the point (–8, 12) with respect to the line mirror $4x + 7y + 13 = 0$.**

Sol. Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$.

Then, PQ is perpendicularly bisected at R.

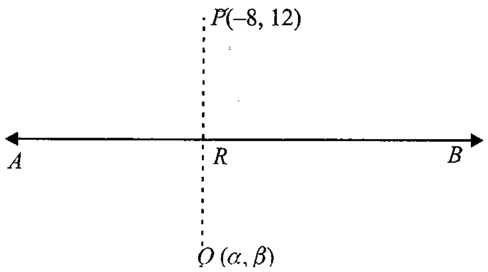


Fig. 1.47

The coordinates of R are $\left(\frac{\alpha-8}{2}, \frac{\beta+12}{2}\right)$

Since R lies on $4x + 7y + 13 = 0$, we get

$2\alpha - 16 + (7\beta + 84)/2 + 13 = 0$

$\Rightarrow \quad 4\alpha + 7\beta + 78 = 0 \quad$ (i)

Since $PQ \perp AB$, therefore (slope of AB) × (slope of PQ) $= -1$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad$ (ii)

Solving (i) and (ii), we get

$$\alpha = -16, \beta = -2$$

Hence, the image of (–8, 12) in the line mirror $4x + 7y + 13 = 0$ is (–16, – 2).

Example 1.67 **A ray of the light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected. Find the equation of the line containing the reflected ray.**

Sol. Solving the equations of LM and PA coordinates of A can be obtained.

If slope of AQ is determined, then the equation of AQ can be determined.

If slope of AQ is m, then equating the two values of $\tan\theta$ (considering the angles between AL and AP and between AM and AQ, m can be found.

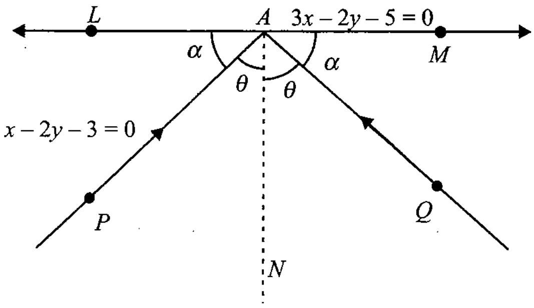


Fig. 1.48

Equation of line LM is

$$3x - 2y - 5 = 0 \quad \text{(i)}$$

Equation of PA is

$$x - 2y - 3 = 0 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = 1, y = -1$$

$$\therefore \quad A \equiv (1, -1)$$

Let slope of $AQ = m$, slope of $LM = 3/2$, slope of $PA = 1/2$

Let $\quad \angle LAP = \angle QAM = \alpha$

As $\quad \angle LAP = \alpha$

$$\Rightarrow \quad \tan\alpha = \left|\frac{\frac{3}{2} - \frac{1}{2}}{1 + \frac{3}{2} \times \frac{1}{2}}\right| = \frac{4}{7} \quad \text{(iii)}$$

Again $\quad \angle QAM = \alpha$

$$\therefore \quad \tan\alpha = \left|\frac{m - \frac{3}{2}}{1 + \frac{3}{2}m}\right| = \left|\frac{2m-3}{2+3m}\right| \quad \text{(iv)}$$

From Eq (iii) and (iv), we have

$$\left|\frac{2m-3}{2+3m}\right| = \frac{4}{7}$$

or $\quad \frac{2m-3}{2+3m} \pm \frac{4}{7}$

$$\therefore \quad m = \frac{1}{2}, \frac{29}{2}$$

But slope of $AP = \frac{1}{2}$

$\therefore$ Slope of $AQ = \frac{29}{2}$

Now, the equation of AQ will be

$$y + 1 = \frac{29}{2}(x - 1)$$

or $\quad 29x - 2y - 31 = 0$

Example 1.68 **A ray of light is sent along the line $2x - 3y = 5$. After refracting across the line $x + y = 1$ it enters**

the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.

Sol.

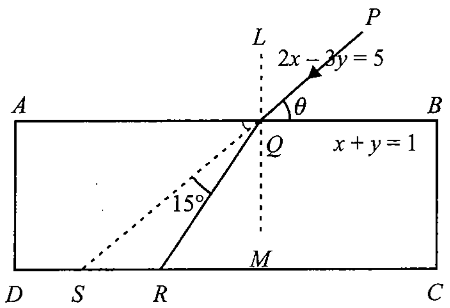


Fig. 1.49

Equation of line AB is

$$x + y = 1 \quad \text{(i)}$$

Equation of line QP is

$$2x - 3y = 5 \quad \text{(ii)}$$

QR is the refracted ray. According to question $\angle SQR = 15°$

Solving Eqs. (i) and (ii), we get

$$x = \frac{8}{5} \text{ and } y = -\frac{3}{5}$$

$$\Rightarrow \quad Q \equiv \left(\frac{8}{5}, -\frac{3}{5}\right)$$

$$\text{Slope of } QP = \frac{2}{3}$$

$$\Rightarrow \quad \text{Slope of } QS = \frac{2}{3}$$

Let slope of $\quad QR = m$

But $\quad \angle SQR = 15°$

$$\Rightarrow \quad \tan 15° = \left|\frac{\frac{2}{3} - m}{1 + \frac{2}{3}m}\right|$$

$$\Rightarrow \quad 2 - \sqrt{3} = \left|\frac{2 - 3m}{3 + 2m}\right|$$

$$\Rightarrow \quad \frac{2 - 3m}{3 + 2m} = \pm (2 - \sqrt{3})$$

$$\Rightarrow \quad 2 - 3m = \pm [6 - 3\sqrt{3} + (4 - 2\sqrt{3})m]$$

$$\text{or} \quad 2 - 3m = \begin{cases} 6 - 3\sqrt{3} + (4 - 2\sqrt{3})m \\ -6 + 3\sqrt{3} - (4 - 2\sqrt{3})m \end{cases}$$

$$\Rightarrow \quad m = \frac{3\sqrt{3} - 4}{7 - 2\sqrt{3}}, \frac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$$

For required line $m = \dfrac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$

Concept Application Exercise 1.3

1. Find the angle between lines $x = 2$ and $x - 3y = 6$.
2. If the coordinates of the points A, B, C, and D, be (a, b), (a', b'), $(-a, b)$, and $(a', -b')$, respectively, then find the equation of the line bisecting the line segments AB and CD.
3. If the coordinates of the vertices of the triangle ABC are $(-1, 6)$, $(-3, -9)$, and $(5, -8)$, respectively, then find the equation of the median through C.
4. Find the equation of the line perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ and passing through a point at which it cuts x-axis.
5. If the middle points of the sides BC, CA, and AB of the triangle ABC are $(1, 3)$, $(5, 7)$ and $(-5, 7)$, respectively, then find the equation of the side AB.
6. Find the equations of the lines which pass through the origin and are inclined at an angle $\tan^{-1} m$ to the line $y = mx + c$.
7. If $(-2, 6)$ is the image of the point $(4, 2)$ with respect to line $L = 0$, then find the equation of line L.
8. If the lines $x + (a - 1)y + 1 = 0$ and $2x + a^2y - 1 = 0$ are perpendicular, then find the values of a.
9. Find the area bounded by the curves $x + 2|y| = 1$ and $x = 0$.
10. Find the equation of the straight line passing through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$.
11. A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. Then find the equation to the line AB so that the triangle OAB is equilateral.
12. Find the equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is -1.
13. A straight line through the point $A(3, 4)$ is such that its intercept between the axis is bisected at A. Find its equation.
14. The diagonals AC and BD of a rhombus intersect at $(5, 6)$. If $A \equiv (3, 2)$, then find the equation of diagonal BD.
15. If the foot of the perpendicular from the origin to a straight line is at the point $(3, -4)$. Then find the equation of the line.
16. If we reduce $3x + 3y + 7 = 0$ to the form $x \cos \alpha + y \sin \alpha = p$, then find the value of p.
17. Find the equation of the straight line which passes through the origin and makes angle 60° with the line $x + \sqrt{3}\,y + 3\sqrt{3} = 0$.

18. A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at A and B, respectively. If a point $P(1, 5)$ on the line AB is such that $AP : PB = 2 : 1$ (internally), find point A.

19. In the given figure PQR is an equilateral triangle and $OSPT$ is a square. If $OT = 2\sqrt{2}$ units, find the equation of lines OT, OS, SP, QR, PR, and PQ.

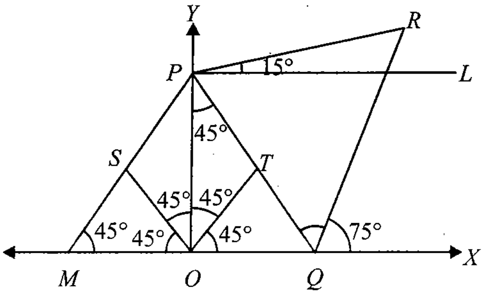


Fig. 1.50

20. Find the obtuse angle between the lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$.

21. Two fixed points A and B are taken on the coordinates axes such that $OA = a$ and $OB = b$. Two variable point A' and B' are taken on the same axes such that $OA' + OB' = OA + OB$. Find the locus of the point of intersection of AB' and $A'B$.

DISTANCE FORM OF A LINE (PARAMETRIC FORM)

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

where r is the distance of the point (x, y) on the line from point (x_1, y_1).

Proof: Let the given line meets x-axis at A, y-axis at B and passes through the point $Q(x_1, y_1)$.

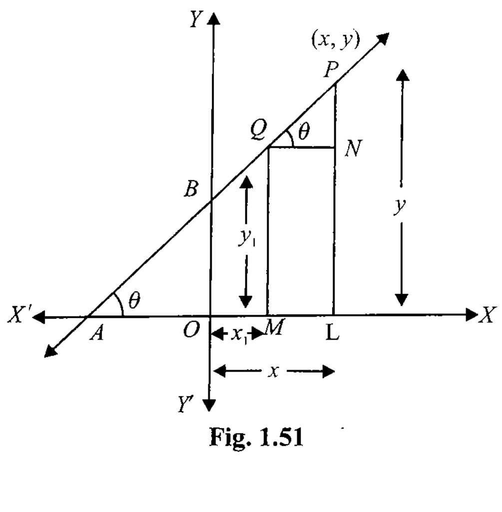


Fig. 1.51

Let $P(x, y)$ be any point on the line at a distance r from $Q(x_1, y_1)$ i.e., $PQ = r$.

In ΔPQN, we have

$$\cos\theta = \frac{QN}{PQ} = \frac{x - x_1}{r}$$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \qquad \text{(i)}$$

and

$$\sin\theta = \frac{PN}{PQ}$$

$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance form.

$\Rightarrow \; x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$

$\Rightarrow \; x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If P is on the right side of (x_1, y_1), then r is positive and if P is on the left side of (x_1, y_1), then r is negative. Since different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

In the parametric form, one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point (x_1, y_1) on the line $(x - x_1)/\cos\theta = (y - y_1)/\sin\theta$ there are two points viz., $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Example 1.69 **A straight line is drawn through the point $P(2, 3)$ and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P.**

Sol. Here $(x_1, y_1) = (2, 3)$, $\theta = 30°$, the equation of the line is

$$\frac{x - 2}{\cos 30°} = \frac{y - 3}{\sin 30°}$$

$$\Rightarrow \quad \frac{x - 2}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{1}{2}}$$

$\Rightarrow \quad x - 2 = \sqrt{3}(y - 3)$

$\Rightarrow \quad x - \sqrt{3}y = 2 - 3\sqrt{3}$

Points on the line at a distance 4 from $P(2, 3)$ are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

or $(2 \pm 4\cos 30°, 3 \pm 4\sin 30°)$

or $(2 \pm 2\sqrt{3}, 3 \pm 2)$ or $(2 + 2\sqrt{3}, 5)$ and $(2 - 2\sqrt{3}, 1)$

Example 1.70 **Find the equation of the line passing through the point $A(2,3)$ and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line $x + y + 1 = 0$.**

Sol.

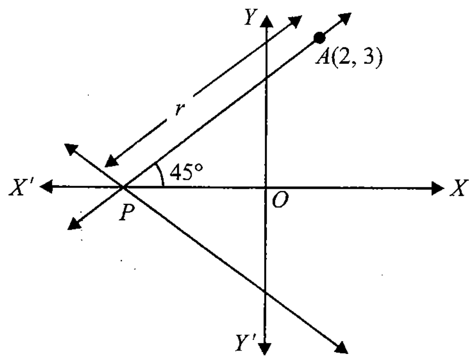


Fig. 1.52

The equation of a line passing through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°}$$

$$\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

$\Rightarrow \quad x - y + 1 = 0$

Suppose this line meets the line $x + y + 1 = 0$ at P such that $AP = r$.

Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$$

$\Rightarrow \quad x = 2 + r\cos 45°, y = 3 + r\sin 45°$

$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}}, y = 3 + \frac{r}{\sqrt{2}}$

Thus, the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

Since P lies on $x + y + 1 = 0$. Therefore,

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$$

$\Rightarrow \quad \sqrt{2}\,r = -6$

$\Rightarrow \quad r = -3\sqrt{2}$

Therefore, length $AP = |r| = 3\sqrt{2}$

Thus, the length of the intercept $= 3\sqrt{2}$

Example 1.71 **The line joining two points $A(2, 0)$, $B(3, 1)$ is rotated about A in anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?**

Sol. The slope m of the line AB is given by

$$m = \frac{1-0}{3-2} = 1$$

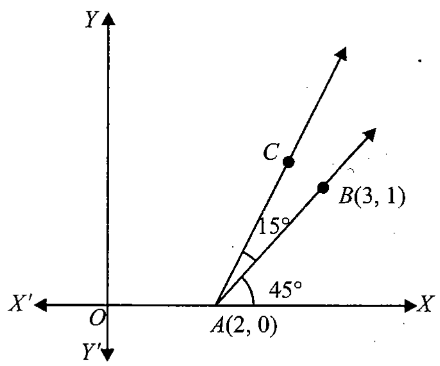


Fig. 1.53

So, AB makes an angle of 45° with x-axis. Now AB is rotated through 15° in anticlockwise direction and so it makes an angle of 60° with x-axis in its new position AC.

Clearly AC passes through $A(2, 0)$ and makes an angle of 60° with x-axis, therefore the equation of AC is

$$\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°}$$

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$

We have

$$AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}.$$

So, the coordinates of C are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2}$$

$\Rightarrow \quad x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}}$ and $y = \frac{\sqrt{3}}{2}\sqrt{2} = \frac{\sqrt{6}}{2}$

Hence, the coordinates of C are $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$.

Example 1.72 **Find the distance of the point (1, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$.**

Sol. The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line passing through (1, 3) and making an angle of 45° is

$$\frac{x-1}{\cos 45°} = \frac{y-3}{\sin 45°} = r$$

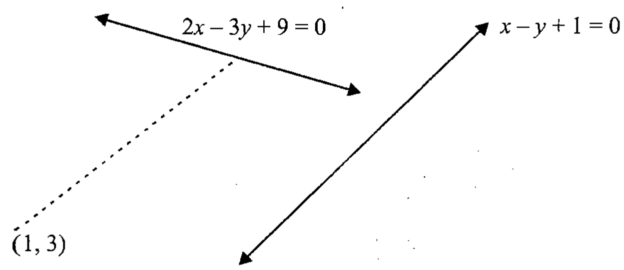


Fig. 1.54

Coordinates of any point on this line are $(1 + r\cos 45°, 3 + r\sin 45°) = \left(1 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

If this point lies on the line $2x - 3y + 9 = 0$, then

$$2 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0$$

$$\Rightarrow \quad r = 2\sqrt{2}$$

Hence, the required distance $= 2\sqrt{2}$

Concept Application Exercise 1.4

1. Two particles start from the point (2, − 1), one moves 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing, then find their new positions.
2. Find the distance between $A(2, 3)$ on the line of gradient 3/4 and the point of intersection P of this line with $5x + 7y + 40 = 0$.
3. The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the coordinates of other vertices of the square.

CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point.

Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \quad \text{(i)}$$

$$a_2x + b_2y + c_2 = 0 \quad \text{(ii)}$$

$$a_3x + b_3y + c_3 = 0 \quad \text{(iii)}$$

Then the point of intersection of Eqs. (i) and (ii) must lie on the third.

The coordinates of the point of intersection of Eqs. (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

This point lies on line (iii). Therefore, we get

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Alternative Method:

Three lines $L_1 \equiv a_1x + b_1y + c_1 = 0$; $L_2 \equiv a_2x + b_2y + c_2 = 0$; $L_3 \equiv a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero at the same time so that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$, i.e., $\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$.

Example 1.73 **Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33$ and $2x - 3y + \lambda = 0$ are concurrent.**

Sol. The given lines are concurrent if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0$$

$$\Rightarrow \quad -\lambda - 7 = 0 \Rightarrow \lambda = -7$$

Alternative Method: The given equations are

$$3x - 4y - 13 = 0 \quad \text{(i)}$$

$$8x - 11y - 33 = 0 \quad \text{(ii)}$$

and

$$2x - 3y + \lambda = 0 \quad \text{(iii)}$$

Solving Eqs. (i) and (ii), we get

$$x = 11 \text{ and } y = 5$$

Thus, (11, 5) is the point of intersection of Eqs. (i) and (ii).

The given lines will be concurrent if they pass through the common point, i.e., the point of intersection of any two lines lies on the third.

Therefore, (11, 5) lies on Eq. (iii),

i.e., $2 \times 11 - 3 \times 5 + \lambda = 0$

$\Rightarrow \quad \lambda = -7$

Example 1.74 **If the lines $a_1x + b_1y + 1 = 0$, $a_2x + b_2y + 1 = 0$ and $a_3x + b_3y + 1 = 0$ are concurrent, then show that the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.**

Sol. The given lines are

$$a_1x + b_1y + 1 = 0 \quad \text{(i)},$$
$$a_2x + b_2y + 1 = 0 \quad \text{(ii)}$$
and
$$a_3x + b_3y + 1 = 0 \quad \text{(iii)}$$

If these lines are concurrent, we must have

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

which is the condition of collinearity of three points (a_1, b_1), (a_2, b_2), and (a_3, b_3).

Hence, if the given lies are concurrent, the given points are collinear.

DISTANCE OF A POINT FROM A LINE

The length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Proof: The line $ax + by + c = 0$ meets x-axis at $A\left(\frac{-c}{a}, 0\right)$ and y-axis at $B\left(0, -\frac{c}{b}\right)$.

Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.

Now, area of ΔPAB

$$= \frac{1}{2}\left|x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| \quad \text{(i)}$$

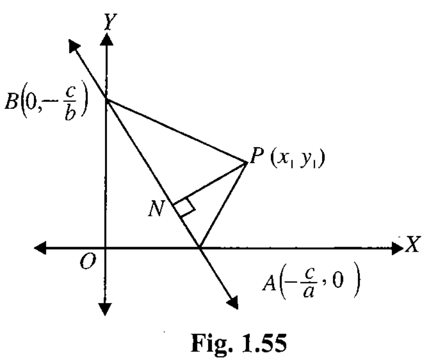


Fig. 1.55

Also, area of ΔPAB

$$= \frac{1}{2} AB \times PN$$

$$= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN$$

$$= \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN \quad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| = \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \quad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Distance between Two Parallel lines

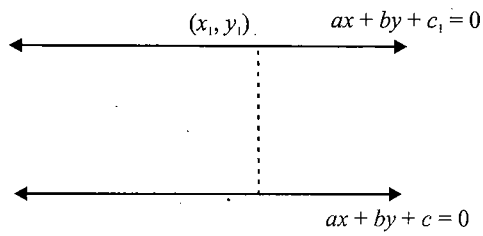


Fig. 1.56

Distance of a point (x_1, y_1) from the line $ax + by + c = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Now point (x_1, y_1) lies on $a_1x + b_1y + c_1 = 0$, then

$$ax_1 + by_1 + c_1 = 0$$

$$\Rightarrow \quad ax_1 + by_1 = -c_1$$

$$\Rightarrow \quad p = \frac{|c - c_1|}{\sqrt{a^2 + b^2}}$$

Example 1.75 **If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.**

Sol. The given line is

$$bx + ay - ab = 0 \quad \text{(i)}$$

It is given that p = length of the perpendicular from the origin to Eq. (i). That is,

$$p = \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}}$$

$$= \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad p^2 = \frac{a^2b^2}{a^2+b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}$$

Example 1.76 **Find the points on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.**

Sol. Let the required point be $P(0, \alpha)$. It is given that the length of the perpendicular from $P(0, \alpha)$ on $4x - 3y - 12 = 0$ is 3

$$\Rightarrow \quad \left|\frac{4(0)-3\alpha-12}{\sqrt{4^2+(-3)^2}}\right| = 3$$

$$\Rightarrow \quad |3\alpha + 12| = 15$$

$$\Rightarrow \quad |\alpha + 4| = 5$$

$$\Rightarrow \quad \alpha + 4 = \pm 5$$

$$\Rightarrow \quad \alpha = 1, -9$$

Hence, the required points are (0, 1) and (0, − 9).

Example 1.77 **Two sides of a squares lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area?**

Sol. Clearly, the length of the side of the square is equal to the distance between the parallel lines.

$$x + y - 1 = 0 \quad \text{(i)}$$

and

$$x + y + 2 = 0 \quad \text{(ii)}$$

Hence side length is $\frac{|2-(-1)|}{\sqrt{(1+1)}} = \frac{3}{\sqrt{2}}$

$\Rightarrow$ Area of square is $\frac{9}{2}$

Example 1.78 **Find the coordinates of a point on $x + y + 3 = 0$, whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.**

Sol. Putting $x = t$ in $x + y + 3 = 0$, we get $y = -3 - t$. So, let the required point be (t, −3 − t). This point is at a distance of $\sqrt{5}$ units from $x + 2y + 2 = 0$. Therefore,

$$\left|\frac{t-6-2t+2}{\sqrt{1^2+2^2}}\right| = \sqrt{5}$$

$$\Rightarrow \quad \left|\frac{-t-4}{\sqrt{5}}\right| = \sqrt{5}$$

$$\Rightarrow \quad t + 4 = \pm 5$$

$$\Rightarrow \quad t = 1, -9$$

Hence, the required points are (1, − 4) and (−9, 6).

Example 1.79 **Find the equations of straight lines passing through (− 2, − 7) and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.**

Sol.

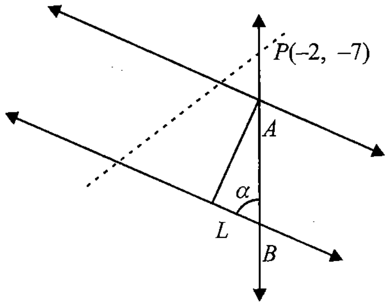


Fig. 1.57

Given lines are

$$4x + 3y - 12 = 0 \quad \text{(i)}$$

and

$$4x + 3y - 3 = 0 \quad \text{(ii)}$$

Given $\quad AB = 3$

AL = distance between parallel lines Eq. (i) and Eq. (ii)

$$= \frac{|-12+3|}{\sqrt{4^2+3^2}} = \frac{9}{5}$$

From $\Delta\, ALB$, we get

$$LB^2 = AB^2 - AL^2$$

$$= 3^2 - 81/25 = (9 \times 16/25)$$

$$\therefore \quad LB = 12/5$$

$\tan\alpha = AL/LB = 3/4$. Also $\tan\theta$ = slope of line (i) $= -4/3$. Let Slope of PB is m.

Now $\quad \tan\alpha = \frac{3}{4} = \left|\frac{m-\left(-\frac{4}{3}\right)}{1+m\left(-\frac{4}{3}\right)}\right|$

$$\Rightarrow \quad m = \infty \text{ or } m = -\frac{7}{24}$$

$\Rightarrow$ Equation of line is $x + 2 = 0$ and $y + 7 = \frac{-7}{24}(x + 2)$

or $x + 2 = 0$ and $7x + 24y + 182 = 0$

Example 1.80 **Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\frac{2c^2}{(ab)}$.**

Sol.

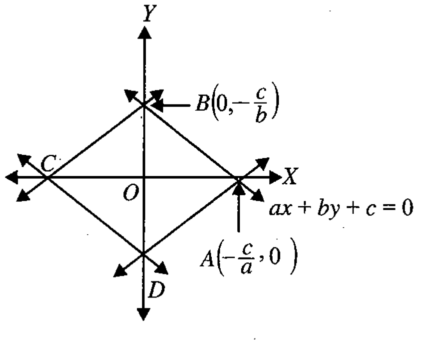


Fig. 1.58

Clearly from the above figure,

Area of parallelogram

$$= 4 \times (\text{area of } \Delta AOB)$$

$$= 4 \times \frac{1}{2}\left|\frac{c}{a}\right|\left|\frac{c}{b}\right|$$

$$= 2\frac{c^2}{ab}$$

Concept Application Exercise 1.5

1. Find the equation of a straight line passing through the point (−5, 4) and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$.
2. Find the ratio in which the line $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$.
3. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.
4. If p and p' are the distances of origin from the lines $x \sec \alpha + y \operatorname{cosec} \alpha = k$ and $x \cos \alpha - y \sin \alpha = k \cos 2\alpha$, then prove that $4p^2 + p'^2 = k^2$.

POSITION OF POINTS RELATIVE TO A LINE

Let the equation of the given line be

$$ax + by + c = 0 \qquad \text{(i)}$$

and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The coordinates of the point P which divides the line joining P and Q in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) \qquad \text{(ii)}$$

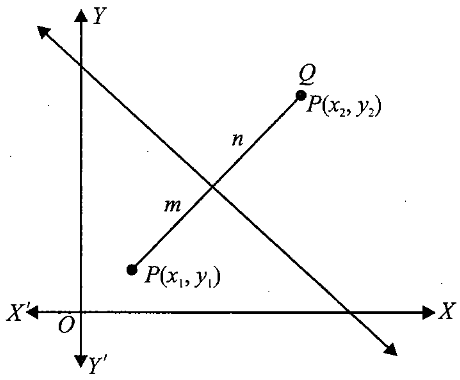


Fig. 1.59

If this point lies on Eq. (i), then

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c = 0$$

$$\Rightarrow \quad m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0$$

$$\Rightarrow \quad \frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \qquad \text{(iii)}$$

If point R is between the points P and Q, i.e., points P and Q are on the opposite sides of the given line, then the ratio $m : n$ is positive.

So, from Eq. (iii), we get

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$$

$$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs

If the points R is not between P and Q, i.e., points P and Q are on the same side of the given line, then the ratio $m : n$ is negative.

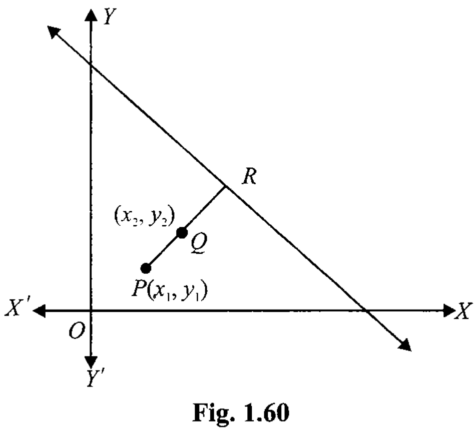


Fig. 1.60

So, from Eq. (iii), we get

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$$

$$\Rightarrow \qquad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.

Thus, the two points (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

Example 1.81 **Are the points (3, 4) and (2, – 6) on the same or opposite sides of the line $3x - 4y = 8$?**

Sol. Let $L = 3x - 4y - 8$. Then the value of L at (3, 4) is $L_1 = -15$ and the value of L at (2, – 6) is $L_2 = 22$.

Since L_1 and L_2 are of opposite signs, therefore the two points are on the opposite sides of the given line.

Example 1.82 **If the point (a, a) is placed in between the lines $|x + y| = 4$, then find the values of a.**

Sol. Lines $x + y = 4$ and $x + y = -4$ is parallel and points (2, 2) and (–2, –2) lie on these lines.

If point (a, a) lies between the lines, then $a > -2$ and $a < 2$ i.e. $-2 < a < 2$.

$$\Rightarrow \qquad |a| < 2$$

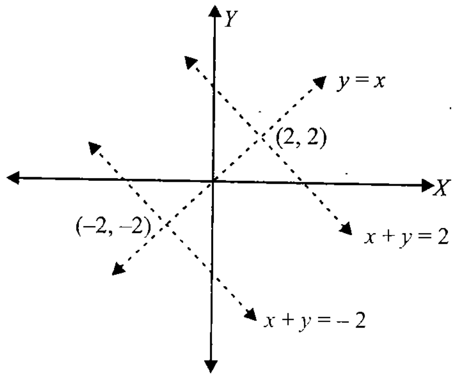


Fig. 1.61

Example 1.83 **Find the range of values of the ordinate of a point moving on the line $x = 1$, and always remaining in the interior of the triangle formed by the lines $y = x$, the x-axis and $x + y = 4$.**

Sol. From the figure, y-coordinates of P vary from 0 to 1. So, to be an interior point, 0 < ordinate of P < 1.

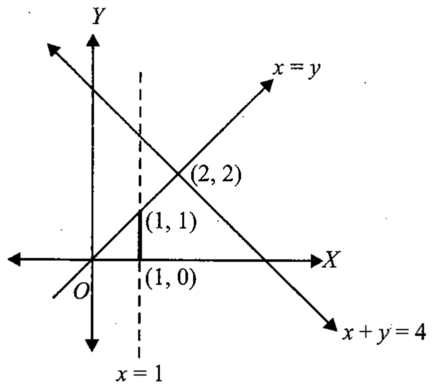


Fig. 1.62

Example 1.84 **If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then find the values of a.**

Sol. Acute angle is formed by lines in first and third quadrants

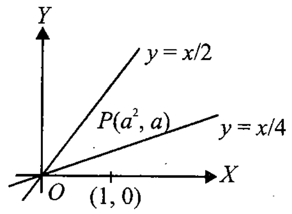


Fig. 1.63

But $a^2 > 0$, hence point $P(a^2, a)$ lies in first quadrant.

We have $a - (a^2/4) > 0$ and $a - (a^2/2) < 0$ ($\because$ (1, 0) and P lies on same side of $x - 2y = 0$ and (1, 0) and P lies opposite sides of $x - 4y = 0$)

$$\Rightarrow \qquad 0 < a < 4 \text{ and } a \in (-\infty, 0) \cup (2, \infty)$$

$$\Rightarrow \qquad a \in (2, 4)$$

Concept Application Exercise 1.6

1. The point (8, –9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ lies on the
   - **a.** same side of the lines
   - **b.** different of sides the line
   - **c.** one of the line
   - **d.** none of these
2. Which pair of points lies on the same side of $3x - 8y - 7 = 0$?
   - **a.** (0, –1) and (0, 0)
   - **b.** (4, –3) and (0, 1)
   - **c.** (–3, –4) and (1, 2)
   - **d.** (–1, –1) and (3, 7)
3. Find the range of α for which the points $(\alpha, 2 + \alpha)$ and $(3\alpha/2, \alpha^2)$ lie on the opposite sides of the line $2x + 3y = 6$.

EQUATIONS OF BISECTORS OF THE ANGLES BETWEEN THE LINES

Angle Bisectors

A bisector of angle between lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$ is the locus of a point, which moves such that the length of perpendiculars drawn from it to the two given lines, are equal.

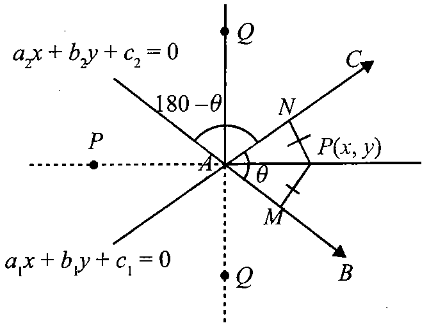


Fig. 1.64

From Fig. $PN = PM$

Then equations of the bisectors are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

i. *To determine the acute angle bisector and the obtuse angle bisector*

AP is the bisector of an acute angle if,

$\tan(\angle PAN) = \tan(\theta/2)$ is such that $|\tan \theta/2| < 1$

$\Rightarrow \quad \theta/2 < \pi/4$

$\Rightarrow \quad \theta < \pi/2$

AP is an obtuse angle bisector if,

$\tan(\angle PAN) = \tan(\theta/2)$ is such that $|\tan \theta/2| > 1$

$\Rightarrow \quad \theta/2 > \pi/4$

$\Rightarrow \quad \theta > \pi/2$

ii. *To determine the bisector of the angle contaning the origin and that of the angle not containing the origin*

Rewrite the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ such that the constant terms c_1, c_2 are positive.

Then, we have

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

it gives the equation of the bisector of the angle containing the origin and

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

gives the equation of the bisector of the angle not containing the origin.

Proof:

$L_1 : a_1x + b_1y + c_1 = 0$

and $L_2 : a_2x + b_2y + c_2 = 0$ (where $c_1, c_2 > 0$)

Since any point $P(h, k)$ and O (origin) are on the same side of the line $L_1 = 0$ and $L_2 = 0$ or on opposite sides of $L_1 = 0$ and $L_2 = 0$ $P(h, k)$ will give the same sign with respect to $L_1 = 0$ and $L_2 = 0$ ($\because c_1, c_2 > 0$)

Therefore, the equation $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

gives the bisector of that angle which contains the origin.

Now, consider the point $Q(h, k)$ on other bisector.

Note that Q and O are on different side of $L_2 = 0$, whereas Q and O are on the same side of $L_1 = 0$ (or Q and O lie on same side of $L_2 = 0$ and on different sides of $L_1 = 0$)

$\dfrac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}}$ and $\dfrac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}$ have opposite signs.

Therefore, the equation $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

will give the bisector of that angle which does not contain the origin.

Bisector of the angle containing the point (α, β) is

$$\therefore \quad \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ have the same sign.

Shortcut Method for Identifying Acute and Obtuse Angle Bisectors

Equations of the bisectors of the lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$

$(a_1b_2 \neq a_2b_1)$ where $c_1 > 0$ and $c_2 > 0$ are

$$\frac{(a_1x + b_1y + c_1)}{\sqrt{(a_1^2 + b_1^2)}} = \pm\frac{(a_2x + b_2y + c_2)}{\sqrt{(a_2^2 + b_2^2)}}$$

Conditions	Acute angle bisector	Obtuse angle bisector
$a_1a_2 + b_1b_2 > 0$	$-$	$+$
$a_1a_2 + b_1b_2 < 0$	$+$	$-$

Note:

A line which is equally inclined to given two lines is parallel to the angle bisectors of the given lines.

Example 1.85 **For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the**

i. bisector of the obtuse angle between them,

ii. bisector of the acute angle between them,

iii. bisector of the angle which contains (1, 2).

Sol. Equations of bisectors of the angles between the given lines are

$$\frac{4x + 3y - 6}{\sqrt{4^2 + 3^2}} = \pm\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

$\Rightarrow$ $9x - 7y - 41 = 0$

and $7x + 9y - 3 = 0$

If θ is the angle between the line $4x + 3y - 6 = 0$ and the bisector $9x - 7y - 41 = 0$, then

$$\tan\theta = \left|\frac{-\frac{4}{3} - \frac{9}{7}}{1 + \left(-\frac{4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence,

i. The bisector of the obtuse angle is $9x - 7y - 41 = 0$.

ii. The bisector of the acute angle is $7x + 9y - 3 = 0$.

iii. The bisector of the angle containing the point (1, 2).

For the point (1, 2), we have

$$4x + 3y - 6 = 4(1) + 3(2) - 6 > 0$$

$$5x + 12y + 9 = 5(1) + 12(2) + 9 > 0$$

Hence, the equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x + 3y - 6}{5} = \frac{5x + 12y + 9}{13}$$

$\Rightarrow$ $9x - 7y - 41 = 0$

Example 1.86 **Find the equation of the bisector of the obtuse angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.**

Sol. Firstly, make the constant terms (c_1, c_2) positive

$$3x - 4y + 7 = 0$$

and $-12x - 5y + 2 = 0$

$\Rightarrow$ $a_1a_2 + b_1b_2 = (3)(-12) + (-4)(-5)$

$$= -36 + 20 = -16$$

Hence, "–" sign gives the obtuse bisector.

$\Rightarrow$ Obtuse bisector is

$$\frac{(3x - 4y + 7)}{\sqrt{(3)^2 + (-4)^2}} = -\frac{(-12x - 5y + 2)}{\sqrt{(-12)^2 + (-5)^2}}$$

$\Rightarrow$ $13(3x - 4y + 7) = -5(-12x - 5y + 2)$

$\Rightarrow$ $21x + 77y - 101 = 0$

is the obtuse angle bisector

Example 1.87 **The vertices of a triangle ABC are (1, 1), (4, – 2) and (5, 5) respectively. Then find the equation of perpendicular dropped from C to the internal bisector of angle A.**

Sol. The internal bisector AD of angle BAC will divide the opposite side BC in the ratio of arms of the angle i.e., $AB : AC$ or $3\sqrt{2} : 4\sqrt{2}$ or 3 : 4.

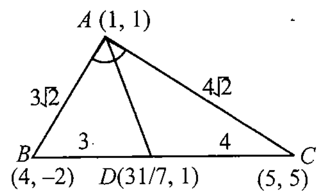


Fig. 1.65

Hence, the point D on BC by section formula is (31/7, 1) and A is (1, 1).

Therefore, slope of $AD = 0$. Hence, slope of CL which is perpendicular from C on AD is ∞.

Therefore, equation of CL is $x - 5 = 0$.

Example 1.88 **Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Determine the equation of the third side.**

Sol. Since triangle is isosceles, the third side is equally inclined to the lines $7x - y + 3 = 0$ and $x + y - 3 = 0$.

Hence, the third side is parallel to angle bisectors of the given lines.

The equations of the two bisectors of the angle are

$$\frac{7x - y + 3}{\sqrt{50}} = \pm\frac{x + y - 3}{\sqrt{2}}$$

or $\quad 3x + y - 3 = 0 \quad$ (i)

and $\quad x - 3y + 9 = 0 \quad$ (ii)

The line through (1, –10) parallel to Eq. (i) is $3x + y + 7 = 0$ and parallel to Eq. (ii) is $x - 3y - 31 = 0$.

Image of a Point with Respect to the Line Mirror

Let the image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(x_2, y_2)$ then it is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

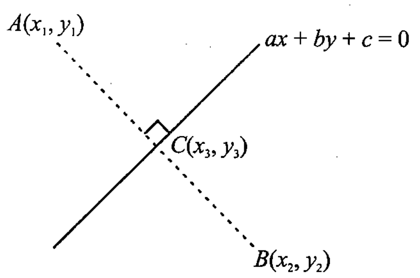


Fig. 1.66

Also let the foot of perpendicular from point $A(x_1, y_1)$ on the line $ax + by + c = 0$ be $C(x_3, y_3)$ which is given by

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

Proof:

Equation of line AB is

$$ax + by + c = 0 \quad \text{(i)}$$

Let $\quad P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$

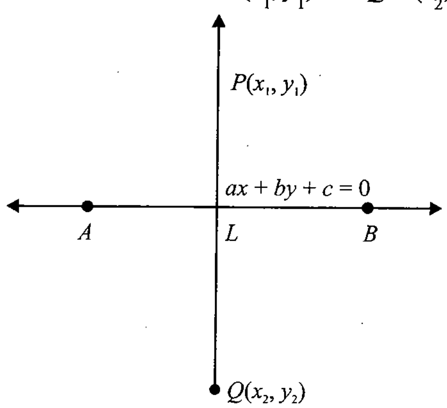


Fig. 1.67

Slope of $AB = -\frac{a}{b}$ and slope of $PQ = (y_2 - y_1)/(x_2 - x_1)$

Since $PQ \perp AB$, we get

$$\left(-\frac{a}{b}\right)\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -1$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = k \text{ (say)} \quad \text{(ii)}$$

From Eq. (ii), $x_2 - x_1 = ka$ and $y_2 - y_1 = kb$

Now,

$$a(x_2 - x_1) + b(y_2 - y_1) = k(a^2 + b^2)$$

or $$ax_2 + by_2 - (ax_1 + by_1) = k(a^2 + b^2)$$

or $$ax_2 + by_2 + c - (ax_1 + by_1 + c) = k(a^2 + b^2) \quad \text{(iii)}$$

Let L be the point of intersection of lines AB and PQ, then

$$L \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Since L lies on line Eq. (i),

$$\Rightarrow \quad a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c = 0$$

or $$ax_1 + by_1 + c + ax_2 + by_2 + c = 0$$

or $$ax_2 + by_2 + c = -(ax_1 + by_1 + c)$$

Therefore, from Eq. (iii), we get

$$-2(ax_1 + by_1 + c) = k(a^2 + b^2)$$

or $$k = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2} \quad \text{(iv)}$$

From Eqs. (ii) and (iv), we have

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

Example 1.89 **Find the image of the point (4, –13) in the line $5x + y + 6 = 0$.**

Sol.

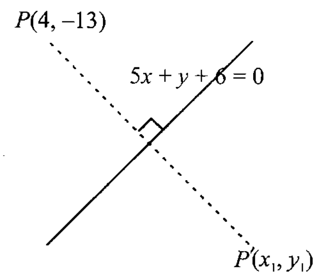


Fig. 1.68

Let $P'(x_1, y_1)$ be the image of (4, –13) with respect to the line mirror $5x + y + 6 = 0$, then

$$\frac{x_1-4}{5}=\frac{y_1+13}{1}=\frac{-2[5(4)-13+6]}{(5^2+1^2)}$$

$$\Rightarrow \quad \frac{x_1-4}{5}=\frac{y_1+13}{1}=-\frac{26}{26}$$

$$\Rightarrow \quad x_1-4=-5 \text{ and } y_1+13=-1$$

$$\Rightarrow \quad x_1=-1 \text{ and } y_1=-14$$

Hence, $P'(-1, -14)$.

Example 1.90 **Find the foot of the perpendicular from the point (2, 4) upon $x+y=1$.**

Sol. Required foot (h, k) of the perpendicular is given by

$$\frac{h-2}{1}=\frac{k-4}{1}=\frac{-(2+4-1)}{1+1}=\frac{-5}{2}$$

$$h=2-(5/2)=-1/2 \text{ and } k=4-(5/2)=3/2$$

Required foot is $(-1/2, 3/2)$.

Example 1.91 **In triangle ABC, equation of the right bisectors of the sides AB and AC are $x+y=0$ and $y-x=0$ respectively. If $A \equiv (5, 7)$ then find the equation of side BC.**

Sol. 'B' and 'C' will be the image of A in $y+x=0$ and $y-x=0$ respectively.

Thus $\quad B \equiv (-7, -5), C \equiv (7, 5)$.

Hence, equation of BC is

$$y-5=\frac{-5-5}{-7-7}(x-7) \text{ i.e., } 14y=10x.$$

FAMILY OF STRAIGHT LINES

Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$

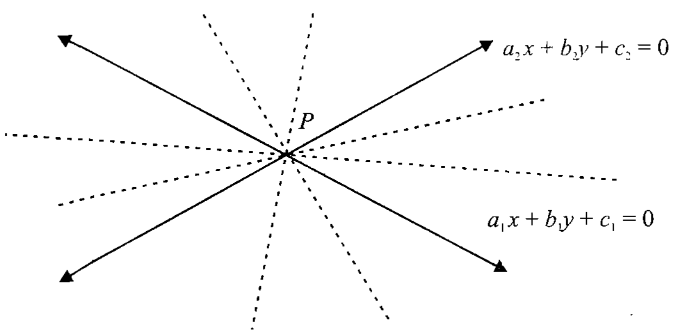


Fig. 1.69

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $L_1+\lambda L_2=0$

where $\lambda \in R$.

These lines form a family of straight line from point P.

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.

Note:

- *Consider variable straight line $ax+by+c=0$, where a, b, c are real. This lines do not form a family of concurrent straight line, as for different values of a, b, c the lines not necessarily concurrent. But if a, b, c are related to each other by any linear relation, i.e., $al+bm+cn=0$, where l, m, n are constants, then lines are concurrent for different values of a, b, c. The given variable line can be adjusted as $ax+by-(al+bm)/n=0$. This is, $a[x-(l/n)]+b[y-(m/n)]=0$, which passes through point $(l/n, m/n)$ for different values of a and b.*
- *A straight line is such that the algebraic sum of the perpendiculars drawn upon it from any number of fixed points is zero; then the line always passes through a fixed point.*

  Proof: Let the fixed points be (x_r, y_r); $r=1, 2, 3, \ldots, n$ and the given line be

$$ax+by+c=0 \tag{i}$$

Given,

$$\sum_{r=1}^{n}\frac{ax_r+by_r+c}{\sqrt{a^2+b^2}}=0;$$

$$or \quad \sum_{r=1}^{n}(ax_r+by_r+c)=0$$

$$\Rightarrow a\sum_{r=1}^{n}x_r+b\sum_{r=1}^{n}y_r+\sum_{r=1}^{n}c=0$$

$$\Rightarrow a(x_1+x_2+\cdots+x_n)+b(y_1+y_2+\cdots+y_n)+cn=0$$

$$\Rightarrow a\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)+b\left(\frac{y_1+y_2+\cdots+y_n}{n}\right)+c=0 \tag{ii}$$

From Eq. (ii) it is clear that Eq. (i) passes through the fixed point

$$\Rightarrow \left(\frac{x_1+x_2+\cdots+x_n}{n}, \frac{y_1+y_2+\cdots+y_n}{n}\right)$$

Example 1.92 **Show that the straight lines $x(a+2b)+y(a+3b)=a+b$ for different values of a and b pass through a fixed point. Find that point.**

Sol. Given equation is

$$x(a+2b)+y(a+3b) = a+b$$

or $$a(x+y-1)+b(2x+3y-1) = 0 \quad \text{(i)}$$

Both a and b cannot be simultaneously zero, therefore at least one of a and b will be non-zero. Let $a \neq 0$.

Now Eq. (i) can be written as

$$x+y-1+(b/a)(2x+3y-1) = 0$$

or $$x+y-1+\lambda(2x+3y-1) = 0, \quad \text{(ii)}$$

where $\lambda = b/a$

From Eq. (ii) it is clear that Eq. (ii) passes through the point of intersection of lines

$$x+y-1 = 0 \quad \text{(iii)}$$

and

$$2x+3y-1 = 0 \quad \text{(iv)}$$

Solving Eqs. (iii) and (iv), we get $x = 2, y = -1$.

Hence, lines represented by Eq. (i) pass through the fixed point $(2, -1)$ for all values of a and b.

Example 1.93 **If a, b, c are in AP, then prove that $ax + by + c = 0$ will always pass through a fixed point. Find that fixed point.**

Sol. Since a, b, c are in AP, therefore $2b = a + c$. Putting the value of b in $ax + by + c = 0$, we get

$$ax+\left(\frac{a+c}{2}\right)y+c = 0$$

$$\Rightarrow a(2x+y)+c(y+2) = 0$$

$$\Rightarrow (2x+y)+\left(\frac{c}{a}\right)(y+2) = 0$$

This equation represents a family of straight lines passing through the intersection of $2x + y = 0$ and $y + 2 = 0$, i.e; $(1, -2)$.

Alternative method:

Given line is

$$ax+by+c = 0 \quad \text{(i)}$$

where a, b, c are in AP then

$$a-2b+c = 0 \quad \text{(ii)}$$

Now comparing ratio of coefficient of a, b and c in Eq. (i) and (ii), we have

$$\frac{x}{1} = \frac{y}{-2} = \frac{1}{1}$$

$$\Rightarrow \quad x = 1 \text{ and } y = -2$$

Example 1.94 **Find the straight line passing through the point of intersection of $2x+3y+5=0$, $5x-2y-16=0$ and through the point $(-1, 3)$.**

Sol. The equation of any line through the point of intersection of the given line is

$$2x+3y+5+\lambda(5x-2y-16) = 0 \quad \text{(i)}$$

But the required line passes through $(-1, 3)$, hence we get

$$-2+9+5+\lambda(-5-6-16) = 0$$

Hence, $\lambda = 4/9$. Use this value of λ in Eq. (i) and the required line is

$$9(2x+3y+5)+4(5x-2y-16) = 0$$

or on simplification, we get

$$2x+y-1 = 0$$

Example 1.95 **Consider a family of straight lines $(x + y) + \lambda(2x - y + 1) = 0$. Find the equation of the straight line belonging to this family that is farthest from $(1, -3)$.**

Sol.

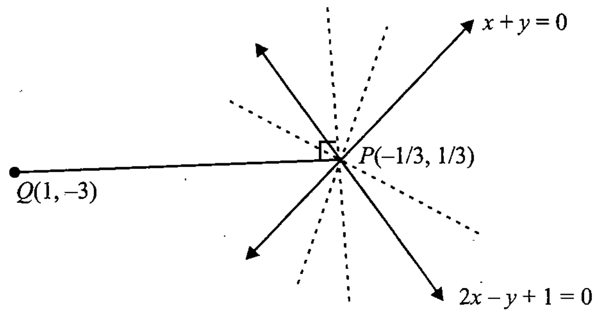


Fig. 1.70

Given family is concurrent at 'P' $(-1/3, 1/3)$. If $Q \equiv (1, -3)$, then $m_{PQ} = [-3-(1/3)]/[1+(1/3)] = -5/2$.

Now, member of the family that is farthest from 'Q' will have its slope as $2/5$. ($\because$ line will be $\perp$ to PQ)

As a result, we get

$$\frac{2}{5} = -\frac{(1+2\lambda)}{(1-\lambda)}$$

$$\Rightarrow \quad \lambda = -\frac{7}{8}$$

Thus, the equation of required line is

$$(x+y)-\frac{7}{8}(2x-y+1) = 0 \text{ i.e. } 15y-6x-7 = 0$$

Example 1.96 **Find the values of non-negative real numbers $h_1, h_2, h_3, k_1, k_2, k_3$ such that the algebraic sum of the perpendiculars drawn from points $(2, k_1)$, $(3, k_2)$, $(7, k_3)$, $(h_1, 4)$, $(h_2, 5)$, $(h_3, -3)$ on a variable line passing through $(2, 1)$ is zero.**

Sol. Let the equation of variable line be $ax + by + c = 0$, it is given that

$$\sum_{i=1}^{6} \frac{ax_i+by_i+c}{\sqrt{a^2+b^2}} = 0$$

$$\Rightarrow \quad a\left(\frac{\Sigma x_i}{6}\right)+b\left(\frac{\Sigma y_i}{6}\right)+c=0$$

So, the fixed point must be $\Sigma x_i/6, \Sigma y_i/6$. But fixed point is (2, 1) so $(2+3+7+h_1+h_2+h_3)/6=2$.

$$\Rightarrow \quad h_1+h_2+h_3=0$$

$$\Rightarrow \quad h_1=0, h_2=0, h_3=0$$

(as h_1, h_2, h_3 are non-negative)

Similarly, we get

$$\frac{k_1+k_2+k_3+4+5-3}{6}=1$$

$$\Rightarrow \quad k_1=k_2=k_3=0$$

Concept Application Exercise 1.7

1. If a and b are two arbitrary constants, then prove that the straight line $(a-2b)x+(a+3b)y+3a+4b=0$ will pass through a fixed point. Find that point.
2. If a, b, c are in harmonic progression, then the straight line $(x/a)+(y/b)+(l/c)=0$ always passes through a fixed point, then find that point.
3. If algebraic sum of distances of a variable line from points (2, 0), (0, 2) and (– 2, – 2) is zero, then the line passes through the fixed point is
   a. (–1 –1) **b.** (0, 0)
   c. (1, 1) **d.** (2, 2)
4. Consider the family of lines $5x+3y-2+\lambda_1(3x-y-4)=0$ and $x-y+1+\lambda_2(2x-y-2)=0$. Find the equation of a straight line that belongs to both the families.

PAIR OF STRAIGHT LINES

If separate equations of two lines be $ax+by+c=0$ and $a_1x+b_1y+c_1=0$, then their combined equation will be

$$(ax+by+c)(a_1x+b_1y+c_1)=0$$

and conversely if the combined equation of two lines be $(ax+by+c)(a_1x+b_1y+c_1)=0$, then their separate equations will be $ax+by+c=0$ and $a_1x+b_1y+c_1=0$.

For example, the combined equation of lines $2x+y+3=0$ and $x-y+4=0$ is $(2x+y+3)(x-y+4)=0$ or $2x^2-xy-y^2+11x+y-12=0$.

Separate equations of lines represented by the equation $6x^2+5xy-4y^2=0$, i.e., $(2x-y)(3x+4y)=0$ are $2x-y=0$ and $3x+4y=0$

Note:

In order to find the combined equation of two lines, make R.H.S. of equation of straight lines equal to zero and then multiply the two equations.

Homogeneous Equation

An equation (whose R.H.S. is zero) in which the sum of the powers of x and y in every term is the same, say n, is called a homogeneous equation of nth degree in x and y.

Thus, $ax^2+2hxy+by^2=0$ is a homogeneous equation of second degree and $cx^3+dxy^2+ey^3=0$ is a homogeneous equation of third degree in x and y.

Pair of Straight Lines through the Origin

To show that any homogeneous equation of second degree in x and y represents two straight lines through the origin.

Lets consider a general homogeneous equation of second degree in x and y as

$$ax^2+2hxy+by^2=0 \quad \text{(i)}$$

Dividing both sides by x^2, we get

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0 \quad \text{(ii)}$$

Since Eq. (ii) is an equation of second degree in y/x, it has two roots. Let the roots be m_1 and m_2. If α, β be the roots of equation $ax^2+bx+c=0$, then $ax^2+bx+c=a(x-\alpha)\times(x-\beta)$. Therefore, Eqs. (ii) can be written as

$$b\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0$$

or $\quad b(y-m_1x)(y-m_2x)=0$

Thus, Eq. (i) represents two straight lines $y-m_1x=0$ and $y-m_2x=0$ both of which pass through the origin.

Comparing $b(y-m_1x)(y-m_2x)=0$ with Eq. (i), we have

$$\frac{bm_1m_2}{a}=-\frac{b(m_1+m_2)}{2h}=\frac{1}{1}$$

$$\Rightarrow \quad m_1+m_2=-\frac{2h}{b}, \; m_1m_2=\frac{a}{b}$$

An Angle between the Line Represented by $ax^2+2hxy+by^2=0$

Let θ be the angle between the lines, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\pm\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$= \pm \frac{\sqrt{\frac{4h^2}{b^2} - 4\frac{a}{b}}}{1 + (a/b)}$$

$$= \pm \frac{\sqrt{4h^2 - 4ab}}{a+b}$$

or $$\tan\theta = \pm \frac{2\sqrt{h^2 - ab}}{a+b}$$

Acute angle between the lines is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

Note:

- *If the lines are perpendicular, then* $\theta = 90°$

$\Rightarrow$ $\cot\theta = 0$

$\Rightarrow$ $\frac{a+b}{2\sqrt{h^2 - ab}} = 0$

or $a + b = 0$ *which is the required condition or* (*coeff. of* x^2) + (*coeff. of* y^2) = 0

- *If the lines are parallel, then* $\theta = 0$

$\Rightarrow$ $\tan\theta = 0$

$\Rightarrow$ $\frac{2\sqrt{h^2 - ab}}{a+b} = 0$

$\Rightarrow$ $\sqrt{h^2 - ab} = 0$

$\Rightarrow$ $h^2 = ab$

which is the condition for the lines to be parallel (coincident).

Since both the lines pass through the origin, therefore they will be coincident if parallel.

Bisectors of Angle between the Lines Represented by $ax^2 + 2hxy + by^2 = 0$

Let the given equation represent the straight lines

$$y - m_1x = 0 \quad \text{(i)}$$

$$y - m_2x = 0 \quad \text{(ii)}$$

then

$$m_1 + m_2 = -2h/b \text{ and } m_1m_2 = a/b \quad \text{(iii)}$$

The equations to the bisectors of the angles between the straight lines in Eqs. (i) and (ii) are

$$\frac{y - m_1x}{\sqrt{1+m_1^2}} = \frac{y - m_2x}{\sqrt{1+m_2^2}}$$

and $$\frac{y - m_1x}{\sqrt{1+m_1^2}} = -\frac{y - m_2x}{\sqrt{1+m_2^2}}$$

Therefore, the combined equation of the bisectors is

$$\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} - \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\}\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} + \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\} = 0$$

or $$\frac{(y - m_1x)^2}{1+m_1^2} - \frac{(y - m_2x)^2}{1+m_2^2} = 0$$

or $$(1+m_2^2)(y^2 - 2m_1xy + m_1^2x^2) - (1+m_1^2)(y^2 - 2m_2xy + m_2^2x^2) = 0$$

or $$(m_1^2 - m_2^2)(x^2 - y^2) + 2(m_1m_2 - 1)(m_1 - m_2)xy = 0$$

or $$(m_1 + m_2)(x^2 - y^2) + 2(m_1m_2 - 1)xy = 0$$

Hence, by Eq. (iii), we get

$$\frac{-2h}{b}(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0,$$

or $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

General Equation of the Second Degree

We have proved that homogeneous equation of the second degree in x and y represents two straight lines passing through the origin. But every quadratic equation in x and y may not always represent two straight lines. The most general form of a quadratic equation in x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Since it is an equation in x and y, therefore it must represent the equation of a locus in a plane. It may represent a pair of straight lines, circle, or other curves in different case. Now we will consider the case when the above equation represents two straight lines.

Condition for General 2nd Degree Equation in x and y Represent Pair of Straight Lines

The given equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

If $a \neq 0$, then writing Eq. (i) as a quadratic equation in x we get

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0$$

Solving, we have

$$x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$= \frac{-(hy+g) \pm \sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}}{a}$$

Equation (i) will represent two straight lines if left-hand side of Eq. (i) can be resolved into two linear factors; therefore, the expression under the square roots should be a perfect square.

Hence,

$$4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0 \quad \text{(ii)}$$

(as $Ax^2 + Bx + C$ is a perfect square if $B^2 - 4AC = 0$)

$$\Rightarrow \quad g^2h^2 + a^2f^2 - 2afgh - h^2g^2 + abg^2 + ach^2 - a^2bc = 0$$

$$\Rightarrow \quad a(af^2 + bg^2 + ch^2 - 2fgh - abc) = 0$$

or $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{(iii)}$$

which is the required condition.

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation of the pair of lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two lines, then the left-hand sides can be resolved into two linear factors.

Let the factors be $(l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

Multiplying we see that $(l_1x + m_1y)(l_2x + m_2y)$, the terms of second degree, must be identical with $ax^2 + 2hxy + by^2$. Therefore, $ax^2 + 2hxy + by^2 = 0$ is identical with $(l_1x + m_1y) \times (l_2x + m_2y) = 0$.

Thus, equation $ax^2 + 2hxy + by^2$ represents two lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$, which are parallel to lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$, respectively.

Note:

The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is equal to the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$. That is,

$$\tan^{-1}\left(\pm \frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

Point of Intersection of Pair of Straight Lines

The point of intersection of the two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

The point of intersection can also be determined with the help of partial differentiation as follows.

Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Differentiating ϕ with respect to x, keeping y constant, we get

$$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g$$

Similarly, differentiating ϕ with respect to y, keeping x constant, we get

$$\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f$$

For point of intersection, we get

$$\frac{\partial \phi}{\partial x} = 0 \text{ and } \frac{\partial \phi}{\partial y} = 0$$

Thus, we have $ax + hy + g = 0$ and $hx + by + f = 0$.

Solving the two equations, we get

$$\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$$

$$\therefore \quad (x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

Example 1.97 **Find the value of λ if $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represents a pair of straight lines.**

Sol. The given equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, if $abc + 2fgh - af^2 - bg^2 - bc^2 = 0$, i.e., if

$$6\lambda + 2(7)(4)\left(\frac{7}{2}\right) - 2(7)^2 - 3(4)^2 - \lambda\left(\frac{7}{2}\right)^2 = 0$$

$$\Rightarrow \quad 6\lambda + 196 - 98 - 48 - \frac{49\lambda}{4} = 0$$

$$\Rightarrow \quad \frac{49\lambda}{4} - 6\lambda = 196 - 146 = 50$$

$$\Rightarrow \quad \frac{25\lambda}{4} = 50$$

$$\Rightarrow \quad \lambda = \frac{200}{25} = 8$$

Example 1.98 **If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1}(m)$, then find the value of m.**

Sol. The angle between the lines $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is given by

$$\tan\theta = \tan^{-1}\left(\pm \frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

where $a = 2$, $b = 3$, $h = 5/2$

$$\Rightarrow \quad \tan\theta = \frac{\pm 2\sqrt{\frac{25}{4} - 6}}{2 + 3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\pm \frac{1}{5}\right) \Rightarrow m = \pm \frac{1}{5}$$

Example 1.99 **Find the value of 'a' for which the lines represented by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular.**

Sol. The lines given by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular if $a + 2 = 0$, i.e., $a = -2$.

Example 1.100 **If $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ bisect angles between each other, then find the condition.**

Sol. The equation of the bisectors of the angles between the lines $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$$

$$\Rightarrow \quad \frac{x^2-y^2}{2} = \frac{-xy}{p}$$

$$\Rightarrow \quad px^2 + 2xy - py^2 = 0$$

This is same as $x^2 - 2qxy - y^2 = 0$, therefore

$$\frac{p}{1} = \frac{2}{-2q} = \frac{-p}{-1}$$

$$\Rightarrow \quad pq + 1 = 0$$

Example 1.101 **If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is rotated about the origin through 90°, then find the equations in the new position.**

Sol. Let $y = m_1 x$, $y = m_2 x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then, we have

$$m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

Let $y = m'_1 x$ and $y = m'_2 x$ be new positions of $y = m_1 x$ and $y = m_2 x$, respectively.

Then, $y = m_1 x$ is perpendicular to $y = m'_1 x$

$$\therefore \quad m_1 m'_1 = -1$$

$$\Rightarrow \quad m'_1 = -\frac{1}{m_1}\cdot \text{ Similarly, } m'_2 = -\frac{1}{m_2}$$

So, the new lines are $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$ and so their combined equation is

$$(m_1 y + x)(m_2 y + x) = 0$$

$$\Rightarrow m_1 m_2 y^2 + x^2 + xy(m_1 + m_2) = 0$$

$$\Rightarrow \frac{a}{b}y^2 + x^2 + xy\left(\frac{-2h}{b}\right) = 0$$

$$\Rightarrow \quad bx^2 - 2hxy + ay^2 = 0$$

Example 1.102 **If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis, then prove that $2fgh = bg^2 + ch^2$.**

Sol. We must have

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{(i)}$$

Let the point of intersection on the y-axis be $(0, \lambda)$. Then, from $\left(\frac{bg-hf}{h^2-ab}, \frac{af-gh}{h^2-ab}\right)$, we have

$$\therefore \quad fh = bg \quad \text{(ii)}$$

The equation of the y-axis is x = 0. Solving this and the equation of the pair of lines, we get

$$by^2 + 2fy + c = 0$$

which must have equal roots.

$$\therefore \quad f^2 = bc \quad \text{(iii)}$$

Equation (ii) can be written as $fgh = bg^2$

Putting $f^2 = bc, fgh = bg^2$ in Eq. (i), we get

$$bg^2 = ch^2$$

$$\therefore \quad 2fgh = bg^2 + ch^2$$

Example 1.103 **Find the distance between the pair of parallel lines $x^2 + 4xy + 4y^2 + 3x + 6y - 4 = 0$.**

Sol. Given lines are

$$(x + 2y)^2 + 3(x + 2y) - 4 = 0$$

Therefore, $$x + 2y = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2}$$

$$= -4, 1$$

Therefore, lines are

$$x + 2y + 4 = 0$$

$$x + 2y - 1 = 0$$

$$\text{Required distance} = \frac{|4-(-1)|}{\sqrt{1+4}}$$

$$= \frac{5}{\sqrt{5}} = \sqrt{5}$$

Combined Equation of Pair of Lines Joining Origin and the Points of Intersection of a Curve and a Line

Lets find the equation of the straight lines joining the origin and the points of intersection of the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

and the line

$$lx + my + n = 0 \quad \text{(ii)}$$

Equation (ii) can be written as

$$lx + my = -n \text{ or } \frac{lx+my}{-n} = 1 \quad \text{(iii)}$$

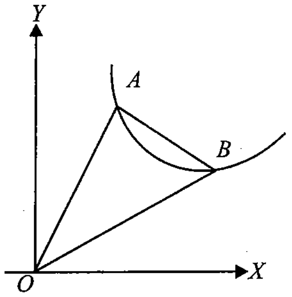


Fig. 1.71

Let A and B be the points of intersection of the (ii) and Eq. (i)

In order to make Eq. (i) homogeneous with the help of Eq. (iii), we write Eq. (i) as

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)1 + c(1)^2 = 0$$

$$\Rightarrow ax^2 + 2hxy + by^2 + (2gx + 2fy) \times \left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0 \quad \text{(iv)}$$

Since coordinates of A and B satisfy Eqs. (i) and (ii), therefore these satisfy Eqs. (iv) which is formed by combining Eqs. (i) and (ii).

Thus, Eq. (iv) is a locus through points A and B.

Also Eq. (iv) being a homogeneous equation of second degree in x and y represents two straight lines through the origin.

Hence, Eq. (iii) is the joint equation of OA and OB.

Example 1.104 **Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x - k)^2 + (y - h)^2 = c^2$ are right angles, if $h^2 + k^2 = c^2$.**

Sol. Making the equation of the curve homogeneous with the help of that of the line, we get

$$x^2 + y^2 - 2(kx + hy)\left(\frac{hx + ky}{2hk}\right) + (h^2 + k^2 - c^2)\left(\frac{hx + ky}{2hk}\right)^2 = 0$$

This is the equation of the pair of lines joining the origin to the points of intersection of the given line and the curve. These will be at right angles if

coefficient of x^2 + coefficient of $y^2 = 0$, i.e.,

$$(h^2 + k^2)(h^2 + k^2 - c^2) = 0$$

$$\Rightarrow \quad h^2 + k^2 = c^2 \quad (\because h^2 + k^2 \neq 0)$$

Example 1.105 **Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}(2\sqrt{2}/3)$.**

Sol. Equation of the given curve is

$$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$$

and equation of the given straight line is

$$y - 3x = 2$$

$$\therefore \quad \frac{y - 3x}{2} = 1$$

Making Eq. (i) homogeneous equation of the second degree in x any y with the help of Eq. (i), we have

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y - 3x}{2}\right) + 8y\left(\frac{y - 3x}{2}\right) - 11\left(\frac{y - 3x}{2}\right)^2 = 0$$

or $7x^2 - 2xy - y^2 = 0$

This is the equation of the lines joining the origin to the point of intersection of Eqs. (i) and (ii).

Comparing Eqs. (iii) with the equation $ax^2 + 2hxy + by^2 = 0$, we get

$a = 7$, $b = -1$ and $2h = -2$, i.e., $h = 1$

If θ be the acute angle between pair of lines as shown in Eq. (iii), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{1 + 7}}{7 - 1}\right|$$

$$= \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$$\therefore \quad \theta = \tan^{-1}\frac{2\sqrt{2}}{3}$$

Example 1.106 **If the lines joining origin and point of intersection of curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ are mutually perpendicular, then prove that $g(a_1 + b_1) = g_1(a + b)$.**

Sol. $\quad ax^2 + 2hxy + by^2 = -2gx$

$$a_1x^2 + 2h_1xy + b_1y^2 = -2g_1x$$

$$\Rightarrow \quad \frac{ax^2 + 2hxy + by^2}{a_1x^2\ 2h_1xy + b_1y^2} = \frac{g}{g_1}$$

$$(ag_1 - a_1g)x^2 + 2(hg_1 - h_1g)xy + (bg_1 - b_1g)y^2 = 0$$

The lines will be perpendicular if $ag_1 - a_1g + bg_1 - b_1g = 0$

i.e., $\quad$ if $(a + b)g_1 = (a_1 + b_1)g$

Concept Application Exercise 1.8

1. Find the combined equation of the pair of lines through the point (1, 0) and parallel to the lines represented by $2x^2 - xy - y^2 = 0$.
2. If the slope of one line is double the slope of another line and the combined equation of the pair of lines is $(x^2/a) + (2xy/h) + (y^2/b) = 0$, then find the ratio $ab : h^2$.
3. Show that straight lines $(A^2 - 3B^2)x^2 + 8AB\,xy$ $(B^2 - 3A^2)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle of area $C^2/\sqrt{3(A^2+B^2)}$.
4. If one of the lines denoted by the line pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes, then prove that $(a + b)^2 = 4h^2$.
5. Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and also the angle between them.
6. A line L passing through the point (2,1) intersects the curve $4x^2 + y^2 - x + 4y - 2 = 0$ at the points A and B. If the lines joining origin and the points A, B are such that the coordinate axes are the bisectors between them, then find the equation of line L.
7. If the equation $x^2 + (\lambda + \mu)xy + \lambda\mu y^2 + x + \mu y = 0$ represents two parallel straight lines, then prove that $\lambda = \mu$.
8. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive direction of the axes, then find the relation for a, b, and h.
9. If the pair of lines $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ are rotated about the origin by $\pi/6$ in the anticlockwise sense, then find the equation of the pair in the new position.
10. If the equation $2x^2 + kxy + 2y^2 = 0$ represents a pair of real and distinct lines, then find the values of k.
11. Find the point of intersection of the pair and straight lines represented by the equations $6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.
12. Find the angle between the lines represented by $x^2 + 2xy \sec\theta + y^2 = 0$.

EXERCISES

Subjective Type

Solutions on page 1.70

1. Show that the reflection of the line $ax + by + c = 0$ in the line $x + y + 1 = 0$ is the line $bx + ay + (a + b - c) = 0$, where $a \neq b$.
2. A straight line cuts intercepts from the axis of coordinates, the sum of the reciprocals of which is a constant. Show that it always passes through a fixed point.
3. If the equal sides AB and AC of a right angled isosceles triangle be produced to P and Q so that $BP \times CQ = AB^2$, then show that the line PQ always passes through a fixed point.
4. Straight lines $y = mx + c_1$ and $y = mx + c_2$, where $m \in R^+$, meet the x-axis at A_1 and A_2, respectively, and y-axis at B_1 and B_2, respectively. It is given that points A_1, A_2, B_1 and B_2 are concyclic. Find the locus of intersection of lines A_1B_2 and A_2B_1.
5. A line $L_1 \equiv 3y - 2x - 6 = 0$ is rotated about its point of intersection with y-axis in clockwise direction to make it L_2 such that the area formed by L_1, L_2, x-axis and line $x = 5$ is 49/3 sq. units if its point of intersection with $x = 5$ lies below x-axis. Find the equation of L_2.
6. Find the locus of the circumcentre of a triangle whose two sides are along the coordinate axes and the third side passes through the point of intersection of the lines $ax + by + c = 0$ and $lx + my + n = 0$.
7. A regular polygon has two of its consecutive diagonals as the lines $\sqrt{3}x + y = \sqrt{3}$ and $2y = \sqrt{3}$. Point $(1, c)$ is one of its vertices. Find the equation of the sides of the polygon and also find the coordinates of the vertices.
8. A diagonal of rhombus $ABCD$ is member of both the families of lines $(x + y - 1) + \lambda(2x + 3y - 2) = 0$ and $(x - y + 2) + \lambda(2x - 3y + 5) = 0$ and one of the vertices of the rhombus is (3, 2). If the area of the rhombus is $12\sqrt{5}$ sq. units, then find the remaining vertices of the rhombus.
9. A triangle has the lines $y = m_1x$ and $y = m_2x$ as two of its sides, with m_1, m_2 being roots of the equation $bx^2 + 2hx + a = 0$. If $H(a, b)$ is the orthocentre of the triangle, then show that equation of the third side is $(a + b)(ax + by) = ab(a + b - 2h)$.
10. Let $A = (6, 7)$, $B = (2, 3)$ and $C = (-2, 1)$ be the vertices of a triangle. Find the point P in the interior of the triangle such that ΔPBC is an equilateral triangle.
11. Find all the values of θ for which the point $(\sin^2\theta, \sin\theta)$ lies inside the square formed by the lines $xy = 0$ and $4xy - 2x - 2y + 1 = 0$.

12. Two sides of a triangle have the joint equation $x^2 - 2xy - 3y^2 + 8y - 4 = 0$. The third side, which is variable, always passes through the point (5, – 1). Find the range of the values of the slope of the third side, so that the origin is an interior point of the triangle.

13. Let ABC be a given isosceles triangle with $AB = AC$. Sides AB and AC are extended up to E and F, respectively, such that $BE \times CF = AB^2$. Prove that the line EF always passes through a fixed point.

14. Let $L_1 = 0$ and $L_2 = 0$ be two fixed lines. A variable line is drawn through the origin to cut the two lines at R and S. P is a point on the line AB such that $(m + n)/OP = m/OR + n/OS$. Show that the locus of P is a straight line passing through the point of intersection of the given lines (R, S, P are on the same side of O).

15. A variable line cuts n given concurrent straight lines at $A_1, A_2, \ldots, A_n$ such that $\sum_{i=1}^{n} \frac{1}{OA_i}$ is a constant. Show that it always passes through a fixed point, O being the point of intersection of the lines.

16. Two sides of a rhombus lying in the first quadrant are given by $3x - 4y = 0$ and $12x - 5y = 0$. If the length of the longer diagonal is 12, then find the equations of the other two sides of rhombus.

17. If D, E, F are three points on the sides BC, AC and AB of a triangle ABC such that AD, BE and CF are concurrent, then show that $BD \times CE \times AF = DC \times EA \times FB$.

18. Let the sides of a parallelogram be $U = a$, $U = b$, $V = a'$ and $V = b'$ where $U = lx + my + n$, $V = l'x + m'y + n'$. Show that the equation of the diagonal through the point of intersection of $U = a$ and $V = a'$ and $U = b$ and $U = b'$ is given by

$$\begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix} = 0$$

19. Consider two lines L_1 and L_2 given by $x - y = 0$, and $x + y = 0$, respectively, and a moving point $P(x, y)$. Let $d(P, L_i)$, $i = 1, 2$ represents the distance of point 'P' from the line L_i. If point 'P' moves in certain region 'R' in such a way that $2 \le d(P, L_1) + d(P, L_2) \le 4$. Find the area of region R.

20. If (x, y) and (X, Y) be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$, where u and v are independent of x and y, becomes $VX + UY$, show that $u^2 + v^2 = U^2 + V^2$.

21. Consider two lines L_1 and L_2 given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively, where $c_1, c_2 \neq 0$, intersecting at point P. A line L_3 is drawn through the origin meeting the lines L_1 and L_2 at A and B, respectively, such that $PA = PB$. Similarly, one more line L_4 is drawn through the origin meeting the lines L_1 and L_2 at A_1 and B_1, respectively, such that $PA_1 = PB_1$. Obtain the combined equation of lines L_3 and L_4.

22. Show that if any line through the variable point $A(k + 1, 2k)$ meets the lines $7x + y - 16 = 0$, $5x - y - 8 = 0$, $x - 5y + 8 = 0$ at B, C, D, respectively, then AC, AB and AD are in harmonic progression. (The three lines lie on the same side of point A.)

23. Show that the lines $4x + y - 9 = 0$, $x - 2y + 3 = 0$, $5x - y - 6 = 0$ make equal intercepts on any line of slope 2.

24. Having given the bases and the sum of the areas of a number of triangles which have a common vertex, show that the locus of the vertex is a straight line.

25. Find the locus of the point at which two given portions of the straight line subtend equal angle.

26. A right angled triangle ABC having C as right angle is of given magnitude and the angular points A and B slide along two given perpendicular axes. Show that the locus of C is the pair of straight lines whose equations are $y = \pm (b/a)\, x$.

27. Let $2x + 3y = 6$ be a line meeting the coordinate axes at A and B, respectively. A variable line $x/a + y/b = 1$ meets the axes at P and Q, respectively, in such a way that the lines BP and AQ always meet at right angle at R. Find the locus of the orthocentre of the triangle ARB.

28. If θ is an angle between the lines given by the equation $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$, then find the equation of the line passing through the point of intersection of these lines and making an angle θ with the positive x-axis.

Objective Type

Solutions on page 1.79

Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct. Find the correct answer.

1. Equations of diagonals of square formed by lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are

a. $y = x, y + x = 1$ **b.** $y = x, x + y = 2$
c. $2y = x, y + x = 1/3$ **d.** $y = 2x, y + 2x = 1$

2. If each of the points $(x_1, 4)$, $(-2, y_1)$ lies on the line joining the points $(2, -1)$, $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line

a. $6(x + y) - 25 = 0$ **b.** $2x + 6y + 1 = 0$
c. $2x + 3y - 6 = 0$ **d.** $6(x + y) + 25 = 0$

3. If

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a, b_3) are

a. equal in area
b. similar
c. congruent
d. none of these

4. The area of a parallelogram formed by the lines $ax \pm bx \pm c = 0$ is
a. $c^2/(ab)$
b. $2c^2/(ab)$
c. $c^2/2ab$
d. none of these

5. The straight line $ax + by + c = 0$ where $abc \neq 0$ will pass through the first quadrant if
a. $ac > 0, bc > 0$
b. $c > 0$ and $bc < 0$
c. $bc > 0$ and/or $ac > 0$
d. $ac < 0$ and/or $bc < 0$

6. If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) internally, then
a. $t < 0$
b. $0 < t < 1$
c. $t > 1$
d. $t = 1$

7. A square of side a lies above the x-axis and has one vertex at the origin. The side passing trough the origin makes an angle $\alpha (0 < \alpha < \pi/4)$ with the positive direction of x-axis. The equation of its diagonal not passing through the origin is
a. $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$
b. $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$
c. $y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$
d. $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$

8. If sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
a. a square
b. a circle
c. a straight line
d. two intersecting lines

9. The area enclosed by $2|x| + 3|y| \leq 6$ is
a. 3 sq. units
b. 4 sq. units
c. 12 sq. units
d. 24 sq. units

10. If two vertices of a triangle are (–2, 3) and (5, – 1), orthocentre lies at the origin and centroid on the line $x + y = 7$, then the third vertex lies at
a. (7, 4)
b. (8, 14)
c. (12, 21)
d. none of these

11. $OPQR$ is a square and M, N are the middle points of the sides PQ and QR, respectively, then the ratio of the areas of the square and the triangle OMN is
a. 4:1
b. 2:1
c. 8:3
d. 7:3

12. The vertices of a triangle are $(pq, 1/(pq))$, $(qr, 1/(qr))$ and $(rq, 1/(rp))$ where p, q, r are the roots of the equation $y^3 - 3y^2 + 6y + 1 = 0$. The coordinates of its centroid are
a. (1, 2)
b. (2, –1)
c. (1, – 1)
d. (2, 3)

13. The lines $y = m_1x$, $y = m_2x$ and $y = m_3x$ make equal intercepts on the line $x + y = 1$, then
a. $2(1 + m_1)(1 + m_3) = (1 + m_2)(2 + m_1 + m_3)$
b. $(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$
c. $(1 + m_1)(1 + m_2) = (1 + m_3)(2 + m_1 + m_3)$
d. $2(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$

14. The condition on a and b, such that the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y = 0$ and $x + by = 0$, subtends a right angle at the origin is
a. $a = b$
b. $a + b = 0$
c. $a = 2b$
d. $2a = b$

15. One diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is (1, 2). Then the equations of the sides of the square passing through this vertex are
a. $23x + 7y = 9$, $7x + 23y = 53$
b. $23x - 7y + 9 = 0$, $7x + 23y + 53 = 0$
c. $23x - 7y - 9 = 0$, $7x + 23y - 53 = 0$
d. none of these

16. If $u = a_1x + b_1y + c_1 = 0$, $v = a_2x + b_2y + c_2 = 0$ and $a_1/a_2 = b_1/b_2 = c_1/c_2$, then the curve $u + kv = 0$ is
a. the same straight line u
b. different straight line
c. not a straight line
d. none of these

17. The point $A(2, 1)$ is translated parallel to the line $x - y = 3$ by a distance 4 units. If the new position A' is in third quadrant, then the coordinates of A' are
a. $(2 + 2\sqrt{2}, 1 + 2\sqrt{2})$
b. $(-2 + \sqrt{2}, -1 - 2\sqrt{2})$
c. $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$
d. none of these

18. The area of the triangle formed by the lines $y = ax$, $x + y - a = 0$ and the y-axis is equal to
a. $\dfrac{1}{2|1 + a|}$
b. $\dfrac{a^2}{|1 + a|}$
c. $\dfrac{1}{2}\left|\dfrac{a}{1 + a}\right|$
d. $\dfrac{a^2}{2|1 + a|}$

19. An equation of a line through the point (1, 2) whose distance from the point (3, 1) has the greatest value is
a. $y = 2x$
b. $y = x + 1$
c. $x + 2y = 5$
d. $y = 3x - 1$

20. A rectangle $ABCD$ has its side AB parallel to line $y = x$ and vertices A, B and D lie on $y = 1$, $x = 2$ and $x = -2$, respectively. Locus of vertex 'C' is
a. $x = 5$
b. $x - y = 5$
c. $y = 5$
d. $x + y = 5$

21. The centroid of an equilateral triangle is (0, 0). If two vertices of the triangle lie on $x + y = 2\sqrt{2}$ then one of them will have its coordinates

a. $(\sqrt{2}+\sqrt{6}, \sqrt{2}-\sqrt{6})$ b. $(\sqrt{2}+\sqrt{3}, \sqrt{2}-\sqrt{3})$
c. $(\sqrt{2}+\sqrt{5}, \sqrt{2}-\sqrt{5})$ d. none of these

22. Vertices of a variable triangle are (3, 4), $(5\cos\theta, 5\sin\theta)$ and $(5\sin\theta, -5\cos\theta)$, where $\theta \in R$. Locus of its orthocentre is

a. $(x+y-1)^2+(x-y-7)^2=100$
b. $(x+y-7)^2+(x-y-1)^2=100$
c. $(x+y-7)^2+(x+y-1)^2=100$
d. $(x+y-7)^2+(x-y+1)^2=100$

23. The line $x/3 + y/4 = 1$ meets the y-axis and x-axis at A and B, respectively. A square $ABCD$ is constructed on the line segment AB away from the origin. The coordinates of the vertex of the square farthest from the origin are

a. (7, 3) b. (4, 7)
c. (6, 4) d. (3, 8)

24. A line of fixed length 2 units moves so that its ends are on the positive x-axis and that part of the line $x+y=0$ which lies in the second quadrant. Then the locus of the midpoint of the line has the equation

a. $x^2+5y^2+4xy-1=0$ b. $x^2+5y^2+4xy+1=0$
c. $x^2+5y^2-4xy-1=0$ d. $4x^2+5y^2+4xy+1=0$

25. Consider the points $A(0, 1)$ and $B(2, 0)$, and P be a point on the line $4x+3y+9=0$. Coordinates of P such that $|PA-PB|$ is maximum are

a. $(-12/5, 17/5)$ b. $(-84/5, 13/5)$
c. $(-6/5, 17/5)$ d. $(0, -3)$

26. Consider points $A(3, 4)$ and $B(7, 13)$. If P be a point on the line $y=x$ such that $PA+PB$ is minimum, then coordinates of P are

a. (12/7, 12/7) b. (13/7, 13/7)
c. (31/7, 31/7) d. (0, 0)

27. A line 'L' is drawn from $P(4, 3)$ to meet the lines L_1 and L_2 given by $3x+4y+5=0$ and $3x+4y+15=0$ at points A and B, respectively. From 'A', a line perpendicular to L is drawn meeting the line L_2 at A_1. Similarly from point 'B', a line perpendicular to L is drawn meeting the line L_1 at B_1. Thus a parallelogram AA_1BB_1 is formed. Then the the equation of 'L' so that the area of the parallelogram AA_1BB_1 is least is

a. $x-7y+17=0$ b. $7x+y+31=0$
c. $x-7y-17=0$ d. $x+7y-31=0$

28. Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle keeping the origin fixed, the same line L has intercepts p and q. Then,

a. $a^2+b^2=P^2+q^2$ b. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$
c. $a^2+p^2=b^2+q^2$ d. $\frac{1}{a^2}+\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{q^2}$

29. If the straight lines $2x+3y-1=0$, $x+2y-1=0$ and $ax+by-1=0$ form a triangle with origin as orthocentre, then (a, b) is given by

a. (6, 4) b. (−3, 3)
c. (−8, 8) d. (0, 7)

30. A light ray emerging from the point source placed at $P(2, 3)$ is reflected at a point 'Q' on the y-axis and then passes through the point $R(5, 10)$. Coordinates of 'Q' are

a. (0, 3) b. (0, 2)
c. (0, 5) d. none of these

31. A straight line passing through $P(3, 1)$ meets the coordinate axes at A and B. It is given that distance of this straight line from the origin 'O' is maximum. Area of triangle OAB is equal to

a. 50/3 sq. units b. 25/3 sq. units
c. 20/3 sq. units d. 100/3 sq. units

32. Let O be the origin. If $A(1, 0)$ and $B(0, 1)$ and $P(x, y)$ are points such that $xy > 0$ and $x+y<1$, then

a. P lies either inside the triangle OAB or in the third quadrant
b. P cannot lie inside the triangle OAB
c. P lies inside the triangle OAB
d. P lies in the first quadrant only

33. The graph $y^2+2xy+40|x|=400$ divides the plane into regions. Then the area of bounded region is

a. 200 sq. units b. 400 sq. units
c. 800 sq. units d. 500 sq. units

34. In a triangle ABC, $A \equiv (\alpha, \beta)$, $B \equiv (2, 3)$ and $C \equiv (1, 3)$ and point A lies on line $y=2x+3$ where $\alpha \in I$. Area of ΔABC, Δ, is such that $[\Delta]=5$. Possible coordinates of A are (where $[\cdot]$ represents greatest integer function)

a. (2, 3) b. (5, 13)
c. (−5, −7) d. (−3, −5)

35. If $a/bc - 2 = \sqrt{b/c} + \sqrt{c/b}$, where $a, b, c > 0$, then family of lines $\sqrt{a}x+\sqrt{b}y+\sqrt{c}=0$ passes though the fixed point given by

a. (1, 1) b. (1, −2)
c. (−1, 2) d. (−1, 1)

36. $P(m, n)$ (where m, n are natural numbers) is any point in the interior of the quadrilateral formed by the pair of lines $xy=0$ and the lines $2x+y-2=0$ and $4x+5y=20$. The possible number of positions of the point P is

a. 7 b. 5
c. 4 d. 6

37. If the area of the rhombus enclosed by the lines $lx \pm my \pm n = 0$ be 2 sq. units, then

a. l, m, n are in G.P. b. l, n, m are in G.P.
c. $lm = n$ d. $ln = m$

38. P, Q, R and S are the points of intersection with the coordinate axes of the lines $px + qy = pq$ and $qx + py = pq$, then $(P, Q > 0)$
a. P, Q, R, S from a parallelogram
b. P, Q, R, S from a rhombus
c. P, Q, R, S are concyclic
d. none of these

39. In a triangle ABC if $A \equiv (1, 2)$ and internal angle bisectors through B and C are $y = x$ and $y = -2x$, then the inradius r of the ΔABC is
a. $1/\sqrt{3}$ b. $1/\sqrt{2}$
c. 2/3 d. none of these

40. In a triangle ABC, the bisectors of angles B and C lie along the lines $x = y$ and $y = 0$. If A is (1, 2), then the equation of line BC is
a. $2x + y = 1$ b. $3x - y = 5$
c. $x - 2y = 3$ d. $x + 3y = 1$

41. If the vertices of a triangle are $(\sqrt{5}, 0)$, $(\sqrt{3}, \sqrt{2})$ and (2, 1), then the orthocentre of the triangle is
a. $(\sqrt{5}, 0)$ b. (0, 0)
c. $(\sqrt{5} + \sqrt{3} + 2, \sqrt{2} + 1)$ d. none of these

42. One of the diagonals of a square is the portion of the line $x/2 + y/3 = 2$ intercepted between the axes. Then the extremities of the other diagonal are
a. (5, 5), (−1, 1) b. (0, 0), (4, 6)
c. (0, 0), (−1, 1) d. (5, 5), (4, 6)

43. Two sides of a triangle are along the coordinate axes and the medians through the vertices (other than the origin) are mutually perpendicular. The number of such triangles is/are
a. zero b. two c. four d. infinite

44. A rectangular billiard table has vertices at P (0, 0), Q (0, 7), R (10, 7) and S (10, 0). A small billiard ball starts at M (3, 4) and moves in a straight line to the top of the table, bounces to the right side of the table, then comes to rest at N (7, 1). The y-coordinate of the point where it hits the right side, is
a. 3.7 b. 3.8 c. 3.9 d. 4

45. In ΔABC the coordinates of the vertex A are (4, −1) and lines $x - y - 1 = 0$ and $2x - y = 3$ are internal bisectors of angles B and C. Then, radius of incircle of triangle ABC is
a. $4/\sqrt{5}$ b. $3/\sqrt{5}$
c. $6/\sqrt{5}$ d. $7/\sqrt{5}$

46. Distance of origin from line $(1 + \sqrt{3})y + (1 - \sqrt{3})x = 10$ along the line $y = \sqrt{3}x + k$ is
a. $5/\sqrt{2}$ b. $5\sqrt{2} + k$
c. 10 d. 0

47. If it is possible to draw a line which belongs to all the given family of lines $y - 2x + 1 + \lambda_1(2y - x - 1) = 0$, $3y - x - 6 + \lambda_2(y - 3x + 6) = 0$, $ax + y - 2 + \lambda_3(6x + ay - a) = 0$, then
a. $a = 4$ b. $a = 3$
c. $a = -2$ d. $a = 2$

48. If the equation of any two diagonals of a regular pentagon belongs to family of lines $(1 + 2\lambda)y - (2 + \lambda)x + 1 - \lambda = 0$ and their lengths are sin 36°, then locus of centre of circle circumscribing the given pentagon (the triangles formed by these diagonals with sides of pentagon have no side common) is
a. $x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$
b. $x^2 + y^2 - 2x - 2y + \cos^2 72° = 0$
c. $x^2 + y^2 - 2x - 2y + 1 + \cos^2 72° = 0$
d. $x^2 + y^2 - 2x - 2y + \sin^2 72° = 0$

49. ABC is a variable triangle such that A is (1, 2), B and C lie on line $y = x + \lambda$ (where λ is a variable), then locus of the orthocentre of triangle ABC is
a. $(x - 1)^2 + y^2 = 4$ b. $x + y = 3$
c. $2x - y = 0$ d. none of these

50. Locus of the image of the point (2, 3) in the line $(x - 2y + 3) + \lambda(2x - 3y + 4) = 0$ is $(\lambda \in R)$
a. $x^2 + y^2 - 3x - 4y - 4 = 0$
b. $2x^2 + 3y^2 + 2x + 4y - 7 = 0$
c. $x^2 + y^2 - 2x - 4y + 4 = 0$
d. none of these

51. If one side of a rhombus has end points (4, 5) and (1, 1) then the maximum area of the rhombus is
a. 50 sq. units b. 25 sq. units
c. 30 sq. units d. 20 sq. units

52. The equations of the sides of a triangle are $x + y - 5 = 0$, $x - y + 1 = 0$ and $y - 1 = 0$. Then the coordinates of the circumcentre are
a. (2, 1) b. (1, 2)
c. (2, −2) d. 1, −2)

53. Two vertices of a triangle are (4, −3) and (−2, 5). If the orthocentre of the triangle is at (1, 2), then the third vertex is
a. (−33, −26) b. (33, 26)
c. (26, 33) d. none of these

54. If $\sum_{i=1}^{4} (x_i^2 + y_i^2) \le 2x_1x_3 + 2x_2x_4 + 2y_2y_3 + 2y_1y_4$ the points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are
a. the vertices of a rectangle

b. collinear

c. trapezium

d. none of these

55. Locus of a point that is equidistant from the lines $x + y - 2\sqrt{2} = 0$ and $x + y - \sqrt{2} = 0$ is

a. $x + y - 5\sqrt{2} = 0$ **b.** $x + y - 3\sqrt{2} = 0$

c. $2x + 2y - 3\sqrt{2} = 0$ **d.** $2x + 2y - 5\sqrt{2} = 0$

56. If the intercept made on the line $y = mx$ by lines $y = 2$ and $y = 6$ is less than 5, then the range of values of m is

a. $(-\infty, -4/3) \cup (4/3, +\infty)$

b. $(-4/3, 4/3)$

c. $(-3/4, 4/3)$

d. none of these

57. If $P\,(1 + t/\sqrt{2},\ 2 + t/\sqrt{2})$ be any point on a line, then the range of the values of t for which the point P lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$ is

a. $-4\sqrt{2}/3 < t < 5\sqrt{2}/6$

b. $0 < t < 5\sqrt{2}/6$

c. $4\sqrt{2}/ < t < 0$

d. none of these

58. P is a point on the line $y + 2x = 1$ and, Q and R are two points on the line $3y + 6x = 6$ such that triangle PQR is an equilateral triangle. The length of the side of the triangle is

a. $2/\sqrt{5}$ **b.** $3/\sqrt{5}$

c. $4/\sqrt{5}$ **d.** none of these

59. The coordinates of two consecutive vertices A and B of a regular hexagon $ABCDEF$ are (1, 0) and (2, 0), respectively. The equation of the diagonal CE is

a. $\sqrt{3}x + y = 4$ **b.** $x + \sqrt{3}y + 4 = 0$

c. $x + \sqrt{3}y = 4$ **d.** none of these

60. The foot of the perpendicular on the line $3x + y = \lambda$ drawn from the origin is C. If the line cuts the x-axis and y-axis at A and B, respectively, then $BC{:}CA$ is

a. 1:3 **b.** 3:1

c. 1:9 **d.** 9:1

61. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with same common ratio, then the points $P\,(x_1, y_1)$, $Q\,(x_2, y_2)$ and $R\,(x_3, y_3)$

a. lie on a straight line **b.** lie on an ellipse

c. lie on a circle **d.** are vertices of a triangle

62. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c' = 0$ have perpendicular diagonals, then

a. $b^2 + c^2 = b'^2 + c'^2$ **b.** $c^2 + a^2 = c'^2 + a'^2$

c. $a^2 + b^2 = a'^2 + b'^2$ **d.** none of these

63. The straight lines $7x - 2y + 10 = 0$ and $7x + 2y - 10 = 0$ form an isosceles triangle with the line $y = 2$. Area of this triangle is equal to

a. 15/7 sq. units **b.** 10/7 sq. units

c. 18/7 sq. units **d** none of these

64. A rectangle $ABCD$, where $A \equiv (0, 0)$, $B \equiv (4, 0)$, $C \equiv (4, 2)$, $D \equiv (0, 2)$, undergoes the following transformations successively: (i) $f_1(x, y) \to (y, x)$, (ii) $f_2(x, y) \to (x + 3y, y)$, (iii) $f_3(x, y) \to ((x - y)/2, (x + y)/2)$. The final figure will be

a. a square **b.** a rhombus

c. a rectangle **d.** a parallelogram

65. θ_1 and θ_2 are the inclination of lines L_1 and L_2 with x-axis. If L_1 and L_2 pass through $P\,(x_1, y_1)$, then equation of one of the angle bisector of these lines is

a. $\dfrac{x - x_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\sin\left(\frac{\theta_1 - \theta_2}{2}\right)}$

b. $\dfrac{x - x_1}{-\sin\left(\frac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$

c. $\dfrac{x - x_1}{\sin\left(\frac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$

d. $\dfrac{x - x_1}{-\sin\left(\frac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$

66. A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then,

a. $a = 64/115$, $b = 112/15$

b. $a = 14/15$, $b = -8/115$

c. $a = 64/115$, $b = -8/115$

d. $a = 64/15$, $b = 14/15$

67. Line $ax + by + p = 0$ makes angle $\pi/4$ with $x \cos\alpha + y \sin\alpha = p$, $p \in R^+$. If these lines and the line $x \sin\alpha - y \cos\alpha = 0$ are concurrent, then

a. $a^2 + b^2 = 1$ **b.** $a^2 + b^2 = 2$

c. $2(a^2 + b^2) = 1$ **d.** none of these

68. A line is drawn perpendicular to line $y = 5x$, meeting the coordinate axes at A and B. If the area of triangle OAB is 10 sq. units where 'O' is the origin, then the equation of drawn line is

a. $3x - y - 9$ **b.** $x - 5y = 10$

c. $x + 4y = 10$ **d.** $x - 4y = 10$

69. If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of a isosceles triangle having area 10 sq. units the equation of third side is

a. $3x - y = -9$ b. $3x - y + 11 = 0$
c. $x - 3y = 19$ d. $3x - y + 15 = 0$

70. The number of values of 'a' for which the lines
$2x + y - 1 = 0$,
$ax + 3y - 3 = 0$, and
$3x + 2y - 2 = 0$
are concurrent is

a. 0 b. 1 c. 2 d. infinite

71. The extremities of the base of an isosceles triangle are (2, 0) and (0, 2). If the equation of one of the equal side is $x = 2$, then equation of other equal side is

a. $x + y = 2$ b. $x - y + 2 = 0$
c. $y = 2$ d. $2x + y = 2$

72. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is

a. $a_1^2 - a_2^2 + b_1^2 - b_2^2$ b. $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
c. $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ d. $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

73. The line PQ whose equation is $x - y = 2$ cuts the x-axis at P and Q is (4, 2). The line PQ is rotated about P through 45° in the anticlockwise direction. The equation of the line PQ in the new position is

a. $y = -\sqrt{2}$ b. $y = 2$ c. $x = 2$ d. $x = -2$

74. The equation of straight line passing through $(-a, 0)$ and making the triangle with axes of area 'T' is

a. $2Tx + a^2y + 2aT = 0$ b. $2Tx - a^2y + 2aT = 0$
c. $2Tx - a^2y - 2aT = 0$ d. none of these

75. If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the sides of the triangle is

a. $\sqrt{\frac{20}{3}}$ b. $\frac{2}{\sqrt{15}}$ c. $\sqrt{\frac{8}{15}}$ d. $\sqrt{\frac{15}{2}}$

76. The number of integral values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

a. 2 b. 0 c. 4 d. 1

77. Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the x-axis is

a. $x\sqrt{3} + y + 8 = 0$ b. $x\sqrt{3} - y = 8$
c. $x\sqrt{3} - y = 8$ d. $x - \sqrt{3}y + 8 = 0$

78. The equation to the straight line passing through the point $(a \cos^3 \theta, a \sin^3 \theta)$ and perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is

a. $x \cos \theta - y \sin \theta = a \cos 2\theta$
b. $x \cos \theta + y \sin \theta = a \cos 2\theta$
c. $x \sin \theta + y \cos \theta = a \cos 2\theta$
d. none of these

79. The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$, is

a. above the x-axis at a distance of 3/2 units from it
b. above the x-axis at a distance of 2/3 units from it
c. below the x-axis at a distance of 3/2 units from it
d. below the x-axis at a distance of 2/3 units from it

80. If a straight line through origin bisects the line passing through the given points $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$, then the lines

a. are perpendicular
b. are parallel
c. have an angle between them of $\pi/4$
d. none of these

81. The straight lines $4ax + 3by + c = 0$, where $a + b + c = 0$, are concurrent at the point

a. (4, 3) b. (1/4, 1/3)
c. (1/2, 1/3) d. none of these

82. The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point

a. (a, b) b. (b, a) c. $(-a, -b)$ d. none of these

83. The coordinates of the foot of the perpendicular from the point (2, 3) on the line $-y + 3x + 4 = 0$ are given by

a. (37/10, – 1/10) b. (– 1/10, 37/10)
c. (10/37, – 10) d. (2/3, – 1/3)

84. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle is

a. $5a^2$ sq. units b. $5a^2/2$ sq. units
c. $25a^2/2$ sq. units d. none of these

85. The combined equation of straight lines that can be obtained by reflecting the lines $y = |x - 2|$ in the y-axis is

a. $y^2 + x^2 + 4x + 4 = 0$ b. $y^2 + x^2 - 4x + 4 = 0$
c. $y^2 - x^2 + 4x - 4 = 0$ d. $y^2 - x^2 - 4x - 4 = 0$

86. If the slope of one line represented by $a^3x^2 - 2hxy + b^3y^2 = 0$ is square of the slope of another line, then

a. $h = 2ab(a + b)$ b. $h = ab(a + b)$
c. $3h = 2ab(a + b)$ d. $2h = ab(a + b)$

87. $A \equiv (-4, 0)$, $B \equiv (4, 0)$. M and N are the variable points of y-axis such that M lies below N and $MN = 4$. Line joining AM and BN intersect at 'P'. Locus of 'P' is

a. $2xy - 16 - x^2 = 0$ **b.** $2xy + 16 - x^2 = 0$
c. $2xy + 16 + x^2 = 0$ **d.** $2xy - 16 + x^2 = 0$

88. Let A_r, $r = 1, 2, 3, \ldots$ be points on the number line such that $OA_1, OA_2, OA_3, \ldots$ are in G.P. where O is the origin and the common ratio of the G.P. be a positive proper fraction. Let M_r be the middle point of the line segment A_rA_{r+1}. Then the value of $\sum_{r=1}^{\infty} OM_r$ is equal to

a. $\frac{OA_1(OSA_1 - OA_2)}{2(OA_1 + OA_2)}$ **b.** $\frac{OA_1(OA_1 - OA_2)}{2(OA_1 + OA_2)}$

c. $\frac{OA_1}{2(OA_1 - OA_2)}$ **d.** ∞

89. The number of triangles that the four lines $y = x + 3$, $y = 2x + 3$, $y = 3x + 2$ and $y + x = 3$ form is

a. 4 **b.** 2 **c.** 3 **d.** 1

90. Let $A = (3, -4)$, $B = (1, 2)$, let $P = (2k - 1, 2k + 1)$ be a variable point such that $PA + PB$ is the minimum. Then k is

a. 7/9 **b.** 0 **c.** 7/8 **d.** none of these

91. L_1 and L_2 are two lines. If the reflection of L_1 in L_2 and the reflection of L_2 in L_1 coincide, then the angle between the lines is

a. 30° **b.** 60° **c.** 45° **d.** 90°

92. If the straight lines $x + y - 2 = 0$, $2x - y + 1 = 0$ and $ax + by - c = 0$ are concurrent, then the family of lines $2ax + 3by + c = 0$ (a, b, c are nonzero) is concurrent at

a. (2, 3) **b.** (1/2 , 1/3)
c. (–1/6 , –5/9) **d.** (2/3 , –7/5)

93. Two medians drawn from acute angles of a right angled triangle intersect at an angle $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then area of the triangle (in sq. units) is

a. $\sqrt{3}$ **b.** 3 **c.** $\sqrt{2}$ **d.** 9

94. A variable line $x/a + y/b = 1$ moves in such a way that harmonic mean of a and b is 8. Then the least area of triangle made by the line with the coordinate axes is

a. 8 sq. unit **b.** 16 sq. unit
c. 32 sq. unit **d.** 64 sq. unit

95. The number of integral points (x, y) (i.e., x and y both are integers) which lie in the first quadrant but not on the coordinate axes and also on the straight line $3x + 5y = 2007$ is equal to

a. 133 **b.** 135 **c.** 138 **d.** 140

96. The number of straight lines equidistant from three non-collinear points in the plane of the points is equal to

a. 0 **b.** 1 **c.** 2 **d.** 3

97. The locus of the point which moves such that its distance from the point (4, 5) is equal to its distance from the line $7x - 3y - 13 = 0$ is

a. a straight line **b.** a circle
c. a parabola **d.** an ellipse

98. In ΔABC if orthocentre be (1, 2) and circumcentre be (0, 0), then centroid of ΔABC is

a. (1/2, 2/3) **b.** (1/3 , 2/3)
c. (2/3 , 1) **d.** none of these

99. The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has equation $x - 7y + 5 = 0$. The equation of the other line is

a. $3x + 3y - 1 = 0$ **b.** $x - 3y + 2 = 0$
c. $5x + 5y - 3 = 0$ **d.** none of these

100. Through a point A on the x-axis a straight line is drawn parallel to y-axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ in B and C. If $AB = BC$, then

a. $h^2 = 4ab$ **b.** $8h^2 = 9ab$ **c.** $9h^2 = 8ab$ **d.** $4h^2 = ab$

101. If $A(1, p^2)$, $B(0, 1)$ and $C(p, 0)$ are the coordinates of three points, then the value of p for which the area of the triangle ABC is minimum is

a. $1/\sqrt{3}$ **b.** $-1/\sqrt{3}$
c. $1/\sqrt{2}$ **d.** none of these

102. m, n are integers with $0 < n < m$. A is the point (m, n) on the Cartesian plane. B is the reflection of A in the line $y = x$. C is the reflection of B in the y-axis, D is the reflection of C in the x-axis and E is the reflection of D in the y-axis. The area of the pentagon $ABCDE$ is

a. $2m(m + n)$ **b.** $m(m + 3n)$
c. $m(2m + 3n)$ **d.** $2m(m + 3n)$

103. If the ends of the base of an isosceles triangle are at (2, 0) and (0, 1) and the equation of one side is $x = 2$, then the orthocentre of the triangle is

a. (3/2, 3/2) **b.** (5/4, 1) **c.** (3/4, 1) **d.** (4/3, 7/12)

104. Vertices of a parallelogram $ABCD$ are A (3, 1), B (13, 6), C (13, 21) and D (3, 16). If a line passing through the origin divides the parallelogram into two congruent parts, then the slope of the line is

a. 11/12 **b.** 11/8 **c.** 25/8 **d.** 13/8

105. The distance between the two parallel lines is 1 unit. A point 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is

a. $(2/3)\sqrt{d^2 + d + 1}$ **b.** $2\sqrt{(d^2 - d + 1)/3}$

c. $2\sqrt{d^2-d+1}$ d. $\sqrt{d^2-d+1}$

106. Given A (0, 0) and B (x, y) with $x \in (0, 1)$ and $y > 0$. Let the slope of the line AB equal to m_1. Point C lies on the line $x = 1$ such that the slope of BC equal to m_2 where $0 < m_2 < m_1$. If the area of the triangle ABC can be expressed as $(m_1 - m_2) f(x)$, then the largest possible value of x is

a. 1 b. 1/2 c. 1/4 d. 1/8

107. If the vertices P and Q of a triangle PQR are given by (2, 5) and (4, –11), respectively, and the point R moves along the line N given by $9x + 7y + 4 = 0$, then the locus of the centroid of the triangle PQR is a straight line parallel to

a. PQ b. QR c. RP d. N

108. Given $A \equiv (1, 1)$ and AB is any line through it cutting the x-axis in B. If AC is perpendicular to AB and meets the y-axis in C, then the equation of locus of midpoint P of BC is

a. $x+y=1$ b. $x+y=2$

c. $x+y=2xy$ d. $2x+2y=1$

109. The number of possible straight lines, passing through (2, 3) and forming a triangle with coordinate axes, whose area is 12 sq. units, is

a. one b. two c. three d. four

110. In a triangle ABC, if A is (2, –1), and $7x - 10y + 1 = 0$ and $3x - 2y + 5 = 0$ are equations of an altitude and an angle bisector, respectively, drawn from B, then equation of BC is

a. $x+y+1=0$ b. $5x+y+17=0$

c. $4x+9y+30=0$ d. $x-5y-7=0$

111. A triangle is formed by the lines $x + y = 0$, $x - y = 0$ and $lx + my = 1$. If l and m vary subject to the condition $l^2 + m^2 = 1$, then the locus of its circumcentre is

a. $(x^2-y^2)^2 = x^2+y^2$ b. $(x^2+y^2)^2 = (x^2-y^2)$

c. $(x^2+y^2) = 4x^2y^2$ d. $(x^2-y^2)^2 = (x^2+y^2)^2$

112. The line $x + y = p$ meets the x- and y-axes at A and B, respectively. A triangle APQ is inscribed in the triangle OAB, O being the origin, with right angle at Q. P and Q lie, respectively, on OB and AB. If the area of the triangle APQ is 3/8th of the area of the triangle OAB, then AQ/BQ is equal to

a. 2 b. 2/3 c. 1/3 d. 3

113. A is a point on either of two lines $y + \sqrt{3}|x| = 2$ at a distance of $4\sqrt{3}$ units from their point of intersection. The coordinates of the foot of perpendicular from A on the bisector of the angle between them are

a. $(2/\sqrt{3}, 2)$ b. (0, 0)

c. $(2/\sqrt{3}, 2)$ d. (0, 4)

114. A pair of perpendicular straight lines is drawn through the origin forming with the line $2x + 3y = 6$ an isosceles triangle right angled at the origin. The equation to the line pair is

a. $5x^2-24xy-5y^2=0$ b. $5x^2-26xy-5y^2=0$

c. $5x^2+24xy-5y^2=0$ d. $5x^2+26xy-5y^2=0$

115. Points A and B are in the first quadrant; point 'O' is the origin. If the slope of OA is 1, slope of OB is 7 and $OA = OB$, then the slope of AB is

a. –1/5 b. –1/4

c. –1/3 d. –1/2

116. $OPQR$ is a square and M, N are the midpoints of the sides PQ and QR, respectively. If the ratio of the areas of the square and the triangle OMN is $\lambda : 6$, then $\lambda/4$ is equal to

a. 2 b. 4

c. 2 d. 16

117. A triangle ABC with vertices $A(-1, 0)$, $B(-2, 3/4)$ and $C(-3, -7/6)$ has its orthocentre H. Then the orthocentre of triangle BCH will be

a. (–3, –2) b. (1, 3)

c. (–1, 2) d. none of these

118. If a pair of perpendicular straight lines drawn through the origin forms an isosceles triangle with the line $2x + 3y = 6$, then area of the triangle so formed is

a. 36/13 b. 12/17

c. 13/5 d. 17/13

119. The image of $P(a, b)$ in the line $y = -x$ is Q and the image of Q in the $y = x$ is R. Then the midpoint of PR is

a. $(a+b, b+a)$ b. $((a+b)/2, (b+2)/2)$

c. $(a-b, b-a)$ d. (0, 0)

120. Let ABC be a triangle. Let A be the point (1, 2), $y = x$ be the perpendicular bisector of AB and $x - 2y + 1 = 0$ be the angle bisector of $\angle C$. If equation of BC is given by $ax + by - 5 = 0$, then the value of $a + b$ is

a. 1 b. 2 c. 3 d. 4

121. If in triangle ABC, $A \equiv (1, 10)$, circumcentre $\equiv (-1/3, 2/3)$ and orthocentre $\equiv (11/4, 4/3)$, then the coordinates of mid-point of side opposite to A is

a. (1, –11/3) b. (1, 5)

c. (1, –3) d. (1, 6)

122. In the ΔABC, the coordinates of B are (0, 0), $AB = 2$, $\angle ABC = \pi/3$ and the middle point of BC has the coordinates (2, 0). The centroid of the triangle is

a. $(1/2, \sqrt{3}/2)$ b. $(5/3, 1/\sqrt{3})$

c. $(4+\sqrt{3}/3, 1/3)$ d. none of these

123. A beam of light is sent along the line $x - y = 1$, which after refracting from the x-axis enters the opposite side by turning through 30° towards the normal at the point of incidence on the x-axis. Then the equation of the refracted ray is

a. $(2 - \sqrt{3})x - y = 2 + \sqrt{3}$

b. $(2 + \sqrt{3})x - y = 2 + \sqrt{3}$

c. $(2 - \sqrt{3})x + y = (2 + \sqrt{3})$

d. $y = (2 - \sqrt{3})(x - 1)$

124. The equation of the straight line which passes through the point $(-4, 3)$ such that the portion of the line between the axes is divided internally by the point in the ratio 5 : 3 is

a. $9x - 20y + 96 = 0$ **b.** $9x + 20y = 24$

c. $20x + 9y + 53 = 0$ **d.** none of these

125. $ABCD$ is a square $A \equiv (1, 2)$, $B \equiv (3, -4)$. If line CD passes through $(3, 8)$ then midpoint of CD is

a. (2, 6) **b.** (6, 2) **c.** (2, 5) **d.** (24/5, 1/5)

126. If the equations $y = mx + c$ and $x \cos \alpha + y \sin \alpha = p$ represent the same straight line, then

a. $p = c\sqrt{1 + m^2}$ **b.** $c = p\sqrt{1 + m^2}$

c. $cp = \sqrt{1 + m^2}$ **d.** $p^2 + c^2 + m^2 = 1$

127. The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is

a. $x + y + 5 = 0$ **b.** $x - y + 1 = 0$

c. $x - y = 5$ **d.** none of these

128. The equation of the line segment AB is $y = x$. If A and B lie on the same side of the line mirror $2x - y = 1$, then the image of AB has the equation

a. $x + y = 2$ **b.** $8x + y = 9$

c. $7x - y = 6$ **d.** none of these

129. The line $L_1 \equiv 4x + 3y - 12 = 0$ intersects the x- and the y-axes at A and B, respectively. A variable line perpendicular to L_1 intersects the x- and the y-axes at P and Q, respectively. Then the locus of the circumcentre of triangle ABQ is

a. $3x - 4y + 2 = 0$ **b.** $4x + 3yy + 7 = 0$

c. $6x - 8y + 7 = 0$ **d.** none of these

130. Consider 3 lines as follows.

$L_1: 5x - y + 4 = 0$

$L_2: 3x - y + 5 = 0$

$L_3: x + y + 8 = 0$

If these lines enclose a triangle ABC and sum of the squares of the tangent to the interior angles can be expressed in the form p/q where p and q are relatively prime numbers, then the value of $p + q$ is

a. 500 **b.** 450

c. 230 **d.** 465

131. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b, c being distinct and different from 1) are concurrent, then $\left(\frac{1}{1-a}\right) + \left(\frac{1}{1-b}\right) + \left(\frac{1}{1-c}\right) =$

a. 0 **b.** 1

c. $1/(a + b + c)$ **d.** none of these

132. If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common, then the joint equation of the other two lines is given by

a. $3x^2 + 8xy - 3y^2 = 0$ **b.** $3x^2 + 10xy + 3y^2 = 0$

c. $y^2 + 2xy - 3x^2 = 0$ **d.** $x^2 + 2xy - 3y^2 = 0$

133. If the origin is shifted to the point $(ab/(a - b), 0)$ without rotation, then the equation $(a - b)(x^2 + y^2) - 2abx = 0$ becomes

a. $(a - b)(x^2 + y^2) - (a + b)xy + abx = a^2$

b. $(a + b)(x^2 + y^2) = 2ab$

c. $(x^2 + y^2) = (a^2 + b^2)$

d. $(a - b)^2 (x^2 + y^2) = a^2b^2$

134. The straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ and $(y - nx)^2 = a^2(1 + n^2)$ form a

a. rectangle **b.** rhombus

c. trapezium **d.** none of these

135. The condition that one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ may coincide with one of those given by the equation $a'x^2 + 2h'xy + b'y^2 = 0$ is

a. $(ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$

b. $(ab' - a'b)^2 = (ha' - h'a)(bh' - b'h)$

c. $(ha' - h'a)^2 = 4(ab' - a'b)(bh' - b'h)$

d. $(bh' - b'h)^2 = 4(ab' - a'b)(ha' - h'a)$

136. The angle between the pair of lines whose equation is $4x^2 + 10xy + my^2 + 5x + 10y = 0$ is

a. $\tan^{-1}(3/8)$

b. $\tan^{-1}(3/4)$

c. $\tan^{-1}(2\sqrt{25 - 4m}/m + 4)$, $m \in R$

d. none of these

137. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is

a. 3 **b.** 2 **c.** $-1/2$ **d.** -1

138. The pair of lines represented by $3ax^2 + 5xy + (a^2 - 2)y^2 = 0$ are perpendicular to each other for

a. two values of a **b.** $\forall\, a$

c. for one value of a **d.** for no value of a

139. If two of the lines represented by $x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = 0$ bisect the angle between the other two, then the value of c is

a. 0 **b.** -1 **c.** 1 **d.** -6

140. If the lines represented by the equation $3y^2 - x^2 + 2\sqrt{3}x - 3 = 0$ are rotated about the point $(\sqrt{3}, 0)$ through an angle 15°, one clockwise direction and other in anti clockwise direction, so that they become perpendicular, then the equation of the pair of lines in the new position is

a. $y^2 - x^2 + 2\sqrt{3}x + 3 = 0$ **b.** $y^2 - x^2 + 2\sqrt{3}x - 3 = 0$

c. $y^2 - x^2 - 2\sqrt{3}x + 3 = 0$ **d.** $y^2 - x^2 + 3 = 0$

141. If the equation of the pair of straight lines passing through the point (1, 1), one making an angle θ with the positive direction of x-axis and the other making the same angle with the positive direction of y-axis, is $x^2 - (a + 2)xy + y^2 + a(x + y - 1) = 0$, $a \neq -2$, then the value of $\sin 2\theta$ is

a. $a - 2$ **b.** $a + 2$

c. $2/(a + 2)$ **d.** $2/a$

142. Equation of a line which is parallel to the line common to the pair of lines given by $6x^2 - xy - 12y^2 = 0$ and $15x^2 + 14xy - 8y^2 = 0$ and at a distance 7 from it is

a. $3x - 4y = -35$ **b.** $5x - 2y = 7$

c. $3x + 4y = 35$ **d.** $2x - 3y = 7$

143. The equation $x^2y^2 - 9y^2 + 6x^2y + 54y = 0$ represents

a. a pair of straight lines and a circle

b. a pair of straight lines and a parabola

c. a set of four straight lines forming a square

d. none of these

144. The combined equation of the lines l_1, l_2 is $2x^2 + 6xy + y^2 = 0$ and that of the lines m_1, m_2 is $4x^2 + 18xy + y^2 = 0$. If the angle between l_1 and m_2 be α, then the angle between l_2 and m_1 will be

a. $\pi/2 - \alpha$ **b.** 2α

c. $\pi/4 + \alpha$ **d.** α

145. The equation $x - y = 4$ and $x^2 + 4xy + y^2 = 0$ represent the sides of

a. an equilateral triangle **b.** a right angled triangle

c. an isosceles triangle **d.** none of these

146. The equation $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ represent

a. two pairs of perpendicular straight lines

b. two pairs of parallel straight lines

c. two pairs of straight lines which are equally inclined to each other

d. none of these

147. The distance between the two lines represented by the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is

a. 8/5 **b.** 6/5

c. 11/5 **d.** none of these

Multiple Correct Answers Type

Solutions on page 1.100

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. If P is a point (x, y) on the line $y = -3x$ such that P and the point (3, 4) are on the opposite sides of the line $3x - 4y = 8$, then

a. $x > 8/15$ **b.** $x > 8/5$

c. $y < -8/5$ **d.** $y < -8/15$

2. If (–6, –4), (3, 5), (–2, 1) are the vertices of a parallelogram, then remaining vertex can be

a. (0, –1) **b.** (7, 9)

c. (–1, 0) **d.** (–11, –8)

3. The lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ are the sides of a square. Equation of the remaining side of the square can be

a. $2x - y + 6 = 0$ **b.** $2x - y + 8 = 0$

c. $2x - y - 10 = 0$ **d.** $2x - y - 14 = 0$

4. Let $0 \equiv (0, 0)$, $A \equiv (0, 4)$, $B \equiv (6, 0)$. 'P' be a moving point such that the area of triangle POA is two times the area of triangle POB. Locus of 'P' will be straight line whose equation can be

a. $x + 3y = 0$ **b.** $x + 2y = 0$

c. $2x - 3y = 0$ **d.** $3y - x = 0$

5. If (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$, then

a. $2\alpha + 3\alpha^2 - 1 > 0$ **b.** $\alpha + 2\alpha^2 - 3 < 0$

c. $\alpha + 2\alpha^2 - 3 < 0$ **d.** $6\alpha^2 - 5\alpha + 1 > 0$

6. Angles made with x-axis by a straight line drawn through (1, 2) so that it intersects $x + y = 4$ at a distance $\sqrt{6}/3$ from (1, 2) are

a. 105° **b.** 75° **c.** 60° **d.** 15°

7. Given three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$. Then,

a. they form a triangle

b. they are concurrent

c. one line bisects the angle between the other two

d. two of them are parallel

8. If (–4, 0) and (1, –1) are two vertices of a triangle of area 4 sq. units, then its third vertex lies on

a. $y = x$ b. $5x + y + 12 = 0$

c. $x + 5y - 4 = 0$ d. $x + 5y + 12 = 0$

9. The equation $x^3 + x^2y - xy^2 = y^3$ represents

a. three real straight lines

b. lines in which two of them are perpendicular to each other

c. lines in which two of them are coincident

d. none of these

10. The straight line represented by $x^2 + mxy - 2y^2 + 3y - 1 = 0$ meets at

a. $(-1/3, 2/3)$ b. $(-1/3, -2/3)$

c. $(1/3, 2/3)$ d. none of these

11. If m_1 and m_2 are the roots of the equation $x^2 - ax - a - 1 = 0$, then the area of the triangle formed by the three straight lines $y = m_1x$, $y = m_2x$ and $y = a$ $(a \neq -1)$ is

a. $\dfrac{a^2(a+2)}{2(a+1)}$ if $a > -1$

b. $\dfrac{-a^2(a+2)}{2(a+1)}$ if $a < -1$

c. $\dfrac{-a^2(a+2)}{2(a+1)}$ if $-2 < a - 1$

d. $\dfrac{a^2(a+2)}{2(a+1)}$ if $a < -2$

12. If the points $\left(a^3/(a-1), (a^2-3)/(a-1)\right)$, $(b^3/(b-1), (b^2-3)/(b-1))$ and $(c^3/(c-1), (c^2-3)/(c-1))$, where a, b, c are different from 1, lie on the line $lx + my + n = 0$, then

a. $a + b + c = -\dfrac{m}{l}$

b. $ab + bc + ca = \dfrac{n}{l}$

c. $abc = \dfrac{(m+n)}{l}$

d. $abc - (bc + ca + ab) + 3(a + b + c) = 0$

13. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is

a. 1 b. 2 c. $-1/2$ d. -1

14. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle, then the value of m is

a. 2 b. 1 c. -1 d. none of these

15. Two sides of a rhombus $OABC$ (lying entirely in first quadrant or third quadrant) of area equal to 2 sq. units are $y = x/\sqrt{3}$, $y = \sqrt{3}\,x$. Then possible coordinates of B is/are (O being origin)

a. $(1 + \sqrt{3}, 1 + \sqrt{3})$ b. $(-1 - \sqrt{3}, -1 - \sqrt{3})$

c. $(3 + \sqrt{3}, 3 + \sqrt{3})$ d. $(\sqrt{3} - 1, \sqrt{3} - 1)$

16. If $(x/a) + (y/b) = 1$ and $(x/c) + (y/d) = 1$ intersect the axes at four concyclic points and $a^2 + c^2 = b^2 + d^2$, then these lines can intersect at $(a, b, c, d > 0)$

a. $(1, 1)$ b. $(1, -1)$ c. $(2, -2)$ d. $(3, 3)$

17. The straight line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. An equilateral triangle ABC is constructed. The possible coordinates of vertex 'C' are

a. $\left(2\left(1 - \dfrac{3\sqrt{3}}{4}\right), \dfrac{3}{2}\left(1 - \dfrac{4}{\sqrt{3}}\right)\right)$

b. $(-2(1 + \sqrt{3}), 3/2\,(1 - \sqrt{3}))$

c. $(2(1 + \sqrt{3}), 3/2\,(1 + \sqrt{3}))$

d. $\left(2\left(1 + \dfrac{3\sqrt{3}}{4}\right), \dfrac{3}{2}\left(1 + \dfrac{4}{\sqrt{3}}\right)\right)$

18. The equation of the lines passing through the point $(1, 0)$ and at a distance $\sqrt{3}/2$ from the origin are

a. $\sqrt{3}x + y - \sqrt{3} = 0$ b. $x + \sqrt{3}y - \sqrt{3} = 0$

c. $\sqrt{3}x - y - \sqrt{3} = 0$ d. $x - \sqrt{3}y - \sqrt{3} = 0$

19. The sides of a triangle are the straight lines $x + y = 1$, $7y = x$ and $\sqrt{3}y + x = 0$. Then which of the following is an interior point of the triangle?

a. circumcentre b. centroid

c. incentre d. orthocentre

In an obtused-angle triangle orthocentre and cicumcentre are exterior to the triangle.

20. If (x, y) be a variable point on the line $y = 2x$ lying between the lines $2(x + 1) + y = 0$ and $x + 3(y - 1) = 0$, then

a. $x \in (-1/2, 6/7)$ b. $x \in (-1/2, 3/7)$

c. $y \in (-1, 3/7)$ d. $y \in (-1, 6/7)$

21. The x-coordinates of the vertices of a square of unit area are the roots of the equation $x^2 - 3|x| + 2 = 0$ and the y-coordinates of the vertices are the roots of the equation $y^2 - 3y + 2 = 0$. Then the possible vertices of the square is/are

a. $(1, 1), (2, 1), (2, 2), (1, 2)$

b. $(-1, 1), (-2, 1), (-2, 2), (-1, 2)$

c. $(2, 1), (1, -1), (1, 2), (2, 2)$

d. $(-2, 1), (-1, -1), (-1, 2), (-2, 2)$

22. The area of triangle ABC is 20 cm^2. The coordinates of vertex A are $(-5, 0)$ and those of B are $(3, 0)$. The vertex C lies on the line $x - y = 2$. The coordinates of C are

a. $(5, 3)$ b. $(-3, -5)$ c. $(-5, -7)$ d. $(7, 5)$

23. If the straight line $ax + cy = 2b$ where $a, b, c > 0$ makes a triangle of area 2 sq. units with coordinate axes, then

a. a, b, c are in G.P. b. $a, -b, c$ are in G.P.

c. $a, 2b, c$ are in G.P. d. $a, -2b, c$ are in G.P.

24. If $(a \cos \theta_1, a \sin \theta_1)$, $(a \cos \theta_2, a \sin \theta_2)$ and $(a \cos \theta_3, a \sin \theta_3)$ represents the vertices of an equilateral triangle inscribed in a circle, then

a. $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$
b. $\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$
c. $\tan \theta_1 + \tan \theta_2 + \tan \theta_3 = 0$
d. $\cot \theta_1 + \cot \theta_2 + \cot \theta_3 = 0$

25. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, then

a. the lines will pass through a fixed point
b. there will be a set of parallel lines
c. all the lines intersect the line $x = x_1$
d. all the lines will be parallel to the line $y = x_1$

26. The points $A(0, 0)$, $B(\cos \alpha, \sin \alpha)$ and $C(\cos \beta, \sin \beta)$ are the vertices of a right-angled triangle if

a. $\sin \frac{\alpha - \beta}{2} = \frac{1}{\sqrt{2}}$ **b.** $\cos \frac{\alpha - \beta}{2} = -\frac{1}{\sqrt{2}}$
c. $\cos \frac{\alpha - \beta}{2} = \frac{1}{\sqrt{2}}$ **d.** $\sin \frac{\alpha - \beta}{2} = -\frac{1}{\sqrt{2}}$

27. The ends of a diagonal of a square are (2, –3) and (–1, 1). Another vertex of the square can be

a. (–3/2, –5/2) **b.** (5/2, 1/2)
c. (1/2, 5/2) **d.** none of these

28. If each of the vertices of a triangle has integral coordinates, then the triangle may be

a. right angled **b.** equilateral
c. isosceles **d.** none of these

29. If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents

a. the median through A
b. the altitude through A
c. the perpendicular bisector of BC
d. the line joining the centroid with a vertex

30. In a ΔABC, $A \equiv (\alpha, \beta)$, $B \equiv (1, 2)$, $C \equiv (2, 3)$ and point 'A' lies on the line $y = 2x + 3$ where $\alpha, \beta \in$ integer and area of the triangle is S such that $[S] = 2$ where $[\cdot]$ denotes the greatest integer function. Then all possible coordinates of A.

a. (–7, –11) **b.** (–6, –9)
c. (2, 7) **d.** (3, 9)

31. Let $P(\sin \theta, \cos \theta)$ $(0 \le \theta \le 2\pi)$ be a point in triangle with vertices (0, 0), $(\sqrt{3/2}, 0)$ and $(0, \sqrt{3/2})$. Then,

a. $0 < \theta < \pi/12$ **b.** $5\pi/2 < \theta < \pi/2$
c. $0 < \theta < 5\pi/2$ **d.** $5\pi/2 < \theta < \pi$

32. Equation(s) of the straight line(s), inclined at 30° to the x-axis such that the length of its (each of their) line segment(s) between the coordinate axes is 10 units is are

a. $x + \sqrt{3}\,y + 5\sqrt{3} = 0$
b. $x - \sqrt{3}y + 5\sqrt{3} = 0$
c. $x + \sqrt{3}y - 5\sqrt{3} = 0$
d. $x - \sqrt{3}y - 5\sqrt{3} = 0$

33. The lines $x + y - 1 = 0$, $(m - 1)x + (m^2 - 7)y - 5 = 0$ and $(m - 2)x + (2m - 5)y = 0$ are

a. concurrent for three values of m
b. concurrent for one value of m
c. concurrent for no value of m
d. are parallel for $m = 3$

34. Equation of a straight line passing through the point (2, 3) and inclined at an angle of $\tan^{-1}(1/2)$ with the line $y + 2x = 5$ is

a. $y = 3$ **b.** $x = 2$
c. $3x + 4y - 18 = 0$ **d.** $4x + 3y - 17 = 0$

35. The equation of the lines on which the perpendiculars from the origin make 30° angle with x-axis and which form a triangle of area $50/\sqrt{3}$ with axes are

a. $\sqrt{3}x + y - 10 = 0$ **b.** $\sqrt{3}x + y + 10 = 0$
c. $x + \sqrt{3}y - 10 = 0$ **d.** $x - \sqrt{3}y - 10 = 0$

36. Sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that diagonals of the rhombus intersect at (1, 3) and one vertex, 'A' of the rhombus lies on the line $y = 2x$. Then the coordinates of the vertex A are

a. (8/5, 16/5) **b.** (7/15, 14/15)
c. (6/5, 12/5) **d.** (4/15, 8/15)

37. The equations of two equal sides AB and AC of an isosceles triangle ABC are $x + y = 5$ and $7x - y = 3$, respectively. Then the equations of the sides BC if ar(ΔABC) = 5 unit^2.

a. $x - 3y + 1 = 0$ **b.** $x - 3y - 21 = 0$
c. $3x + y + 2 = 0$ **d.** $3x + y - 12 = 0$

38. Two straight lines $u = 0$ and $v = 0$ pass through the origin and angle between them is $\tan^{-1}(7/9)$. If the ratio of the slope of $v = 0$ and $u = 0$ is 9/2 then their equations are

a. $y + 3x = 0$ and $3y + 2x = 0$
b. $2y + 3x = 0$ and $3y + x = 0$
c. $2y = 3x$ and $3y = x$
d. $y = 3x$ and $3y = 2x$

39. Let $u \equiv ax + by + a\sqrt[3]{b} = 0$, $v \equiv bx - ay + b\sqrt[3]{a} = 0$, $a, b \in R$ be two straight lines. The equations of the bisectors of the angle formed by $k_1u - k_2v = 0$ and $k_1u + k_2v = 0$ for nonzero real k_1 and k_2 are

a. $u = 0$ **b.** $k_2u + k_1v = 0$

c. $k_2u - k_1v = 0$ **d.** $v = 0$

40. If $x^2 + 2hxy + y^2 = 0$ represents the equation of the straight lines through the origin which make an angle α with the straight line $y + x = 0$, then

a. $\sec 2\alpha = h$ **b.** $\cos\alpha = \sqrt{(1+h)/(2h)}$

c. $2\sin\alpha = \sqrt{(1+h)/h}$ **d.** $\cot\alpha = \sqrt{(h+1)/(h-1)}$

41. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle, then

a. $2 < a < 10/3$ **b.** $-2 < a < 10/3$

c. $-1 < b < 9/2$ **d.** $-1 < b < 1$

42. If one of the lines given by the equation $2x^2 + pxy + 3y^2 = 0$ coincide with one of those given by $2x^2 + qxy - 3y^2 = 0$ and the other lines represented by them be perpendicular, then

a. $p = 5$ **b.** $p = -5$ **c.** $q = -1$ **d.** $q = 1$

43. The lines joining the origin to the point of intersection of $3x^2 + mxy - 4x + 1 = 0$ and $2x + y - 1 = 0$ are at right angles. Then which of the following is not possible value of m?

a. -4 **b.** 4 **c.** 7 **d.** 3

Reasoning Type

Solutions on page 1.107

Each question has four choices, a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are True but Statement 2 is the correct explanation of Statement 1.

b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.

c. Statement 1 is True and Statement 2 is False.

d. Statement 1 is False and Statement 2 is True.

1. Statement 1: The lines $(a + b)x + (a - 2b)y = a$ are concurrent at the point (2/3, 1/3).

Statement 2: The lines $x + y - 1 = 0$ and $x - 2y = 0$ intersect at the point (2/3, 1/3).

2. Statement 1: If the vertices of a triangle are having rational coordinates then its centroid, circumcentre and orthocentre are rational.

Statement 2: In any triangle, orthocentre, centroid and circumcentre are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1.

3. Statement 1: If sum of algebraic distances from points A (1, 1), B (2, 3), C (0, 2) is zero on the line $ax + by + c = 0$, then $a + 3b + c = 0$.

Statement 2: The centroid of triangle is (1, 2).

4. Statement 1: Each point on the line $y - x + 12 = 0$ is equidistant from the lines $4y + 3x - 12 = 0$, $3y + 4x - 24 = 0$.

Statement 2: The locus of a point which is equidistant from two given lines is the angular bisector of the two lines.

5. Statement 1: If the point $(2a - 5, a^2)$ is on the same side of the line $x + y - 3 = 0$ as that of the origin, then $a \in (2, 4)$.

Statement 2: The points (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides of the line $ax + by + c = 0$, as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs.

6. Statement 1: Lines passing through the given point and is equally inclined to the given two lines are always perpendicular.

Statement 2: Angle bisectors of the given two lines are always perpendicular.

7. Statement 1: The area of the triangle formed by the points A (1000, 1002), B (1001, 1004) C (1002, 1003) is same as the area formed by A' (0, 0), B' (1, 2), C' (2, 1).

Statement 2: The area of the triangle is constant with respect to translation of axes.

8. Statement 1: If the diagonals of the quadrilateral formed by the lines $px + qy + r = 0$, $p'x + q'y + r = 0$ are at right angles, then $p^2 + q^2 = p'^2 + q'^2$.

Statement 2: Diagonals of a rhombus are bisected and perpendicular to each other.

9. Statement 1: The joint equation of lines $y = x$ and $y = -x$ is $y^2 = -x^2$, i.e., $x^2 + y^2 = 0$.

Statement 2: The joint equation of lines $ax + by = 0$ and $cx + dy = 0$ is $(ax + by)(cx + dy) = 0$, where a, b, c, d are constant.

10. Statement 1: The internal angle bisector of angle C of a triangle ABC with sides AB, AC and BC as $y = 0$, $3x + 2y = 0$ and $2x + 3y + 6 = 0$, respectively, is $5x + 5y + 6 = 0$.

Statement – 2: Image of point A with respect to $5x + 5y + 6 = 0$ lies on side BC of the triangle.

11. Statement 1: If the lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the axis in A, B and y-axis at C, D, then points A, B, C, D are concyclic.

Statement 2: Since $OA \times OB = OC \times OD$, where O is origin, therefore A, B, C, D are concyclic.

12. Statement 1: Let the vertices of a ΔABC are A (–5, –2), B (7, 6) and C (5, – 4). Then coordinates of circumcentre are (1, 2).

Statement 2: In a right angle triangle, midpoint of hypotenuse is the circumcenter of the triangle.

13. **Statement 1:** The incentre of a triangle formed by the lines $x \cos(\pi/9) + y \sin(\pi/9) = \pi$, $x \cos(8\pi/9) + y \sin(8\pi/9) = \pi$; $x \cos(13\pi/9) + y \sin(13\pi/9) = \pi$ is (0, 0).

    Statement 2: Any point equidistant from the given three non-concurrent straight lines in the plane is incentre of the triangle.

14. **Statement 1:** If $(a_1x + b_1y + c_1) + (a_2x + b_2y + c_2) + (a_3x + b_3y + c_3) = 0$, then lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ cannot be parallel.

    Statement 2: If sum of the equations for three straight lines is identically zero, then they are either concurrent or parallel.

15. **Statement 1:** The lines $(a + b)x + (a - b)y - 2ab = 0$, $(a - b)x + (a + b)y - 2ab = 0$ and $x + y = 0$ form an isosceles triangle.

    Statement 2: If internal bisector of any angle of triangle is perpendicular to the opposite side, then the given triangle is isosceles.

16. **Statement 1:** If $-2h = a + b$, then one line of the pair of lines $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes in positive quadrant.

    Statement 2: If $ax + y(2h + a) = 0$ is a factor of $ax^2 + 2hxy + by^2 = 0$, then $b + 2h + a = 0$.

Linked Comprehension Type

Solutions on page 1.108

Based upon each paragraph, some multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Let L be the line belonging to the family of the straight lines $(a + 2b)x + (a - 3b)y + a - 8b = 0$, $a, b \in R$, which is farthest from the point (2, 2).

1. The equation of line L is

   a. $x + 4y + 7 = 0$ **b.** $2x + 3y + 4 = 0$
   c. $4x - y - 6 = 0$ **d.** none of these

2. Area formed by the line L with coordinate axis is

   a. 4/3 sq. units **b.** 9/2 sq. units
   c. 49/8 sq. units **d.** none of these

3. If L is concurrent with the lines $x - 2y + 1 = 0$ and $3x - 4y + \lambda = 0$, then the value of λ is

   a. 2 **b.** 1 **c.** -4 **d.** 5

For Problems 4–6

The equation of an altitude of an equilateral triangle is $\sqrt{3}x + y = 2\sqrt{3}$ and one of the vertices is $(3, \sqrt{3})$.

4. The possible number of triangles is

   a. 1 **b.** 2 **c.** 3 **d.** 4

5. Which of the following is not one of the possible vertices the triangle?

   a. (0, 0) **b.** $(0, 2\sqrt{3})$
   c. $(3, -\sqrt{3})$ **d.** none of these

6. Which of the following is one of the orthocentres of the triangle?

   a. $(1, \sqrt{3})$ **b.** $(0, \sqrt{3})$
   c. (0, 2) **d.** none of these

For Problems 7–9

For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O = (0, 0)$ and $A = (3, 2)$. Consider the set of points P in the first quadrant which are equidistant (with respect to the new distance) from O and A.

7. The set of points P consists of

   a. one straight line only
   b. union of two line segments
   c. union of two infinite rays
   d. union of a line segment of finite length and an infinite ray

8. The area of the region bounded by locus of P and line $y = 4$ in first quadrant is

   a. 2 sq. units **b.** 4 sq. units
   c. 6 sq. units **d.** none of these

9. The locus of points P is

   a. one-one and onto function
   b. many-one and onto function
   c. one-one and into function
   d. relation but not function

For Problems 10–12

A variable line 'L' is drawn through $O(0, 0)$ to meet the lines L_1 and L_2 given by $y - x - 10 = 0$ and $y - x - 20 = 0$ at the points A and B, respectively.

10. A point P is taken on 'L' such that $2/OP = 1/OA + 1/OB$. Then the locus of 'P' is

    a. $3x + 3y = 40$ **b.** $3x + 3y + 40 = 0$
    c. $3x - 3y = 40$ **d.** $3y - 3x = 40$

11. Locus of P, if $OP^2 = OA \times OB$, is

    a. $(y - x)^2 = 100$ **b.** $(y + x)^2 = 50$
    c. $(y - x)^2 = 200$ **d.** none of these

12. Locus of P, if $(1/OP^2) = (1/OB^2) + (1/OA^2)$, is

    a. $(y - x)^2 = 80$ **b.** $(y - x)^2 = 100$
    c. $(y - x)^2 = 64$ **d.** none of these

For Problems 13–15

The line $6x + 8y = 48$ intersects the coordinate axes at A and B, respectively. A line L bisects the area and the perimeter of the triangle OAB where O is the origin.

13. The number of such lines possible is

a. 1 **b.** 2 **c.** 3 **d.** more than 3

14. The slope of the line L can be

a. $(10 + 5\sqrt{6})/10$ **b.** $(10 - 5\sqrt{6})/10$

c. $(8 + 3\sqrt{6})/10$ **d.** none of these

15. The line L

a. does not intersect AB

b. does not intersect OB

c. does not intersect OA

d. can intersect all the sides

For Problems 16–18

$A(1, 3)$ and $C(-2/5, -2/5)$ are the vertices of a triangle ABC and the equation of the internal angle bisector of $\angle ABC$ is $x + y = 2$.

16. Equation of side BC is

a. $7x + 3y - 4 = 0$ **b.** $7x + 3y + 4 = 0$

c. $7x - 3y + 4 = 0$ **d.** $7x - 3y - 4 = 0$

17. Coordinates of vertex B are

a. (3/10, 17/10) **b.** (17/10, 3/10)

c. (–5/2, 9/2) **d.** (1, 1)

18. Equation of side AB is

a. $3x + 7y = 24$ **b.** $3x + 7y + 24 = 0$

c. $13x + 7y + 8 = 0$ **d.** $13x - 7y + 8 = 0$

For Problems 19–21

Let $ABCD$ be a parallelogram whose equations for the diagonals AC and BD are $x + 2y = 3$ and $2x + y = 3$, respectively. If length of diagonal $AC = 4$ units and area of parallelogram $ABCD = 8$ sq. units, then

19. the length of other diagonal BD is

a. 10/3 **b.** 2 **c.** 20/3 **d.** none of these

20. The length of side AB is equal to

a. $2\sqrt{58}/3$ **b.** $4\sqrt{58}/9$ **c.** $3\sqrt{58}/9$ **d.** $4\sqrt{58}/9$

21. The length of BC is equal to

a. $2\sqrt{10}/3$ **b.** $4\sqrt{10}/3$

c. $8\sqrt{10}/3$ **d.** none of these

For Problems 22–24

Consider a triangle PQR with coordinates of its vertices as $P(-8, 5)$; $Q(-15, -19)$ and $R(1, -7)$. The bisector of the interior angle of P has the equation which can be written in the form $ax + 2y + c = 0$.

22. The distance between the orthocentre and the circumcentre of the triangle PQR is

a. 25/2 **b.** 29/2 **c.** 37/2 **d.** 51/2

23. Radius of the incircle of the triangle PQR is

a. 4 **b.** 5 **c.** 6 **d.** 8

24. The sum of $a + c$ is

a. 129 **b.** 78 **c.** 89 **d.** none of these

For Problems 25–27

Let us consider the situation when axes are inclined at an angle 'ω'. If coordinates of a point P are (x_1, y_1), then $PN = x_1$, $PM = y_1$, where PM is parallel to y-axis and PN is parallel x-axis. Straight line through P that makes an angle θ with x-axis is

$$RQ = y - y_1, PQ = x - x_1$$

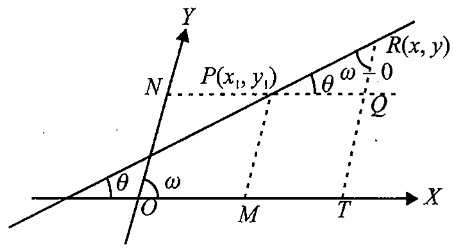


Fig. 1.72

From ΔPQR, we have

$$\frac{PQ}{\sin(\omega - \theta)} = \frac{RQ}{\sin\theta}$$

$$\Rightarrow \quad y - y_1 = \frac{\sin\theta}{\sin(\omega - \theta)}(x - x_1)$$

written in the form of $y - y_1 = m(x - x_1)$ where

$$m = \frac{\sin\theta}{\sin(\omega - \theta)}$$

Therefore, if slope of the line is m, then angle of inclination of the line with x-axis is given by

$$\tan\theta = \left(\frac{m\sin\omega}{1 + m\cos\omega}\right)$$

25. The axes being inclined at an angle of 60°, the inclination of the straight line $y = 2x + 5$ with x-axis is

a. 30° **b.** $\tan^{-1}(\sqrt{3}/2)$

c. $\tan^{-1}2$ **d.** 60°

26. The axes being inclined at an angle of 60°, the angle between the two straight lines $y = 2x + 5$ and $2y + x + 7 = 0$ is

a. 90° **b.** $\tan^{-1}(5/3)$

c. $\tan^{-1}(\sqrt{3}/2)$ **d.** $\tan^{-1}(5/\sqrt{3})$

27. The axes being inclined at an angle of 30°, the equation of straight line which makes an angle of 60° with the positive direction of x-axis and x-intercept 2 is

a. $y-\sqrt{3}x=0$ **b.** $\sqrt{3}y=x$

c. $y+\sqrt{3}x=2\sqrt{3}$ **d.** $y+2x=0$

For Problems 28–29

Consider the triangle having vertices O (0, 0), A (2, 0) and B(1, $\sqrt{3}$). Also $b \le \min\{a_1, a_2, a_3, \ldots, a_n\}$ means $b \le a_1$ when a_1 is least; $b \le a_2$ when a_2 is least and so on. From this we can say $b \le a_1, b \le a_2, \ldots, b \le a_n$.

28. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min [d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Then the area of the region R is

a. $\sqrt{3}$ sq. units **b.** $(2+\sqrt{3})$ sq. units

c. $\sqrt{3}/2$ sq. units **d.** $1/\sqrt{3}$ sq. units

29. Let R be the region consisting of all those points P inside ΔOAB which satisfy $OP \le \min [BP, AP]$. Then the area of the region R is

a. $\sqrt{3}$ sq. units **b.** $1/(2\sqrt{3})$ sq. units

c. $\sqrt{3}/2$ sq. units **d.** none of these

For Problems 30–32

Let $ABCD$ is a square with sides of unit length. Points E and F are taken on sides AB and AD respectively so that $AE = AF$. Let P be a point inside the square $ABCD$.

30. The maximum possible area of quadrilateral $CDFE$ is

a. 1/8 **b.** 1/4

c. 5/8 **d.** 3/8

31. The value of $(PA)^2-(PB)^2+(PC)^2-(PD)^2$ is equal to

a. 3 **b.** 2

c. 1 **d.** 0

32. Let a line passing through point A divides the square $ABCD$ into two parts so that area of one portion is double the other, then the length of portion of line inside the square is

a. $\sqrt{10}/3$ **b.** $\sqrt{13}/3$

c. $\sqrt{11}/3$ **d.** $2/\sqrt{3}$

Matrix-Match Type

Solutions on page 1.112

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, b → q, b → r, c → p, c → q, and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1. Consider the lines represented by equation $(x^2+xy-x) \times (x-y)=0$, forming a triangle. Then match the following.

Column I	Column II
a. Orthocentre of triangle	**p.** (1/6, 1/2)
b. Circumcentre	**q.** $(1/(2+2\sqrt{2}), 1/2)$
c. Centroid	**r.** (0, 1/2)
d. Incentre	**s.** (1/2, 1/2)

2. Consider the triangle formed by the lines

$y+3x+2=0$, $3y-2x-5=0$, $4y+x-14=0$

Column I	Column II
a. Values of α if (0, α) lies inside triangle	**p.** $(-\infty, 7/3)\cup(13/4, \infty)$
b. Values of α if (α, 0) lies inside triangle	**q.** $-4/3 < \alpha < 1/2$
c. Values of α if (α, 2) lies inside triangle	**r.** No value of α
d. Value of α if (1, α) lies outside triangle	**s.** $5/3 < \alpha < 7/2$

3.

Column I	Column II
a. A straight line with negative slope passing through (1, 4) meets the coordinate axes at A and B. The minimum length of $OA + OB$, O being the origin, is	**p.** $5\sqrt{2}$
b. If the point P is symmetric to the point Q(4, −1) with respect to the bisector of the first quadrant, then the length of PQ is	**q.** $3\sqrt{2}$
c. On the portion of the straight line $x + y = 2$ between the axis a square is constructed away from the origin, with this portion as one of its sides. If 'd' denotes the perpendicular distance of a side of this square from the origin then the maximum value of 'd' is	**r.** 9/2
d. If the parametric equation of a line is given by $x = 4 + \lambda/\sqrt{2}$ and $y = -1 + \sqrt{2}\lambda$ where λ is a parameter, then the intercept made by the line on the x-axis is	**s.** 9

4.

Column I	Column II
a. If lines $3x + y - 4 = 0$, $x - 2y - 6 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then value of λ is	**p.** -4
b. If the points $(\lambda + 1, 1)$, $(2\lambda + 1, 3)$ and $(2\lambda + 2, 2\lambda)$ are collinear, then the value of λ is	**q.** $-1/2$
c. If line $x + y - 1 - \lvert \lambda/2 \rvert = 0$, passing through the intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$, is perpendicular to one of them, then the value of λ is	**r.** 4
d. If line $y - x - 1 + \lambda = 0$ is equidistant from the points $(1, -2)$ and $(3, 4)$, then λ is	**s.** 2

5.

Column I	Column II
a. Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$. If orthocentre is the origin, then coordinates of the third vertex are	**p.** $(-4, -7)$
b. A point on the line $x + y = 4$ which lies at a unit distance from the line $4x + 3y = 10$ is	**q.** $(-7, 11)$
c. Orthocentre of the triangle formed by the lines $x + y - 1 = 0$, $x - y + 3 = 0$, $2x + y = 7$ is	**r.** $(2, -2)$
d. If $2a$, b, c are in A.P., then lines $ax + by = c$ are concurrent at	**s.** $(-1, 2)$

6.

Column I	Column II
a. Four lines $x + 3y - 10 = 0$, $x + 3y - 20 = 0$, $3x - y + 5 = 0$ and $3x - y - 5 = 0$ form a figure which is	**p.** a quadrilateral which is neither a parallelogram nor a trapezium
b. The points $A(1, 2)$, $B(2, -3)$, $C(-1, -5)$ and $D(-2, 4)$ in order are the vertices of	**q.** a parallelogram
c. The lines $7x + 3y - 33 = 0$, $3x - 7y + 19 = 0$, $3x - 7y - 10$ and $7x + 3y - 4 = 0$ form a figure which is	**r.** a rectangle of area 10 sq. units
d. Four lines $4y - 3x - 7 = 0$, $3y - 4x + 7 = 0$, $4y - 3x - 21 = 0$, $3y - 4x + 14 = 0$ form a figure which is	**s.** a square

7.

Column I	Column II
a. The distance between the lines $(x + 7y)^2 + 4\sqrt{2}\,(x + 7y) - 42 = 0$ is	**p.** 2
b. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is $\lvert x \rvert + \lvert y \rvert = k$, where k is equal to	**q.** 7
c. If $6x + 6y + m = 0$ is acute angle bisector of line $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$, then m is equal to	**r.** 3
d. Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 6$ is	**s.** 1

8.

Column I	Column II
a. The value k for which $4x^2 + 8xy + ky^2 = 9$ is the equation of a pair of straight lines is	**p.** 3
b. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then the value of c is	**q.** -3
c. If the gradient of one of the lines $x^2 + hxy + 2y^2 = 0$ is twice that of the other, then $h =$	**r.** 2
d. If the lines $ax^2 + 2hxy + by^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$, then the value of λ can be	**s.** 4

9. O is origin and B is a point on the x-axis at a distance of 2 units from the origin.

Column I	Column II
a. If $\Delta\, AOB$ is equilateral triangle, then the coordinates of A can be	**p.** $(-1, \sqrt{3})$
b. If $\Delta\, AOB$ is isosceles such that $\angle OAB$ is 30°, then coordinate of A can be	**q.** $(-1, 2 - \sqrt{3})$
c. If OB is one side of rhombus of area $\sqrt{3}$ units, then other vertices of rhombus can be	**r.** $(-3, -\sqrt{3})$
d. If OB is a chord of circle with radius equal to OB, then coordinates of point A on the circumference of the circle such that ΔOAB is isosceles can be	**s.** $(1, 2 + \sqrt{3})$

10.

Column I	Column II
a. The lines $y = 0$; $y = 1$; $x - 6y + 4 = 0$ and $x + 6y - 9 = 0$ constitute a figure which is	**p.** a cyclic quadratieral

b.	The points $A(a, 0)$, $B(0, b)$, $C(c, 0)$ and $D(0, d)$ are such that $ac = bd$ and a, b, c, d are all non-zero. The points A, B, C and D always constitute	**q.** a rhombus
c.	The figure formed by the four lines $ax \pm by \pm c = 0$ $(a \neq b)$ is	**r.** a square
d.	The line pairs $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ constitue a figure which is	**s.** a trapezium

Integer Type

Solutions on page 1.118

1. The area of the triangular region in the first quadrant bounded on the left by the y-axis, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$ is A, then the value of $3A/10$ is
2. If the area enclosed by the graph of $x^2y^2 - 9x^2 - 25y^2 + 225 = 0$ is A if then value of $A/10$ is
3. The number of values of k for which the lines $(k + 1)x + 8y = 4k$ and $kx + (k + 3)y = 3k - 1$ are coincident
4. The sides of a triangle ABC lie on the lines $3x + 4y = 0$, $4x + 3y = 0$ and $x = 3$. Let (h, k) be the centre of the circle inscribed in ΔABC. The value of $(h + k)$ equals
5. Number of value of b for which in an acute triangle ABC, if the coordinates of orthocentre 'H' are $(4, b)$, centroid 'G' are $(b, 2b - 8)$ and circumcentre 'S' are $(-4, 8)$, is
6. The piont A divided the join of $P(-5, 1)$, Q $(3, 5)$ in the ratio $k : 1$, then the integral value of k for which the area of ΔABC where B is $(1, 5)$ and C is $(7, -2)$ is equal to 2 units in magnitude is
7. The sides of a triangle have the combined equation $x^2 - 3y^2 - 2xy + 8y - 4 = 0$. The third side, which is variable always passes through the point $(-5, -1)$. If the range of values of the slope of the third line is such that the origin is an interior point of the triangle is (a, b) then the value of $\left(a + \frac{1}{b}\right)$ is
8. If area of the triangle formed by the line $x + y = 3$ and the angle bisectors of the pair of lines $x^2 - y^2 + 4y - 4 = 0$ is A, then the value of $16A$ is
9. The points (x, y) lies on the line $2x + 3y = 6$. The smallest value of the quantity $\sqrt{x^2 + y^2}$ is m then the value of $\sqrt{13}\, m$ is
10. The distance between the circumcnetre and orthocenter of the triangle whose vertices are $(0, 0)$, $(6,8)$ and $(-4, 3)$ is L, then the value of $\frac{2}{\sqrt{5}} L$ is
11. Absolute value of the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$ is
12. The line $x = C$ cuts the triangle with vertices $(0, 0)$, $(1, 1)$ and $(9, 1)$ into two regions. For the area of the two regions to be the same, C must be equal to
13. A man starts from the point $P(-3, 4)$ and reaches point $Q(0, 1)$ touching x-axis at R $(\alpha, 0)$ such that $PR + RQ$ is minimum, then $5|\alpha| =$
14. If the area of triangle formed by the points $(2a, b)$ $(a + b, 2b + a)$ and $(2b, 2a)$ be 2 sq. units, then the area of the triangle whose vertices are $(a + b, a - b)$, $(3b - a, b + 3a)$ and $(3a - b, 3b - a)$ will be
15. For all real values of a and b, lines $(2a + b)\, x + (a + 3b)\, y + (b - 3a) = 0$ and $mx + 2y + 6 = 0$ are concurrent, then $|m|$ is equal to
16. The line $3x + 2y = 24$ meets the y-axis at A and the x-axis at B. The perpendicular bisector of AB meets the line through $(0, -1)$ parallel to x-axis at C. If the area of the triangle ABC is A then the value of $A/13$ is
17. Consider a ΔABC whose sides AB, BC and CA are represented by the straight lines $2x + y = 0$, $x + py = q$ and $x - y = 3$, respectively. The point P is $(2, 3)$ is orthocenter then the value of $(p + q)/10$ is
18. Triangle ABC with $AB = 13$, $BC = 5$ and $AC = 12$ slides on the coordinates axis with A and B on the positive x-axis and positive y-axis respectively, the locus of vertex C is a line $12x - ky = 0$, then the value of k is

Archives

Solutions on page 1.121

Subjective Type

1. A straight line segment of length l moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line segment in the ratio 1 : 2.
(IIT-JEE, 1978)
2. The area of a triangle is 5. Two of its vertices are $A(2, 1)$ and $B(3, -2)$. The third vertex C is on $y = x + 3$. Find C.
(IIT-JEE, 1978)
3. One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Find the equations of the other three sides. **(IIT-JEE, 1978)**
4. **a.** Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$. If the orthocentre of the triangle is the origin, find the coordinates of the third point.

   b. Find the equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$. **(IIT-JEE, 1979)**

5. A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the coordinate axes is 5. Find the equation of the line L.

(IIT-JEE, 1980)

6. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining vertices.

(IIT-JEE, 1981)

7. The ends A, B of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY, respectively. If the rectangle $OAPB$ be completed, then show that the locus of the foot of the perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = c^{2/3}$. **(IIT-JEE, 1983)**

8. The vertices of a triangle are $(at_1t_2, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$, $(at_3t_1, a(t_3 + t_1))$. Find the orthocentre of the triangle. **(IIT-JEE, 1983)**

9. The coordinates of A, B, C are (6, 3), (–3, 5), (4, – 2), respectively, and P is any point (x, y). Show that the ratio of the area of the triangles ΔPBC and ΔABC is $|x + y - 2|/7$. **(IIT-JEE, 1983)**

10. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Determine the equation of the third side. **(IIT-JEE, 1984)**

11. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4), respectively, then find the area of the rectangle. **(IIT-JEE, 1985)**

12. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis, then find possible coordinates of A.

(IIT-JEE, 1985)

13. The equation of the perpendicular bisectors of the sides AB and AC of triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the point A is (1, – 2), then find the equation of the line BC. **(IIT-JEE, 1986)**

14. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line different from L_2 which passes through P and makes the same angle θ with L_1.

(IIT-JEE, 1988)

15. Let ABC be a triangle with $AB = AC$. If D is the mid-point of BC, is the foot of the perpendicular drawn from D to AC and F is the midpoint of DE, then prove that AF is perpendicular to BE. **(IIT-JEE, 1989)**

16. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Points B and C are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point (1, 2).

(IIT-JEE, 1990)

17. A line cuts the x-axis at $A(7, 0)$ and the y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x-axis at P and the y-axis at Q. If AQ and BP intersect at R, then find the locus of R.

(IIT-JEE, 1990)

18. Find the equation of the line passing through the point (2, 3) and making intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$. **(IIT-JEE, 1991)**

19. Show that all chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point.

(IIT-JEE, 1991)

20. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines

$$2x + 3y - 1 = 0$$
$$x + 2y - 3 = 0$$
$$5x - 6y - 1 = 0$$

(IIT-JEE, 1992)

21. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D, respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line. **(IIT-JEE, 1993)**

22. For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O = (0, 0)$ and $A = (3, 2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.

(IIT-JEE, 2000)

23. A straight line L through the origin meets the lines $x + y = 1$ and $x + y = 3$ at P and Q, respectively. Through P and Q two straight lines L_1 and L_2 are drawn, parallel to $2x - y = 5$ and $3x + y = 5$, respectively. Lines L_1 and L_2 intersect at R. Show that the locus of R, as L varies, is a straight line. **(IIT-JEE, 2002)**

24. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of $OP + OQ$ as L varies, where O is the origin. **(IIT-JEE, 2002)**

25. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through $P(h, k)$ with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point P. **(IIT-JEE, 2005)**

26. A rectangle $PQRS$ has its side PQ parallel to the line $y = mx$ and vertices P, Q and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R.

(IIT-JEE, 1996)

Objective Type

Fill in the blanks

1. The area enclosed within the curve $|x| + |y| = 1$ is_____. **(IIT-JEE, 1981)**
2. The set of lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$, is concurrent at the point__________. **(IIT-JEE, 1982)**
3. If a, b and c are in A.P., then the straight line $ax + by + c = 0$ will always pass through a fixed point whose coordinates are____________. **(IIT-JEE, 1984)**
4. The orthocentre of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in quadrant number________________. **(IIT-JEE, 1985)**
5. Let the algebraic sum of the perpendicular distance from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero. Then the line passes through a fixed point whose coordinates are____________. **(IIT-JEE, 1991)**
6. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the bisector of the angle $\angle ABC$ is_______________. **(IIT-JEE, 1993)**

True or false

1. The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$. **(IIT-JEE, 1983)**
2. The lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the coordinate axes in concyclic points. **(IIT-JEE, 1988)**

Multiple choice questions with one correct answer

1. The point (4, 1) undergoes the following three transformations successively.
   - **i.** Reflection about the line $y = x$.
   - **ii.** Translation through a distance 2 units along the positive direction of x-axis.
   - **iii.** Rotation through an angle $\pi/4$ about the origin in the counterclockwise direction.

   Then the final position of the point is given by the coordinates

   a. $(1/\sqrt{2}, 7/\sqrt{2})$ **b.** $(-\sqrt{2}, 7\sqrt{2})$
   c. $(-1/\sqrt{2}, 7/\sqrt{2})$ **d.** $(\sqrt{2}, 7\sqrt{2})$
   (IIT-JEE, 1980)
2. The straight lines $x + y = 0$, $3x + y - 4 = 0$, $x + 3y - 4 = 0$ from a triangle which is

   a. isosceles **b.** equilateral
   c. right angled **d.** none of these
   (IIT-JEE, 1983)
3. If $P = (1, 0)$, $Q = (-1, 0)$ and $R = (2, 0)$ are three given points, then locus of the point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is

   a. a straight line parallel to x-axis
   b. a circle passing through the origin
   c. a circle with the centre at the origin
   d. a straight line parallel to y-axis **(IIT-JEE, 1988)**
4. Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q then,

   a. $a^2 + b^2 = p^2 + q^2$
   b. $1/a^2 + 1/b^2 = 1/p^2 + 1/q^2$
   c. $a^2 + p^2 = b^2 + q^2$
   d. $1/a^2 + 1/p^2 = 1/b^2 + 1/q^2$ **(IIT-JEE, 1990)**
5. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

   a. square **b.** circle
   c. straight line **d.** two intersecting lines
   (IIT-JEE, 1992)
6. The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is

   a. (1/2, 1/2) **b.** (1/3, 1/3)
   c. (0, 0) **d.** (1/4, 1/4)
   (IIT-JEE, 1995)
7. Let PQR be a right-angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is

   a. $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
   b. $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
   c. $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
   d. $3x^2 - 3y^2 - 8xy - 15y - 20 = 0$
   (IIT-JEE, 1999)
8. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

   a. lie on a straight line
   b. lie on an ellipse
   c. lie on a circle
   d. are vertices of a triangle **(IIT-JEE, 1999)**
9. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

   a. $2x - 9y - 7 = 0$ **b.** $2x - 9y - 11 = 0$
   c. $2x + 9y - 11 = 0$ **d.** $2x + 9y + 7 = 0$
   (IIT-JEE, 2000)
10. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0, 0) and (2, 0) is

a. $(1, \sqrt{3}/2)$ b. $(2/3, 1/\sqrt{3})$
c. $(2/3, \sqrt{3}/2)$ d. $(1, 1/\sqrt{3})$

(IIT-JEE, 2000)

11. The number of integral values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also integer is
a. 2 b. 0 c. 4 d. 1

(II-JEE, 2001)

12. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals
a. $|m + n|/(m - n)^2$ b. $2/|m + n|$
c. $1/(|m + n|)$ d. $1/(|m - n|)$

13. Let $0 < \alpha < \pi/2$ be a fixed angle. If $P = (\cos \theta, \sin \theta)$ and $Q = (\cos (\alpha - \theta), \sin (\alpha - \theta))$, then Q is obtained from P by
a. clockwise rotation around origin through an angle α
b. anticlockwise rotation around origin through an angle α
c. reflection in the line through origin with slope $\tan \alpha$
d. reflection in the line through origin with slope $\tan (\alpha/2)$ **(IIT-JEE, 2002)**

14. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. Then the equation of the bisector of the angle $\angle PQR$ is
a. $\sqrt{3}/2x + y = 0$
b. $x + m\sqrt{3}y = 0$
c. $\sqrt{3}x + y = 0$
d. $x + (\sqrt{3}/2)y = 0$ **(IIT-JEE, 2002)**

15. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. Then the point O divides the segment PQ in the ratio
a. 1 : 2 b. 3 : 4
c. 2 : 1 d. 4 : 3 **(IIT-JEE, 2002)**

16. The number of integral points (integral point means both the coordinates should be integer) exactly in the interior of the triangle with vertices (0, 0), (0, 21) and (21, 0) is
a. 133 b. 190 c. 233 d. 105

(IIT-JEE, 2003)

17. Orthocentre of triangle with vertices (0, 0), (3, 4) and (4, 0) is
a. (3, 5/4) b. (3, 12)
c. (3, 3/4) d. (3, 9) **(IIT-JEE, 2003)**

18. Area of the triangle formed by the line $x + y = 3$ and angle bisectors of the pairs of straight lines $x^2 - y^2 + 2y = 1$ is
a. 2 sq. units b. 4 sq. units
c. 6 sq. units d. 8 sq. units **(IIT-JEE, 2004)**

19. The equation to a pair of opposite sides of parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. The equations to its diagonals are
a. $x + 4y = 13, y = 4x - 7$
b. $4x + y = 13, 4y = x - 7$
c. $4x + y = 13, y = 4x - 7$
d. $y - 4x = 13, y + 4x = 7$ **(IIT-JEE, 1994)**

20. Let $O(0, 0)$, $P(3, 4)$, $Q(6, 0)$ be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangle OPR, PQR, OQR are of equal area. The coordinates of R are
a. (4/3, 3) b. (3, 2/3)
c. (3, 4/3) d. (4/3, 2/3) **(IIT-JEE, 2007)**

21. Consider three points $P = (-\sin (\beta - \alpha), -\cos \beta)$, $Q = (\cos (\beta - \alpha), \sin \beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin (\beta - \theta))$, where $0 < \alpha, \beta, \theta < \pi/4$. Then
a. P lies on the line segment RQ
b. Q lies on the line segment PR
c. R lies on the line segment QP
d. P, Q, R are non-collinear **(IIT-JEE, 2008)**

22. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is
a. a hyperbola b. a parabola
c. an ellipse d. a straight line

(IIT-JEE, 2009)

23. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is
a. $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$
b. $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
c. $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$
d. $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$ **(IIT-JEE, 2011)**

Multiple choice questions with one or more than one correct answer

1. Three lines $px + qy = r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent if
a. $p + q + r = 0$ b. $p^2 + q^2 + r^2 = pr + rp + pq$
c. $p^3 + q^3 + r^3 = 3pqr$ d. none of these

(IIT-JEE, 1985)

2. The points (0, 8/3), (1, 3) and (82, 30) are vertices of
a. an obtuse-angled triangle
b. and acute-angled triangle

c. a right-angled triangle

d. none of these **(IIT-JEE, 1986)**

3. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy

a. $3x + 2y \geq 0$ b. $2x + y - 13 \geq 0$

c. $2x - 3y - 12 \leq$ d. $-2x + y \geq 0$ **(IIT-JEE, 1986)**

4. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then

a. $a = 2, b = 4$ b. $a = 3, b = 4$

c. $a = 2, b = 3$ d. $a = 1$ or $b = -1$

(IIT-JEE, 1998)

5. The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then $PQRS$ must be a

a. rectangle b. square

c. cyclic quadrilateral d. rhombus **(IIT-JEE, 1998)**

6. If the vertices P, Q, R of a triangle PQR are rational points, which of the following points of the triangle PQR is (are) always rational points(s)? (A rational point is a point whose coordinates are rational numbers.)

a. centroid

b. incentre

c. circumcentre

d. orthocentre **(IIT-JEE,1982)**

Assertion and reasoning

1. Lines L_1 L_2 given by $y - x = 0$ and $2x + y = 0$ intersect the line L_3 given by $y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R. **(IIT-JEE, 2007)**

Statement 1: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement 2: In any triangle, bisector of an angle divides the triangle into two similar triangles.

a. Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.

b. Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.

c. Statement 1 is True, Statement 2 is False.

d. Statement 1 is False, Statement 2 is True.

Matrix-match

This question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a → p, a → s, b → q b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows: **(IIT-JEE, 2008)**

	p	q	r	s
a	ⓟ	ⓠ	ⓡ	ⓢ
b	ⓟ	ⓠ	ⓡ	ⓢ
c	ⓟ	ⓠ	ⓡ	ⓢ
d	ⓟ	ⓠ	ⓡ	ⓢ

Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Column I	Column II
a. L_1, L_2, L_3 are concurrent, if	p. $k = -9$
b. One of L_1, L_2, L_3 is parallel to at least one of the other two, if	q. $k = -6/5$
c. L_1, L_2, L_3 form a triangle, if	r. $k = 5/6$
d. L_1, L_2, L_3 do not form a triangle, if	s. $k = 5$

ANSWERS AND SOLUTIONS

Subjective Type

1.

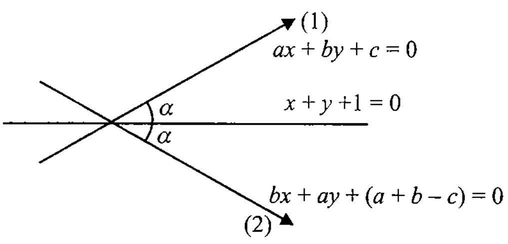


Fig. 1.73

Angle bisectors of line Eqs. (i) and (ii) is

$$\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|bx + ay + a + b - c|}{\sqrt{b^2 + a^2}}$$

or

$$ax + by + c = \pm (bx + ay + a + b - c)$$

Hence,

$$x + y + 1 = 0$$

or

$$(a - b)x + (b - a)y = a + b - 2c$$

Since $x + y + 1 = 0$ is one of the bisectors of lines (i) and (ii), hence the given result is proved.

2. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{(i)}$$

Its intercepts on x and y axes are a and b, respectively. According to the question,

$$\frac{1}{a}+\frac{1}{b}=\text{constant}=k \text{ (say)}$$

$$\therefore \quad \frac{1}{ak}+\frac{1}{bk}=1$$

or

$$\frac{1/k}{a}+\frac{1/k}{b}=1 \quad \text{(ii)}$$

Form (ii), it follows that line (i) passes through the fixed point $(1/k, 1/k)$.

3.

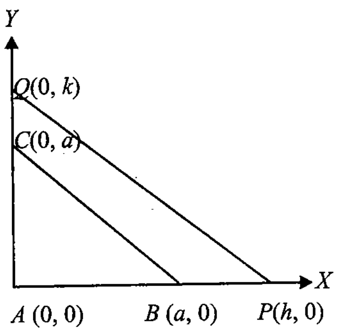


Fig. 1.74

Let ABC be a right-angled isosceles triangle in which $AB = AC$. We take A as the origin and AB and AC as x- and y-axes, respectively.

Let $AB = AC = a$, $AP = h$, $AQ = k$. Now equation of line PQ will be

$$\frac{x}{h}+\frac{y}{k}=1 \quad \text{(i)}$$

Given,

$$BP \times CQ = AB^2$$

or

$$(h-a)(k-a)=a^2$$

or

$$hk-ak-ah=0$$

or

$$ak+ha=hk$$

or

$$\frac{a}{h}+\frac{a}{k}=1 \quad \text{(ii)}$$

From (ii), it follows that line (i), i.e., line PQ passes through the fixed point (a, a).

4. Let the points be $A_1(-c_1/m, 0)$, $A_2(-c_2/m, 0)$, $B_1(0, -c_1)$, $B_2(0, -c_2)$. Now, A_1, A_2, B_1, B_2 are concyclic. Hence,

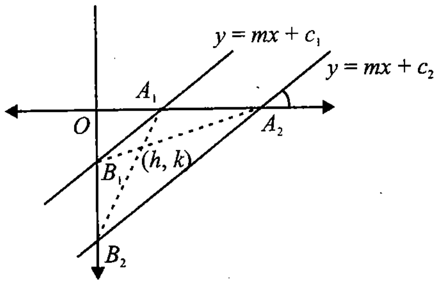


Fig. 1.75

$$OA_1 \times OA_2 = OB_1 \times OB_2$$

$$\Rightarrow \quad \left|-\frac{c_1}{m}\right|\left|-\frac{c_2}{m}\right| = |c_1|\,|c_2| \Rightarrow m^2=1 \Rightarrow m=\pm 1$$

$$\Rightarrow \quad m=1 \ (\because m \in R^+)$$

Hence, equations of A_1B_2 and A_2B_1 are

$$\frac{x}{-c_1}+\frac{y}{c_2}=1$$

and

$$\frac{x}{-c_2}+\frac{y}{c_1}=1$$

or,

$$-c_2x+c_1y=c_1c_2 \text{ and } -c_1x+c_2y=c_1c_2$$

Subtracting, we get

$$x+y=0$$

which is a required locus.

5.

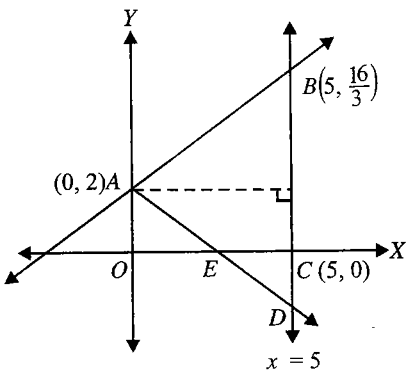


Fig. 1.76

$$L_1 = 3y-2x-6=0$$

Point about which the line is rotated is $A = (0, 2)$. Let equation of L_2 be $y = mx + 2$. As L_2 will be cutting line $x = 5$ below x-axis, so

$A \equiv (0, 2)$, $B\left(5, \frac{16}{3}\right)$, $C \equiv (5, 0)$, $D \equiv (5, 5m + 2)$, $E \equiv \left(\frac{-2}{m}, 0\right)$

Now,

Area of $AECB$ = area of ADB – area of ECD

$$\Rightarrow \quad \frac{49}{3}=\frac{1}{2}BD\times AF-\frac{1}{2}EC\times CD$$

$$=\frac{1}{2}\left[\frac{16}{3}-(5m+2)\right]\times 5-\frac{1}{2}\times\left(5+\frac{2}{m}\right)[-(5m+2)]$$

$$\Rightarrow \quad \frac{49}{3}=\frac{110m+12}{6m}\Rightarrow m=-1$$

Hence, equation of L_2 is $x+y=2$.

6. Let the equation of the third line be

$(ax+by+c)+\lambda(lx+my+n)=0$

where λ is a parameter. It meets the x-axis at A where $y=0$ and $x=-(\lambda n+c)/(\lambda l+a)$ and y-axis at B $\left(0,-\frac{\lambda n+C}{\lambda m+C}\right)$.

The triangle OAB is a right-angled triangle. Its circumcentre is the midpoint of the hypotenuse. Let it be (α,β). Then,

$$2\alpha=-\left(\frac{\lambda n+c}{\lambda l+a}\right)$$

and

$$2\beta=-\left(\frac{\lambda n+c}{\lambda m+a}\right)$$

Hence, $$\lambda=-\frac{2\alpha a+c}{2\alpha l+n}=-\frac{2\beta b+c}{2\beta m+n}$$

Hence, the locus of (α,β) is

$$\frac{c+2ax}{n+2lx}=\frac{c+2by}{n+2my}$$

$$\Rightarrow \quad 2xy(ma-bl)=x(an-lc)+y(mc-bn)$$

7. Two consecutive diagonals are the lines $\sqrt{3}x+y=\sqrt{3}$ and $y=\sqrt{3}/2$.

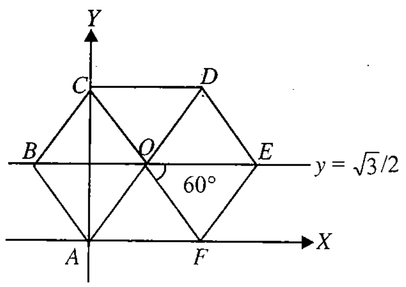


Fig. 1.77

Angle between these two lines is 60°. As the polygon is regular, it will be hexagon. Centre of the polygon is $(1/2,\ \sqrt{3}/2)$ and $(1, 0)$ lies on $\sqrt{3}+y=\sqrt{3}$. Hence,

$$OF=\sqrt{\left(\frac{1}{2}-1\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=1$$

Equation of the third diagonal will be $y=\sqrt{3}x$. The coordinates of the vertices are as follows:

$D(1,\ \sqrt{3})$, $A(0,\ 0)$, $C(0,\ \sqrt{3})$, $F(1,\ 0)$, $E\left(\frac{3}{2},\ \frac{\sqrt{3}}{2}\right)$, $B\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

Equations of the sides are

$$\overleftrightarrow{AF}\equiv y=0$$

$$\overleftrightarrow{CD}\equiv y=\sqrt{3}$$

$$\overleftrightarrow{FE}\equiv y=\sqrt{3}(x-1)$$

$$\overleftrightarrow{AB}\equiv y=-\sqrt{3}x$$

$$\overleftrightarrow{BC}\equiv y=\sqrt{3}(x+1)$$

$$\overleftrightarrow{ED}\equiv y=-\sqrt{3}(x-2)$$

8.

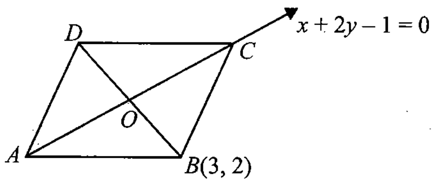


Fig. 1.78

Since diagonal is member of both the given families of lines, So it will pass through the points (1, 0) and (–1, 1). Hence, the equation of diagonal AC is

$$x+2y-1=0$$

Since one of the vertices is (3, 2) (which does not lie on AC), hence diagonal BD is

$$2x-y=4$$

Therefore, the point of intersection of diagonals AC and BD is $(9/5,-2/5)$. Hence, vertex D is $(3/5,-14/5)$ (since O is midpoint of B and D). Now,

$$BO=\frac{6\sqrt{5}}{5}\Rightarrow BD=\frac{12\sqrt{5}}{5}$$

since area of rhombus is

$$\frac{1}{2}d_1\times d_2=12\sqrt{5}$$

or

$$d_2=10 \text{ units}$$

Hence, the length of AC is 10. Using parametric form of straight line on AC, we get

$$A=\left(\frac{9}{5}-2\sqrt{5},-\frac{2}{5}+\sqrt{5}\right),$$

$$C=\left(\frac{9}{5}-2\sqrt{5},-\frac{2}{5}+\sqrt{5}\right)$$

9. Let the equation of OA be $y=m_1x$ and that of OB be $y=m_2x$.

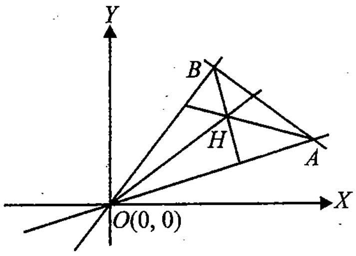


Fig. 1.79

Slope of line $\quad AB = -\dfrac{1}{\text{slope of } OH} = -\dfrac{a}{b}$

m_1, m_2 being roots of equation $bx^2 + 2hx + a = 0$,

$$m_1 + m_2 = \frac{-2h}{b} \qquad \text{(i)}$$

$$m_1 m_2 = \frac{a}{b} \qquad \text{(ii)}$$

Equation of line perpendicular to OA and passing through $H(a, b)$ is

$$y - b = -\frac{1}{m_1}(x - a) \qquad \text{(iii)}$$

Point B is the point of intersection of Eq. (iii) and $y = m_2 x$. Then coordinates of B are

$$\left(\frac{a + bm_1}{1 + m_1 m_2}, m_2\left(\frac{a + bm_1}{1 + m_1 m_2}\right)\right)$$

Hence, the equation of AB is

$$y - m_2\left(\frac{a + bm_1}{1 + m_1 m_2}\right) = \frac{a}{b}\left(x - \frac{a + bm_1}{1 + m_1 m_2}\right)$$

$\Rightarrow$ $b(1 + m_1 m_2)y + ax(1 + m_1 m_2) = ab(m_1 + m_2) + b^2 m_1 m_2 + a^2$

From Eqs. (i) and (ii),

$$(a + b)\,[ax + by] = ab(a + b - 2h)$$

10.

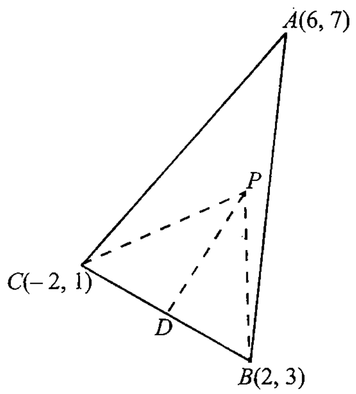


Fig. 1.80

Equation of line BC is

$$(y - 1) = \frac{3 - 1}{2 + 2}(x + 2)$$

$$\Rightarrow \qquad 2y - x - 4 = 0$$

Also,

$$BC = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

If 'D' is the midpoint of BC, then $D \equiv (0, 2)$ and $DP = 2\sqrt{5}\sin(\pi/3) = \sqrt{15}$ (as ΔPBC is equilateral). Also, slope of DP is -2. If P is (x, y), then

$$\frac{x - 0}{\cos\theta} = \frac{y - 2}{\sin\theta} = \sqrt{15}$$

where

$$\tan\theta = -2$$

$$\Rightarrow \qquad \frac{x\sqrt{5}}{-1} = \frac{(y - 2)\sqrt{5}}{2} = \sqrt{15}$$

$$\Rightarrow \qquad x = -\sqrt{3},\ y = 2 + 2\sqrt{3}$$

Hence, $\qquad P \equiv (-\sqrt{3}, 2 + 2\sqrt{3})$.

11.

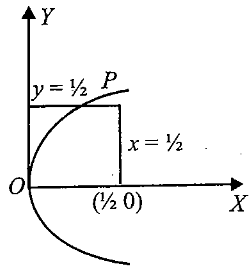


Fig. 1.81

The equation of the lines along the sides of the given square are $x = 0$, $y = 0$, $2x = 1$ and $2y = 1$. The point $(\sin^2\theta, \sin\theta)$ lies on the parabola

$$y^2 = x,\ 0 \le x \le 1 \qquad \text{(i)}$$

The points of intersection of (i) and the given square are (0, 0) and (1/4, 1/2). Hence, $0 < \sin\theta < 1/2$. Therefore,

$$\theta \in \left\{\cdots\left(2n\pi, 2n\pi + \frac{\pi}{6}\right)\right\}\cdots\left\{\cdots\left(2m\pi + \frac{5\pi}{6}, 2m\pi + \pi\right)\right\}$$

(I denotes the set of integers)

12.

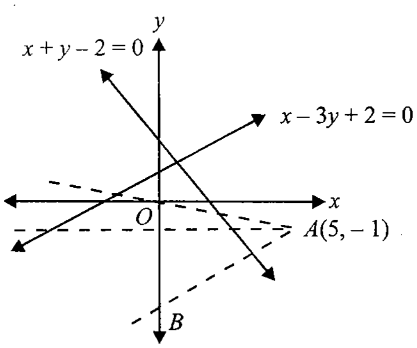


Fig. 1.82

Combined equation of lines is $x^2 - 2xy - 3y^2 + 8y - 4 = 0$

$\Rightarrow (x-y)^2 = 4y^2 - 8y + 4 \Rightarrow x - y = \pm 2(y-1)$

Thus, two sides of the triangle are

$L_1 : 3y - x - 2 = 0$ and $L_2 : y + x - 2 = 0$

From the above figure if line through $A(5, -1)$ (line AB) is parallel to L_1', triangle is not formed.

Slope of line AB is 1/3

Slope line OA is – 1/5

It is clear that origin lies inside triangle formed by L_1, L_2 and line through A(5, –1), if slope of the line lies in the interval (– 1/5, 1/3).

13.

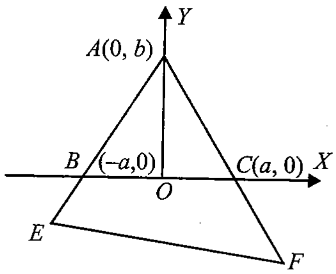


Fig. 1.83

Let ABC be the triangle having vertices $(-a, 0)$, $(0, b)$ and $(a, 0)$. Now,

$$BE \times CF = AB^2 \Rightarrow \frac{BE}{AB} = \frac{AB}{CF} = \lambda \text{ (let)}$$

$$\Rightarrow \quad \frac{BE}{AB} = \frac{AC}{CF} = \lambda$$

Hence, the coordinates of E and F are $(-a(\lambda+1), -\lambda b)$ and $(a(1 + 1/\lambda) - b/\lambda)$. Equation of line EF is

$$y + \lambda b = \frac{-\lambda b + \frac{b}{\lambda}}{-a(\lambda+1) - \frac{a(\lambda+1)}{\lambda}}[x + a(\lambda+1)]$$

or

$$y + \lambda b = \frac{\frac{b}{\lambda}(1-\lambda^2)}{-\frac{a(\lambda+1)}{\lambda}[1+\lambda]}[x + a(\lambda+1)]$$

or

$$y + \lambda b = \frac{b(\lambda-1)}{a(\lambda+1)}[x + a(\lambda+1)]$$

or

$$a(\lambda+1)y + ab\lambda(\lambda+1) = b(\lambda-1)x + ab(\lambda^2-1)$$

or

$$(bx + ay + ab) - \lambda(bx - ay - ab) = 0$$

which is the equation of a family of lines passing through the point of intersection of the lines $bx + ay + ab = 0$ and $bx - ay - ab = 0$, the point of intersection being $(0, -b)$. Hence, the line EF passes through a fixed point.

14.

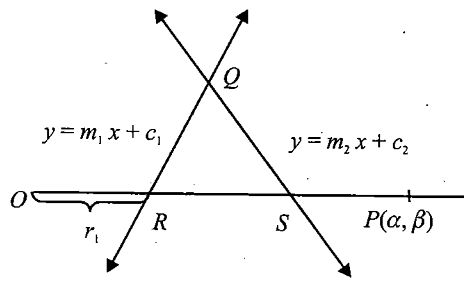


Fig. 1.84

Let the variable line through O make an angle θ with the positive direction of x-axis. Any point on this variable line is $(0 + r\cos\theta, 0 + r\sin\theta)$, i.e., $(r\cos\theta, r\sin\theta)$. Let the fixed lines be

$$y = m_1 x + c_1 \qquad \text{(i)}$$

and

$$y = m_2 x + c_2 \qquad \text{(ii)}$$

Let $OR = r_1$, $OS = r_2$, $OP = r_3$. So according to question,

$$\frac{m+n}{r_3} = \frac{m}{r_1} + \frac{n}{r_2} \qquad \text{(iii)}$$

Since R and S lie on lines (i) and (ii), respectively, so

$$r_1 \sin\theta = m_1 (r_1 \cos\theta) + c_1 \qquad \text{(iv)}$$

and

$$r_2 \sin\theta = m_2 (r_2 \cos\theta) + c_2 \qquad \text{(v)}$$

Let $\quad P \equiv (\alpha, \beta)$. Then,

$$\alpha = r_3 \cos\theta, \beta = r_3 \sin\theta \qquad \text{(vi)}$$

From (iii), (iv) and (v), we have

$$\frac{m+n}{r_3} = m\frac{\sin\theta - m_1\cos\theta}{c_1} + n\frac{\sin\theta - m_2\cos\theta}{c_2}$$

or

$$m + n = \frac{m}{c_1}(\beta - m_1\alpha) + \frac{n}{c_2}(\beta - m_2\alpha) \qquad \text{[From (vi)]}$$

Hence, the locus of P is

$$m + n = \left(\frac{m}{c_1} + \frac{n}{c_2}\right)y - \left(\frac{mm_1}{c_1} + \frac{mm_2}{c_2}\right)x$$

or

$$y - m_1 x - c_1 + \frac{n}{m}\frac{c_1}{c_2}(y - m_2 x - c_2) = 0 \qquad \text{(vii)}$$

Clearly, locus of P is a straight line passing through the point of intersection of the given lines the (i) and (ii).

15. Let the point of intersection O of the given lines be taken as the origin.

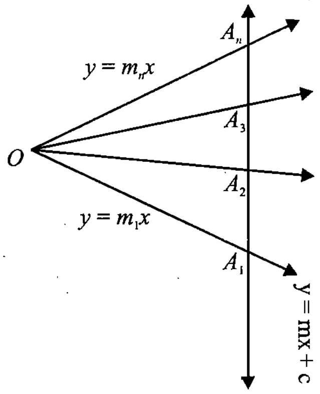


Fig. 1.85

Let the equation of given lines be

$$y = m_r x,\ r = 1, 2, \ldots, n \qquad \text{(i)}$$

and let the variable line be

$$y = mx + c \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = \frac{c}{m_r - m},\ y = \frac{m_r c}{m_r - m}$$

$$\Rightarrow \quad A_r \equiv \left(\frac{c}{m_r - m}, \frac{cm_r}{m_r - m}\right),$$

$$r = 1, 2, \ldots, n$$

$$\Rightarrow \quad OA_r^2 = \frac{c^2}{(m_r - m)^2}(1 + m_r^2)$$

$$\Rightarrow \quad OA_r = \left|\frac{c}{m_r - m}\right| \sqrt{1 + m_r^2}$$

or

$$\frac{1}{OA_r} = \left|\frac{m_r - m}{c}\right| \frac{1}{\sqrt{1 + m_r^2}}$$

Given,

$$\sum_{r=1}^{n} \frac{1}{OA_r} = \frac{1}{OA_1} + \frac{1}{OA_2} + \cdots + \frac{1}{OA_n} = \text{constant} = k \text{ (say)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} \pm \frac{m_r - m}{c} \frac{1}{\sqrt{1 + m_r^2}} = k$$

$$\Rightarrow \quad \frac{1}{k}\sum \pm \frac{m_r}{\sqrt{1 + m_r^2}} = m\frac{1}{k}\sum \pm \frac{1}{\sqrt{1 + m_r^2}} + c$$

Therefore, the line $y = mx + c$ passes through the fixed point

$$\left(\frac{1}{k}\sum \pm \frac{m_r}{\sqrt{1 + m_r^2}}, m\frac{1}{k}\sum \pm \frac{1}{\sqrt{1 + m_r^2}}\right)$$

16.

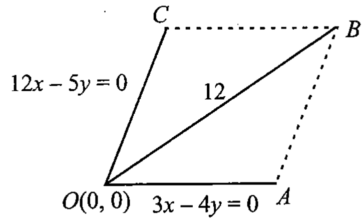


Fig. 1.86

As the sides are in the first quadrant and their inclinations are $\tan^{-1} 3/4$ and $\tan^{-1} 12/5$, the angle at the vertex $O(0, 0)$ is acute. So the longer diagonal passes through $O(0, 0)$. Let the length of each side be a. The slope of OA is $\tan\theta = 3/4$ and the slope of OC is $\tan\phi = 12/5$. Therefore,

$$A = (0 + a\cos\theta, 0 + a\sin\theta)$$

$$= \left(\frac{4a}{5}, \frac{3a}{5}\right)$$

Similarly,

$$C = (0 + a\cos\phi, 0 + a\sin\phi)$$

$$= \left(\frac{5a}{13}, \frac{12a}{13}\right)$$

Now, AB is a line parallel to OC and passing through $A\ (4a/5, 3a/5)$. Therefore, the equation of AB is

$$12x - 5y = \frac{33}{5}a \qquad \text{(i)}$$

The equation of CB is

$$3x - 4y = -\frac{33}{13}a \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\therefore \quad B = \left(\frac{77a}{65}, \frac{99a}{65}\right)$$

17.

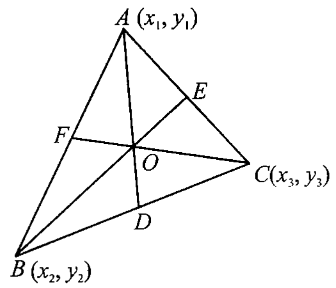


Fig. 1.87

Let AD, BE and CF meet at O. We take O as the origin. Let the coordinates of points A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. Let D divide BC in the ratio k:1, i.e.,

$$\frac{BD}{DC} = \frac{k}{1}$$

Then,

$$D \equiv \left(\frac{kx_3 + x_2}{k+1}, \frac{ky_3 + y_2}{k+1}\right)$$

Also equation of line AD is

$$y - 0 = \frac{y_1 - 0}{x_1 - 0}(x - 0)$$

or

$$y = \frac{y_1}{x_1}x \qquad \text{(i)}$$

Since D lies on AD. Hence,

$$\frac{ky_3 + y_2}{k+1} = \frac{y_1}{x_1}\left(\frac{kx_3 + x_2}{k+1}\right)$$

or

$$k(x_1y_3 - x_3y_1) = x_2y_1 - x_1y_2$$

$\Rightarrow$
$$k = \frac{BD}{DC} = \frac{x_2y_1 - x_1y_2}{x_1y_3 - x_3y_1} \qquad \text{(ii)}$$

Similarly,

$$\frac{CE}{EA} = \frac{x_3y_2 - x_2y_3}{x_2y_1 - x_1y_2} \qquad \text{(iii)}$$

and

$$\frac{AF}{FB} = \frac{x_1y_3 - x_3y_1}{x_3y_2 - x_2y_3} \qquad \text{(iv)}$$

From (i), (ii) and (iii), we get

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$$

or

$$D \times CE \times AF = DC \times EA \times FB$$

18. The parallelogram is shown in figure.

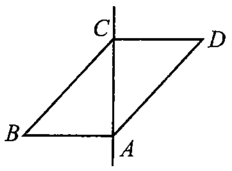


Fig. 1.88

Since one end of the required diagonal passes through the intersection of $U - a = 0$ and $V - a' = 0$, its equation is given by

$$(U - a) + \lambda(V - a') = 0 \qquad \text{(i)}$$

But the other end of the same diagonal passes through the intersection of $U - b = 0$ and $V - b' = 0$. So its equation is also given by

$$(U - b) + \mu(V - b') = 0 \qquad \text{(ii)}$$

Therefore, the parameters λ and μ must satisfy

$$1 = \frac{\lambda}{\mu} = \frac{a + \lambda a'}{b + \mu b'}$$

$\Rightarrow$
$$\lambda = \mu = \left(\frac{a - b}{a' - b'}\right)$$

Putting these values back in (i), the equation of diagonal is

$$\begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix} = 0$$

19.

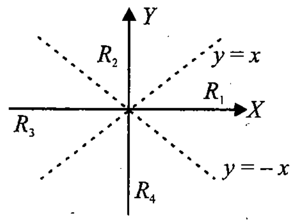


Fig. 1.89

$$d(P, L_1) = \frac{|x - y|}{\sqrt{2}}$$

and

$$d(P, L_2) = \frac{|x + y|}{\sqrt{2}}$$

Now we have,

$$2 \le d(P, L_1) + d(P, L_2) \le 4$$

$\Rightarrow$
$$2\sqrt{2} \le |x - y| + |x + y| \le 4\sqrt{2} \quad \text{(i)}$$

Now let us consider the four regions, namely, R_1, R_2, R_3 and R_4 in the lines L_1 and L_2 dividing the coordinate plane.

In R_1 we have $y < x$, $y > -x$. In R_2, we have $y > x$, $y > -x$. Similarly, in R_3, we have $y > x$, $y < -x$. Finally, in R_4, we have $y < x$, $y < -x$. Thus for R_1 Eq. (i) becomes

$$2\sqrt{2} \le x - y + x + y \le 4\sqrt{2} \Rightarrow \sqrt{2} \le x \le 2\sqrt{2}$$

Similarly, for R_2 Eq. (i) becomes

$$2\sqrt{2} \le y - x + x + y \le 4\sqrt{2} \Rightarrow \sqrt{2} \le y \le 2\sqrt{2}$$

In R_3, Eq. (i) will become

$$2\sqrt{2} \le y - x - x - y \le 4\sqrt{2} \Rightarrow -\sqrt{2} \le y \le -2\sqrt{2}$$

Finally, in R_4, Eq. (i) will become

$$2\sqrt{2} \le x - y - x - y \le 4\sqrt{2}$$

$\Rightarrow$
$$-\sqrt{2} \le y \le -2\sqrt{2}$$

Thus region 'R' will be the region between concentric squares formed by the lines $x = \pm\sqrt{2}$,

$$y = \pm\sqrt{2} \text{ and } x = \pm 2\sqrt{2},$$

$$y = \pm 2\sqrt{2}.$$

Thus the required area is $(4\sqrt{2})^2 - (2\sqrt{2})^2 = 24$ sq. units.

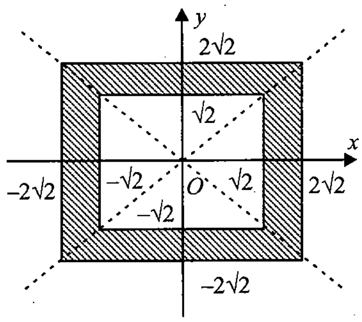


Fig. 1.90

20. Let the axes rotate by an angle θ, and if (x, y) be the point with respect to old axes and (X, Y) be the coordinates with respect to new axes. Then, we get

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

Then,

$ux + vy = u(X\cos\theta - Y\sin\theta) + v(X\sin\theta + Y\cos\theta)$

$= (u\cos\theta + v\sin\theta)X + (-u\sin\theta + v\cos\theta)Y$

But given new curve $VX + UY$, we have

$VX + UY = (u\cos\theta + v\sin\theta)X + (-u\sin\theta + v\cos\theta)Y$

On comparing the coefficients of X and Y, we get

$$u\cos\theta + v\sin\theta = V \qquad \text{(i)}$$

and

$$u\sin\theta + v\cos\theta = U \qquad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$u^2 + v^2 = U^2 + v^2$$

21. Since $PA = PB$ and $PA_1 = PB_1$, it implies that triangles PAB and PA_1B_1 are isosceles. Thus bisectors of the given lines will be parallel to lines L_3 and L_4. Now bisectors of the given lines are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Hence, the equation of lines L_3 and L_4 are

$$\frac{a_1x + b_1y}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y}{\sqrt{a_2^2 + b_2^2}}$$

and their combined equation is

$$(a_2^2 + b_2^2)(a_1x + b_1y)^2 = (a_1^2 + b_1^2)(a_2x + b_2y)^2$$

22.

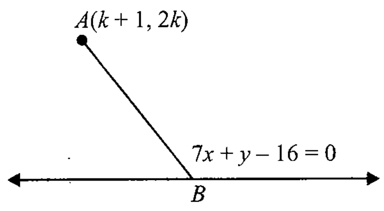


Fig. 1.91

$$A \equiv (k + 1, 2k)$$

Let the line through A making an angle θ with the positive direction of x-axis meet the given lines

$$7x + y - 16 = 0 \qquad \text{(i)}$$
$$5x - y - 8 = 0 \qquad \text{(ii)}$$

and

$$x - 5y + 8 = 0 \qquad \text{(iii)}$$

at B, C and D, respectively. Let $AB = r_1$, $AC = r_2$ and $AD = r_3$. Now,

$B \equiv (k + 1 + r_1\cos\theta, 2k + r_1\sin\theta)$

Since B lies on line (i), so

$7(k + 1 + r_1\cos\theta) + 2k + r_1\sin\theta - 16 = 0$

or

$$r_1(7\cos\theta + \sin\theta) = 9 - 9k$$

$$\therefore \quad r_1 = \frac{9(1-k)}{7\cos\theta + \sin\theta} \qquad \text{(iv)}$$

Again,

$C \equiv (k + 1 + r_2\cos\theta, 2k + r_2\sin\theta)$

Since C lies on line (ii), so

$5(k + 1 + r_2\cos\theta) - (2k + r_2\sin\theta) - 8 = 0$

$$\Rightarrow \quad r_2 = \frac{3(1-k)}{5\cos\theta - \sin\theta} \qquad \text{(v)}$$

Also,

$D \equiv (k + 1 + r_3\cos\theta, 2k + r_3\sin\theta)$

As D lies on line (iii), so

$k + 1 + r_3\cos\theta - 5(2k + r_3\sin\theta) + 8 = 0$

$$\Rightarrow \quad r_3 = \frac{9(1-k)}{5\sin\theta - \cos\theta} \qquad \text{(vi)}$$

Now,

$$\frac{1}{r_2} + \frac{1}{r_3} = \frac{5\cos\theta - \sin\theta}{3(1-k)} + \frac{5\sin\theta - \cos\theta}{9(1-k)}$$

$$= \frac{2(7\cos\theta + \sin\theta)}{9(1-k)} = \frac{2}{r_1}$$

Hence, r_2, r_1, r_3, i.e., AC, AB, AD are in H.P.

23.

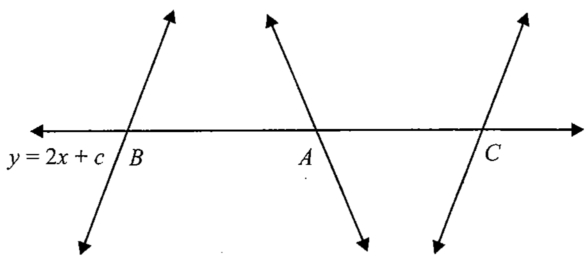


Fig. 1.92

Given lines are

$$4x+y-9=0 \quad \text{(i)}$$
$$x-2y+3=0 \quad \text{(ii)}$$
$$5x-y-6=0 \quad \text{(iii)}$$

Now equation of any line having slope (gradient) 2 will be

$$y=2x+c \quad \text{(iv)}$$

Let line (iv) cuts lines (i), (ii) and (iii) at A, B and C, respectively. Solving (i) and (iv), we get

$$A\equiv\left(\frac{3}{2}-\frac{c}{6}, 3+\frac{2c}{3}\right)$$

Similarly,

$$B\equiv\left(1-\frac{2c}{3}, 2-\frac{c}{3}\right)$$

and

$$C\equiv\left(2+\frac{c}{3}, 4+\frac{5c}{3}\right)$$

Clearly, A is the middle point of BC. Hence, $AB = AC$.

24. Let the coordinates of the vertex be (h, k) and let the lengths of the bases be l_1, l_2, l_3, etc. and their equations respectively are

$$x\cos\alpha+y\sin\alpha=p_1$$
$$x\cos\beta+y\sin\beta=p_2$$

and so on. Then the lengths of perpendicular from (h, k) on base are, respectively, $(h\cos\alpha + k\sin\alpha - p_1)$ $(h\cos\beta + k\cos\beta - p_2)$, etc. Sum of the area of the triangles is constant by hypothesis. Therefore,

$$\frac{1}{2}l_1(h\cos\alpha+k\sin\alpha-p_1)+\frac{1}{2}l_2(h\cos\beta+k\sin\beta-p_2)$$
$$+\cdots=\text{constant}=\frac{c}{2}$$

$$h(l_1\cos\alpha+l_2\cos\beta+\cdots)+k(l_1\sin\alpha+l_2\sin\beta+\cdots)$$
$$-(l_1p_1+l_2p_2+\ldots)=c$$

Generalizing, we get

$$x\,\Sigma\, l_1\cos\alpha+y\,\Sigma\, l_1\sin\alpha-\Sigma\, l_1p_1=c$$

25. Let OA and BC be the two portions of straight line $OABC$. Taking O as origin OC as x-axis, the coordinates of O, A, B and C may be taken as $(0, 0)$, $(a, 0)$, $(b, 0)$ and $(c, 0)$. Consider the point $P(h, k)$.

Slope of OP is

$$m_1=\frac{k}{h}$$

Slope of PA is

$$m_2=\frac{k-0}{h-a}$$

Slope of PB is

$$m_3=\frac{k-0}{h-b}$$

Slope of PC is $\frac{k-0}{h-c}$

Angles between OP and PA is

$$\tan^{-1}\left[\frac{\left(\frac{k}{h}-\frac{k}{h-a}\right)}{1+\frac{k}{h}\times\frac{k}{h-a}}\right]$$

Angle between PB and PC is

$$\tan^{-1}\left[\frac{\left(\frac{k}{h-b}-\frac{k}{h-c}\right)}{1+\frac{k}{h-b}\times\frac{k}{h-c}}\right]$$

By hypothesis,

$$\frac{\frac{k}{h}-\frac{k}{h-a}}{1+\frac{k}{h}\times\frac{k}{h-a}}=\frac{\frac{k}{h-b}-\frac{k}{h-c}}{1+\frac{k}{h-b}\times\frac{k}{h-c}}$$

$$\Rightarrow \quad \frac{-ak}{h^2-ah+k^2}=\frac{kb-kc}{h^2-hc-hb+bc+k^2}$$

$$\Rightarrow \ (a+b-c)(x^2+y^2)-2abx+abc=0$$

26. Let the point A, B and C be, respectively, $(h, 0)$, $(0, k)$ and (x, y). Then,

$$BC^2=x^2+(y-k)^2 \quad \text{(i)}$$
$$AC^2=(x-h)^2+y^2 \quad \text{(ii)}$$

Again slopes of BC and AC are, respectively, $(y-k)/(x-0)$ and $(y-0)/(x-h)$. As they are perpendicular to each other, so

$$\frac{y-k}{x}\times\frac{y}{x-h}=-1$$

Squaring, we get

$$\frac{(y-k)^2y^2}{x^2(x-h)^2}=1 \quad \text{(iii)}$$

Therefore, $BC = a$, $AB = b$. From Eqs. (i) and (ii), we get

$$(y-k)^2=(a^2-x^2)$$

and

$$(x-h)^2=(b^2-y^2)$$

Putting the values in Eq. (iii)

$$(a^2-x^2)y^2/x^2(b^2-y^2)=1$$

or

$$a^2y^2-x^2y^2=b^2x^2-x^2y^2$$

or

$$y^2/x^2=b^2/a^2$$

or

$$y/x=\pm\, b/a$$

Hence, proved.

27. Since AB is fixed for all positions of R and $\angle ARB = 90°$, locus of R is A circle with AB as diameter, i.e., required locus is

$$x(x-3)+y(y-2)=0 \text{ (see equation in circle)}$$

$$\Rightarrow \quad x^2+y^2-3x-2y=0$$

28. Writing the given equation as quadratic in x, we have

$6x^2 + (5y + 7)x - (4y^2 - 13y + 3) = 0$

$$= \frac{-(5y+7) \pm \sqrt{(5y+7)^2 + 24(4y^2 - 13y + 3)}}{12}$$

$$= \frac{-(5y+7) \pm \sqrt{121y^2 - 242y + 121}}{12}$$

$$= \frac{-5(y+7) \pm 11(y-1)}{12} = \frac{6y-18}{12}, \frac{-16y+4}{12}$$

$$\Rightarrow \quad 2x - y + 3 = 0$$

and

$$3x + 4y - 1 = 0$$

which are the two lines represented by the given equation and the point of intersection is $(-1, 1)$, obtained by solving these equations. Also,

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{(5/2)^2 - 6(-4)}}{6-4} = \sqrt{\frac{121}{4}} = \frac{11}{2}$$

$(\because a = 6, b = -4, h = 5/2)$

So the equation of the required line is

$$y - 1 = \frac{11}{2}(x + 1)$$

$$\Rightarrow \quad 11x - 2y + 13 = 0$$

Objective Type

1. a. Coordinates of the vertices of the square are $A(0, 0)$, $B(0, 1)$, $C(1, 1)$ and $D(1, 0)$.

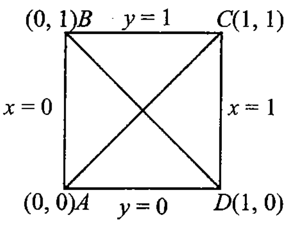


Fig. 1.93

Now the equation of AC is

$$y = x$$

and that of BD is

$$y - 1 = -\frac{1}{1}(x - 0)$$

$$\Rightarrow \quad x + y = 1$$

2. b. The equation of the line joining the points $(2, -1)$ and $(5, -3)$ is given by

$$y + 1 = \frac{-1+3}{2-5}(x - 2)$$

or $\quad 2x + 3y - 1 = 0 \quad$ (i)

Since $(x_1, 4)$ and $(-2, y_1)$ lie on $2x + 3y - 1 = 0$, therefore

$$2x_1 + 12 - 1 = 0 \Rightarrow x_1 = -\frac{11}{2}$$

and

$$-4 + 3y_1 - 1 = 0 \Rightarrow y_1 = \frac{5}{3}$$

Thus, (x_1, y_1) satisfies $2x + 6y + 1 = 0$.

3. a. We have,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Hence, the area of triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is same as the area of triangle with vertices (a_1, b_1), (a_2, b_2), (a_3, b_3). Hence, the two triangles are equal in area.

4. b. The given lines

$$ax \pm by \pm c = 0$$

$$\Rightarrow \quad \frac{x}{\pm c/a} + \frac{y}{\pm c/b} = 1$$

the vertex at A $(c/a, 0)$, C $(-c/a, 0)$, B $(0, c/b)$, D $(0, -c/b)$. Therefore, the diagonals AC and BD of quadrilateral $ABCD$ are perpendicular. Hence, it is a rhombus whose area is given by

$$\frac{1}{2} \times AC \times BD = \frac{1}{2} \times \frac{2c}{a} \times \frac{2c}{b} = \frac{2c^2}{ab}$$

5. d. If the line meets the x-and y-axes at A and B, then $A \equiv (-c/a, 0)$, $B \equiv (0, -c/b)$. The line will pass through the first quadrant if

$$-c/a > 0 \text{ and/or } -c/b > 0$$

$$\Rightarrow \quad ac < 0 \text{ and/or } bc < 0$$

6. b. $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)) = (x_1(1 - t) + tx_2, y_1(1 - t) + ty_2)$ is the point which divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $t : (1 - t)$ which is positive if $0 < t < 1$.

7. c. The coordinates of A and B are as shown in the figure.

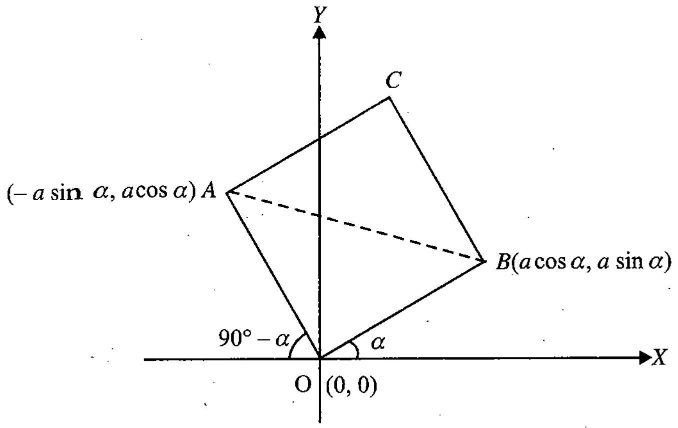


Fig. 1.94

The equation of the diagonal AB is

$$y - a\sin\alpha = \frac{a\cos\alpha - a\sin\alpha}{-a\sin\alpha - a\cos\alpha}(x - a\cos\alpha)$$

$$\Rightarrow \quad y(\cos\alpha + \sin\alpha) - a(\sin\alpha\cos\alpha + \sin^2\alpha)$$

$$= \quad -(\cos\alpha - \sin\alpha)x + a\cos\alpha(\cos\alpha - \sin\alpha)$$

$$\Rightarrow \quad y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a.$$

8. a. Let the two perpendicular lines be taken as the coordinate axes. If (h, k) be any point on the locus, then according to the given condition $|h| + |k| = 1$. Hence, the locus of (h, k) is $|x| + |y| = 1$. This consists of four line segments enclosing a square as shown in the figure below.

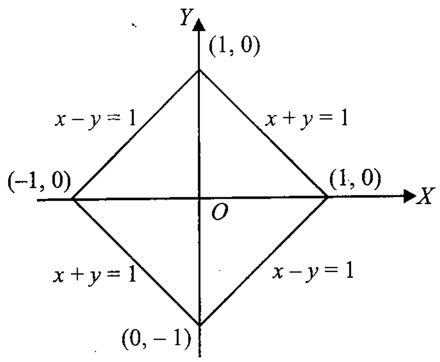


Fig. 1.95

9. c. The given inequality is equivalent to the following system of inequalities.

$$2x + 3y \le 6, \text{ when } x \ge 0, y \ge 0$$
$$2x - 3y \le 6, \text{ when } x \ge 0, y \le 0$$
$$-2x + 3y \le 6, \text{ when } x \le 0, y \ge 0$$
$$-2x - 3y \le 6, \text{ when } x \le 0, y \le 0$$

which represents a rhombus with sides

$$2x \pm 3y = 6 \text{ and } 2x \pm 3y = -6$$

Length of the diagonals is 6 and 4 units along x-and y-axes. Therefore, the required area is $1/2 \times 6 \times 4 = 12$ sq. units.

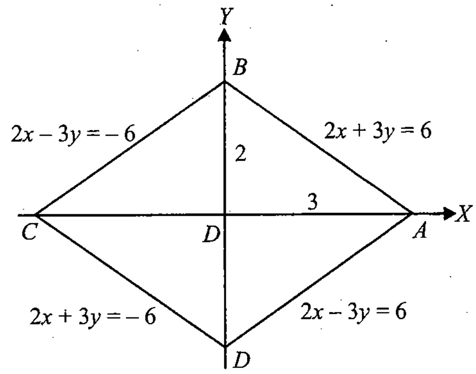


Fig. 1.96

10. d. Given $O(0, 0)$ is the orthocentre. Let $A(h, k)$ be the third vertex, $B(-2, 3)$ and $C(5, -1)$ the other two vertices. Then the slope of the line through A and O is k/h, while the lines through B and C has the slope $-4/7$. By the property of the orthocentre, these two lines must be perpendicular, so we have

$$(k/h)\left(-\tfrac{4}{7}\right) = -1 \Rightarrow k/h = \tfrac{7}{4} \quad \text{(i)}$$

Also,

$$\frac{5 - 2 + h}{3} + \frac{-1 + 3 + k}{3} = 7$$

$$\Rightarrow \quad h + k = 16 \quad \text{(ii)}$$

which is not satisfied by the points given in (a), (b) or (c).

11. c. Let the coordinates of vertices O, P, Q, R be $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$, respectively. Then, we get the coordinates of M as $(a, a/2)$ and those of N as $(a/2, a)$.

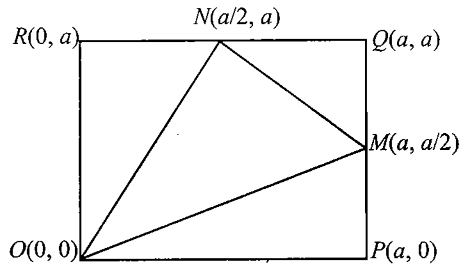


Fig. 1.97

Therefore, area of ΔOMN is

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a & a/2 & 1 \\ a/2 & a & 1 \end{vmatrix} = \frac{3a^2}{8}$$

Area of the square is a^2. Hence, the required ratio is 8 : 3.

12. b. p, q, r are the roots of equation $y^3 - 3y^2 + 6y + 1 = 0$. So, $p + q + r = 3$, $pq + qr + rp = 6$ and $pqr = -1$. Now, the centroid of the triangle is

$$\left(\frac{pq + qr + rp}{3}, \frac{\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}}{3}\right)$$

i.e.,

$$\left(\frac{pq + qr + rp}{3}, \frac{p + q + r}{3pqr}\right) = \left(\frac{6}{3}, \frac{3}{-3}\right) \text{ or } (2, -1)$$

13. a.

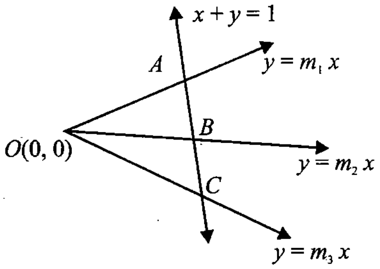


Fig. 1.98

Solving the equations of the lines, we get

$$A = \left(\frac{1}{1+m_1}, \frac{m_1}{1+m_1}\right), C = \left(\frac{1}{1+m_3}, \frac{m_3}{1+m_3}\right)$$

If $AB = BC$, then midpoint of AC lies on

$$y = m_2 x$$

$$\Rightarrow \quad \frac{\frac{m_1}{1+m_1} + \frac{m_3}{1+m_3}}{2} = m_2 \left[\frac{\frac{1}{1+m_1} + \frac{1}{1+m_3}}{2}\right]$$

14. b. Lines $ax + y = 0$ and $x + by = 0$ intersect at $O(0, 0)$.

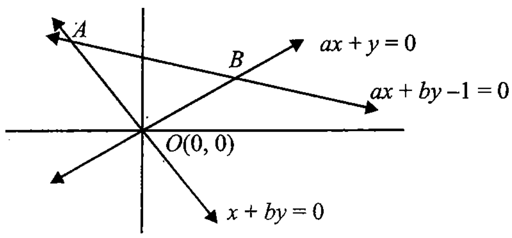


Fig. 1.99

Hence, if AB subtends right angle at $O(0, 0)$, then $ax + y = 0$ and $x + by = 0$ are perpendicular to each other. So,

$$(-a)\left(-\frac{1}{b}\right) = -1$$

$$\Rightarrow \quad a + b = 0$$

15. c.

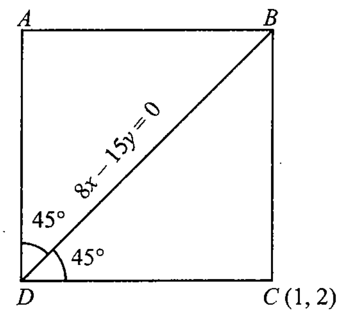


Fig. 1.100

Slope of BD is 8/15 and angle made by BD with DC and BC is 45°. So let slope of DC be m. Then,

$$\tan 45° = \pm \frac{m - \frac{8}{15}}{1 + \frac{8}{15}m}$$

$$\Rightarrow \quad (15 + 8m) = \pm (15m - 8)$$

$$\Rightarrow \quad m = \frac{23}{7} \text{ and } -\frac{7}{23}$$

Hence, the equations of DC and BC are

$$y - 2 = \frac{23}{7}(x - 1)$$

$$\Rightarrow \quad 23x - 7y - 9 = 0$$

and

$$y - 2 = -\frac{7}{23}(x - 1)$$

$$\Rightarrow \quad 7x + 23y - 53 = 0$$

16. a. Since $a_1/a_2 = b_1/b_2 = c_1/c_2$, then $u = 0$ and $v = 0$ are same straight line. Hence, $u + kv = 0$ is also the same straight line.

17. c. Since the point $A(2, 1)$ is translated parallel to $x - y = 3$, therefore AA' has the same slope as that of $x - y = 3$. Therefore, AA' passes through (2, 1) and has the slope of 1. Here,

$$\tan\theta = 1 \Rightarrow \cos\theta = 1/\sqrt{2}, \sin\theta = 1/\sqrt{2}$$

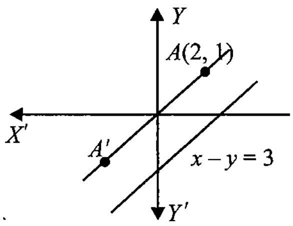


Fig. 1.101

Thus, the equation of AA' is

$$\frac{x - 2}{\cos(\pi/4)} = \frac{y - 1}{\sin(\pi/4)}$$

Since $AA' = 4$, therefore the coordinates of A' are given by

$$\frac{x - 2}{\cos(\pi/4)} = \frac{y - 1}{\sin(\pi/4)} = -4$$

$$\Rightarrow \quad x = 2 - 4\cos\frac{\pi}{4}, y = 1 - 4\sin\frac{\pi}{4}$$

$$\Rightarrow \quad x = 2 - 2\sqrt{2}, y = 1 - 2\sqrt{2}$$

Hence, the coordinates of A' are $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$.

18. d.

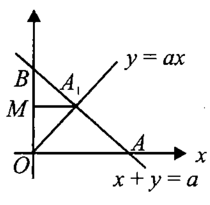


Fig. 1.102

$$B \equiv (0, a), A_1 \equiv \left(\frac{a}{1+a}, \frac{a^2}{1+a}\right)$$

$$\Delta_{OA_1B} = \frac{1}{2}(OB)\,A_1M$$

$$= \frac{1}{2}|a|\left|\frac{a}{1+a}\right| = \frac{1}{2}\frac{a^2}{|1+a|}$$

19. a.

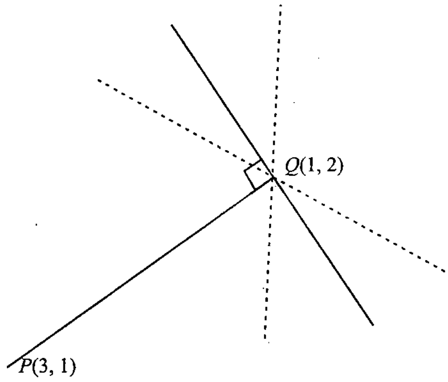


Fig. 1.103

Obviously line though Q is at greatest distance from point P when it is perpendicular to PQ. Now slope of line PQ is $m_{PQ} = -1/2$. Then slope of perpendicular line passing through Q is

$$y - 2 = 2(x - 1)$$

or

$$2x - y = 0$$

20. c. Let the equation of side AB be $y = x + a$. Then, $A \equiv (1 - a, 1), B \equiv (2, 2 + a)$. Equation of side AD is $y - 1 = -(x - (1 - a))$. Hence, $D \equiv (-2, 4 - a)$.

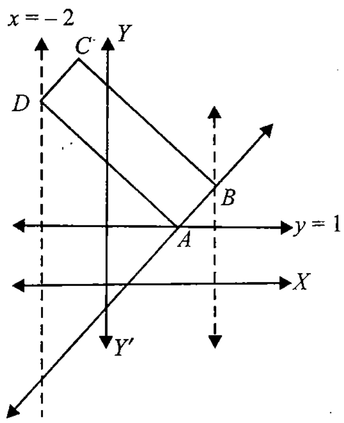


Fig. 1.104

Let $\quad C \equiv (h, k)$. Then,

$$h + 1 - a = 2 - 2$$

$\Rightarrow \quad h = a - 1$ and $k + 1 = 2 + a + 4 - a$

$\Rightarrow \quad k = 5$

Thus locus of C is $\quad y = 5$.

21. a. Let the vertices 'B' and 'C' lie on the given line. Then,

$OD = \dfrac{2\sqrt{2}}{\sqrt{2}} = 2$. Equation of OD is

$$y = x \Rightarrow x = y = \sqrt{2} \text{ (for point } D)$$

Also, $BD = OD \times \tan 60° = 2\sqrt{3}$ for the coordinates of B and C. Using parametric equation of line, we get

$$\frac{x - \sqrt{2}}{-\frac{1}{\sqrt{2}}} = \frac{y - \sqrt{2}}{\frac{1}{\sqrt{2}}} = \pm 2\sqrt{3}$$

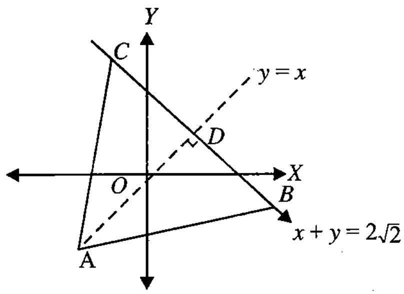


Fig. 1.105

$$\Rightarrow C \equiv (\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6})$$

and $\quad B \equiv (\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6})$

22. d. Distance of all the points from (0, 0) are 5 units. That means cirumcentre of the triangle formed by the given points is (0, 0). If $G \equiv (h, k)$ be the centroid of the triangle, then $3h = 3 + 5(\cos\theta + \sin\theta)$, $3k = 4 + 5(\sin\theta - \cos\theta)$. If $H(\alpha, \beta)$ be the orthocentre, then

$$OG{:}GH = 1{:}2 \Rightarrow \alpha = 3h, \beta = 3k$$

$$\cos\theta + \sin\theta = \frac{\alpha - 3}{5}, \ \sin\theta - \cos\theta = \frac{\beta - 4}{5}$$

$$\Rightarrow \quad \sin\theta = \frac{\alpha + \beta - 7}{10}, \ \cos\theta = \frac{\alpha - \beta + 1}{10}$$

Thus the locus of (α, β) is

$$(x + y - 7)^2 + (x - y + 1)^2 = 100$$

23. b.

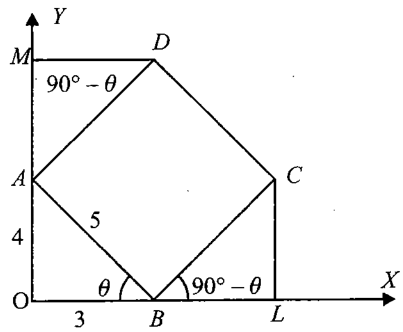


Fig. 1.106

The coordinates of A are (0, 4) and those of B are (3, 0).

$BC = AB = \sqrt{3^2 + 4^2} = 5$

$\Rightarrow \quad BL = BC \sin\theta$ and $CL = BC \cos\theta$

$\Rightarrow \quad BL = 5 \times \frac{4}{5} = 4$ and $CL = 5 \times \frac{3}{5} = 3$

Similarly, $MD = 4$ and $AM = 3$. So, the coordinates of C are $(OB + BL, CL) = (7, 3)$ and those of D are $(MD, OA + AM) = (4, 7)$.

The coordinates of the vertex farthest from the origin is (4, 7).

24. a.

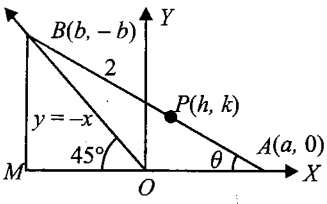


Fig. 1.107

If $\angle BAO = \theta$, then $BM = 2\sin\theta$ and $MO = BM = 2\sin\theta$, $MA = 2\cos\theta$. Hence, $A = (2\cos\theta - 2\sin\theta, 0)$ and $B = (-2\sin\theta, 2\sin\theta)$. Since $P(x, y)$ is the midpoint of AB, so

$$2x = (2\cos\theta) + (-4\sin\theta)$$

or $\quad \cos\theta - 2\sin\theta = x$

$$2y = (2\sin\theta) \text{ or } \sin\theta = y$$

Eliminating θ, we have

$$(x + 2y)^2 + y^2 = 1 \text{ or } x^2 + 5y^2 + 4xy - 1 = 0$$

25. b.

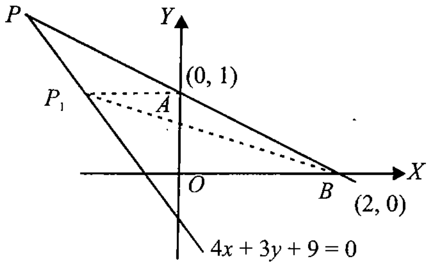


Fig.1.108

Equation of AB is

$$y - 1 = \frac{0-1}{2-1}x \Rightarrow x + 2y - 2 = 0$$

$$|PA - PB| \le AB$$

Thus, $|PA - PB|$ is maximum if points A, B and P are collinear.

Hence, solving $x + 2y - 2 = 0$ and $4x + 3y + 9 = 0$, we get point $P = (-84/5, 13/5)$.

26. c.

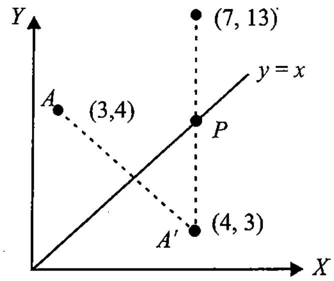


Fig. 1.109

Consider a point A', the image of A in $y = x$. Therefore, the coordinates of A' are (4, 3).

or

(Notice that A and B lie are the same side with respect to $y = x$). Then $PA = PA'$. Thus, $PA + PB$ is minimum, if $PA' + PB$ is minimum, i.e., if P, A', B are collinear. Now, the equation of AB is

$$y - 3 = \frac{13-3}{7-4}(x - 4)$$

$\Rightarrow \quad 3y - 10x + 31 = 0$

It intersects $y = x$ at (31/7, 31/7), which is the required point P.

27. a.

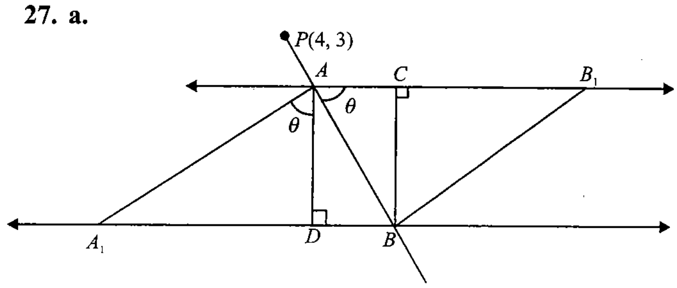


Fig. 1.110

The given lines (L_1 and L_2) are parallel and distance between them (BC or AD) is $(15 - 5)/5 = 2$ units. Let $\angle BCA = \theta \Rightarrow AB = BC \operatorname{cosec}\theta = 2\operatorname{cosec}\theta$ and $AA_1 = AD \sec\theta = 2\sec\theta$. Now area of parallelogram AA_1BB_1 is

$$\Delta = AB \times AA_1 = 4\sec\theta \operatorname{cosec}\theta$$

$$= \frac{8}{\sin 2\theta}$$

Clearly, Δ is least for $\theta = \pi/4$. Let slope of AB be m.

Then, $\quad 1 = \left|\dfrac{m + 3/4}{1 - \frac{3m}{4}}\right|$

$\Rightarrow \quad 4m + 3 = \pm(4 - 3m) \Rightarrow m = 1/7$ or -7

Hence, the equation of 'L' is

$$x - 7y + 17 = 0$$

or

$$7x + y - 31 = 0$$

28. b. Suppose we rotate the coordinate axes in the anticlockwise direction through an angle α. The equation of the line L with respect to old axes is

$$\frac{x}{a}+\frac{y}{b}=1$$

In this equation replacing x by $x\cos\alpha - y\sin\alpha$ and y by $x\sin\alpha + y\cos\alpha$, the equation of the line with respect to new axes is

$$\frac{x\cos\alpha - y\sin\alpha}{a}+\frac{x\sin\alpha + y\cos\alpha}{b}=1$$

$$\Rightarrow x\left(\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}\right)+y\left(\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}\right)=1 \qquad \text{(i)}$$

The intercepts made by (i) on the coordinate axes are given as p and q.

Therefore,

$$\frac{1}{p}=\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}$$

and $$\frac{1}{q}=\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}$$

Squaring and adding, we get

$$\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

29. c.

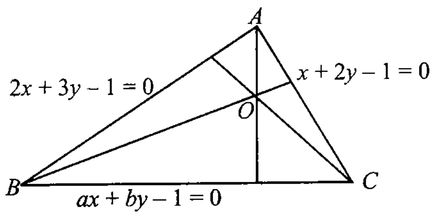


Fig. 1.111

Equation of AO is $2x + 3y - 1 + \lambda(x + 2y - 1) = 0$, where $\lambda = -1$ since the line passes through the origin. So, $x + y = 0$. Since AO is perpendicular to BC, so

$$(-1)\left(-\frac{a}{b}\right)=-1$$

$$\therefore \qquad a=-b$$

Similarly,

$(2x + 3y - 1) + \mu(ax - ay - 1) = 0$

will be the equation of BO for $\mu = -1$. Now, BO is perpendicular to AC. Hence,

$$\left\{-\frac{(2-a)}{3+a}\right\}\left(-\frac{1}{2}\right)=-1$$

$$\therefore \qquad a=-8,\ b=8$$

30. c. If P_1 be the reflection of P in y-axis, then $P_1 \equiv (-2, 3)$.

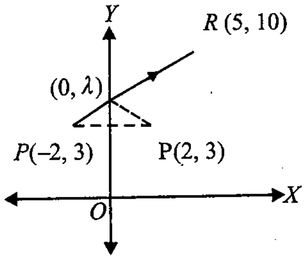


Fig. 1.112

Equation of line P_1R is

$$(y-3)=\frac{10-3}{5+2}(x+2)$$

$$\Rightarrow \qquad y=x+5$$

It meets y-axis at $(0, 5) \Rightarrow Q \equiv (0, 5)$.

31. a.

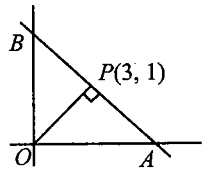


Fig. 1.113

Line AB will be farthest from origin if OP is right angled to the line drawn. Hence,

$$m_{OP}=\frac{1}{3} \Rightarrow m_{AB}=-3$$

Thus equation of AB is

$$(y-1)=-3(x-3)$$

$$\Rightarrow \qquad A\equiv\left(\frac{10}{3},0\right),\ B\equiv(0,10)$$

$$\Rightarrow \qquad \Delta_{OAB}=\frac{1}{2}(OA)(OB)=\frac{1}{2}\times\frac{10}{3}\times 10$$

$$=\frac{100}{6}\text{ sq. units}$$

32. a. Since $xy > 0$, P lies either in the first quadrant or in the third quadrant. The inequality $x + y < 1$ represents all points below the line $x + y = 1$ so that $xy > 0$ and $x + y < 1$ imply that P lies either inside the triangle OAB or in third quadrant.

33. c.

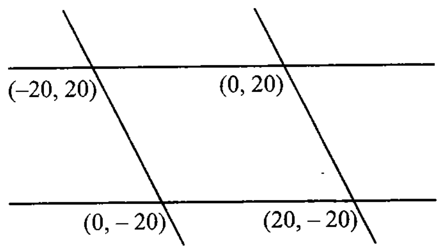


Fig. 1.114

For $x \geq 0$, the equation is

$(y + 20)(y + 2x - 20) = 0$

For $x \leq 0$, the equation is

$(y - 20)(y + 2x + 20) = 0$

Hence, the area is $20 \times 40 = 800$ sq. units.

34. **b.** $A \equiv (\alpha, 2\alpha + 3)$, $BC = 1$ unit. Equation of BC is $y - 3 = 0$.

Distance of A from BC is $p \Rightarrow |2\alpha + 3 - 3|$.

Area of $\Delta ABC = \Delta = |\alpha|$; $5 \leq \Delta < 6 \Rightarrow 5 \leq |\alpha| < 6$

35. **d.** $\frac{a}{\sqrt{bc}} - 2 = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}$

$\Rightarrow a = b + c + 2\sqrt{bc}$

$\Rightarrow a = (\sqrt{b} + \sqrt{c})^2$

$\Rightarrow (\sqrt{a} - \sqrt{b} - \sqrt{c})(\sqrt{a} + \sqrt{b} + \sqrt{c}) = 0$

$\Rightarrow \quad \sqrt{a} - \sqrt{b} - \sqrt{c} = 0$

Since $\sqrt{a} + \sqrt{b} + \sqrt{c} \neq 0 \quad (\because a, b, c > 0)$. Comparing with $\sqrt{a}x + \sqrt{b}y = \sqrt{c} = 0$, we have $x = -1, y = 1$.

36. **b.** There are clearly five points.

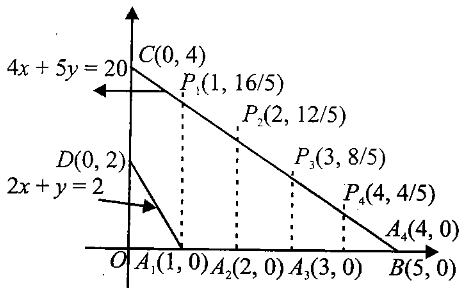


Fig. 1.115

37. **b.** By solving the sides of the rhombus, its vertices are $(0, -n/m)$, $(-n/l, 0)$, $(0, n/m)$ and $(n/l, 0)$. Hence, the

area is $\quad \frac{1}{2} \times \frac{2n}{m} \times \frac{2n}{l} = 2$

$\Rightarrow n^2 = lm$. Therefore, l, n, m are in G.P.

38. **c.** If the point of intersection of two lines with coordinate axes be concyclic, then product of intercepts on x-axis is equal to product of intercepts on y-axis by these lines. This is a geometric property. The intercepts on x-axis are p and q and whose product is pq. Also, the intercepts on y-axis are p, and q, whose product is also pq. Hence, the four points are concyclic

39. **b.**

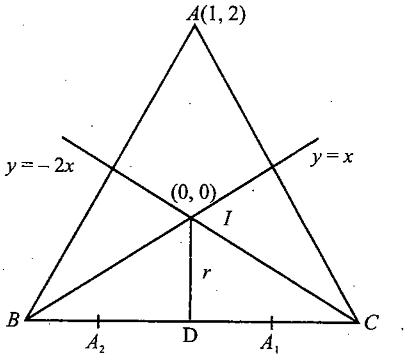


Fig. 1.116

Images of A about $y = x$, $y = -2x$ are A_1 and A_2 which lie on BC. Now, $A_1 \equiv (2, 1)$ and $A_2 \equiv (-11/5, 2/5)$. Equation of BC is $x - 7y + 5 = 0$. Hence,

$$r = ID = \left|\frac{5}{\sqrt{1+49}}\right| = \frac{1}{\sqrt{2}}$$

40. **b.** Reflections of A in the two angle bisectors will lie on the line BC, so $(2, 1)$ and $(1, -2)$ will lie on BC. Equation of BC will be $y + 2 = \left(\frac{1+2}{2-1}\right)(x - 1)$,

i.e., $3x - y = 5$

41. **c.** The circumcentre of the triangle is $(0, 0)$ as all the vertices lie on the circle $x^2 + y^2 = 5$. So the orthocentre will be (sum of x coordinates, sum of y coordinates).

42. **a.** Extremities of the given diagonal are $(4, 0)$ and $(0, 6)$. Hence, slope of this diagonal is $-3/2$ and slope of other diagonal is $2/3$. The equation of the other diagonal is

$$\frac{x-2}{\frac{3}{\sqrt{13}}} = \frac{y-3}{\frac{2}{\sqrt{13}}} = r$$

For the extremities of the diagonal, $r = \pm\sqrt{13}$. Hence,

$$x - 2 = \pm 3, y - 3 = \pm 2$$

$$x = 5, -1 \text{ and } y = 5, 1$$

Therefore, the extremities of the diagonal are $(5, 5)$ and $(-1, 1)$.

43. **a.** No such triangle is possible as the medians through the vertices of a right-angled triangle (other than right angle) cannot be perpendicular to each other.

44. **a.** $\quad \tan\theta = \frac{a-1}{3} = \frac{7-a}{10-b}$

also $\quad \tan\theta = \frac{7-4}{b-3} = \frac{3}{b-3}$

Hence, $\quad \frac{3}{b-3} = \frac{a-1}{3} = \frac{7-a}{10-b}$

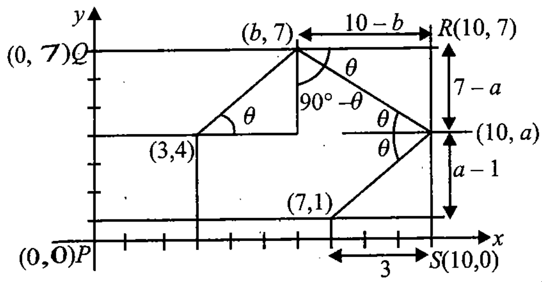


Fig. 1.117

from 1st two relations

$$9 = ab - b - 3a = 3$$

$$3a + 6 = ab - b \quad \text{(i)}$$

from last two

$$10a - ab - 10 + b = 21 - 3a$$

$$13a - ab + b = 31$$

or $$ab - b = 13a - 31 \quad (2)$$

Hence, from (i) and (ii)

$$3a + 6 = 13a - 31 \Rightarrow 10a = 37$$

$\Rightarrow$ $$a = 3.7$$

45. c. Let incentre be/ Then/ (2, 1)

$\Rightarrow$ $$IA = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

Also,

$$AI = r \operatorname{cosec} \frac{1}{2}$$

$$\angle BIC = \frac{\pi}{2} + \frac{A}{2}$$

$\Rightarrow$ $$\tan\left(\frac{\pi}{2} + \frac{A}{2}\right) = \frac{1-2}{1+2}$$

$$= \frac{1}{3} \cot \frac{1}{2} = 1$$

So, $$r = \frac{AI}{\text{coses}\,\frac{A}{2}} = \frac{2\sqrt{2}}{\sqrt{1 + \frac{1}{9}}}$$

$$\frac{2\sqrt{2} \times 3}{\sqrt{10}} = \frac{6}{\sqrt{5}}$$

46. d.

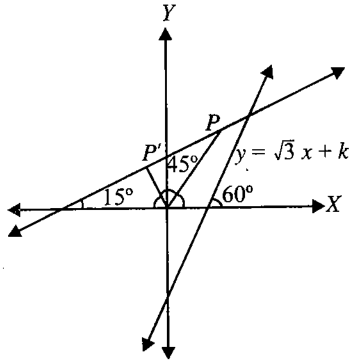


Fig. 1.118

Angle between both the lines is 45°. Hence, $OP = OP'\sqrt{2} = (5/\sqrt{2}) \times \sqrt{2} = 5$.

47. a. First, two family of lines passes through (1, 1) and (3, 3), respectively. The point of intersection of lines belonging to third family of lines will lie on line $y = x$. Hence,

$\Rightarrow$ $$ax + x - 2 = 0 \text{ and } 6x + ax - a = 0$$

or $$\frac{2}{a+1} = \frac{a}{6+a}$$

$\Rightarrow$ $$a^2 - a - 12 = 0 \Rightarrow (a-4)(a+3) = 0$$

48. a.

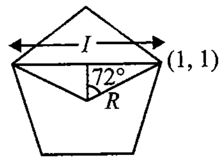


Fig. 1.119

The point of intersection of diagonals, i.e., (1, 1), lies on circumcircle. Hence,

$\Rightarrow$ $$I = 2R \sin 72°$$

$$R = \frac{\sin 36°}{2 \sin 72°} = \cos 72°$$

Therefore, the locus is $(x-1)^2 + (y-1)^2 = \cos^2 72°$. Hence,

$x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$

49. b. As altitude from A is fixed and the orthocentre lies on altitude, hence $x + y = 3$ is the required locus.

50. c. The family of lines $(x - 2y + 3) + \lambda(2x - 3y + 4) = 0$ are concurrent at point $P(1, 2)$

If image of point $A(2, 3)$ in the above variable line is $B(h, k)$ then $AP = BP$

$\Rightarrow (h-1)^2 + (k-2)^2 = (2-1)^2 + (3-2)^2$

Hence, locus of point P is $x^2 + y^2 - 2x - 4y + 4 = 0$

51. b. From the figure

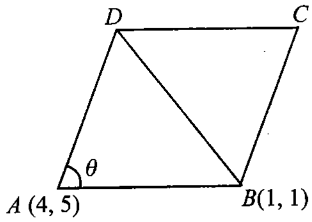


Fig. 1.120

Area of rhombus = 2 × (area of ΔABD)

$$= 2 \times \frac{1}{2} \times 5 \times 5 \sin\theta$$

$$= 25 \sin\theta$$

Hence, maximum area is 25 (when $\sin\theta = 1$)

52. a.

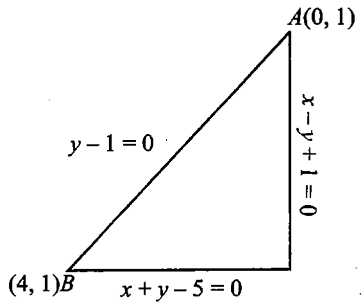


Fig. 1.121

Since the triangle is right angled, so the circumcentre will be the middle point of hypotenuse, i.e., (2, 1).

53. b. Let the third vertex be (h, k).

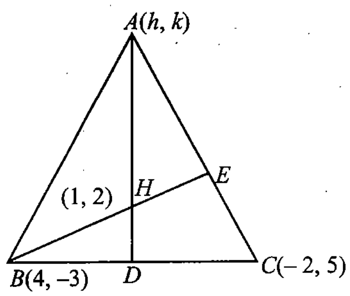


Fig. 1.122

Now slope of AD is $(k - 2)/(h - 1)$ slope of BC is $(5 + 3)/(-2 - 4) = -4/3$, slope of BE is $(-3 - 2)/(4 - 1) = 5/3$ and slope of AC is $(k - 5)/(h + 2)$. Since $AD \perp BC$, so

$$\frac{k-2}{h-1} \times \frac{-4}{3} = -1$$

$\Rightarrow \quad 3h - 4k + 5 = 0 \qquad$ (i)

Again since $BE \perp AC$, so

$$-\frac{5}{3} \times \frac{k-5}{h+2} = -1$$

$\Rightarrow \quad 3h - 5k + 31 = 0 \qquad$ (ii)

On solving (i) and (ii) we get $h = 33$, $k = 26$. Hence, the third vertex is (33, 26).

54. a. Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$, $D \equiv (x_4, y_4)$

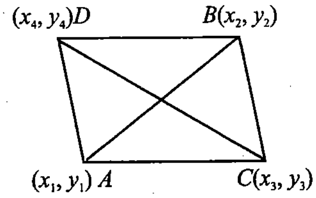


Fig. 1.123

Given,

$x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 - 2x_1x_3 - 2x_2x_4 - 2y_2y_3 - 2y_1y_4 \le 0$

$\Rightarrow (x_1 - x_3)^2 + (x_2 - x_4)^2 + (y_2 - y_3)^2 + (y_1 - y_4)^2 \le 0$

$\Rightarrow \quad x_1 = x_3, x_2 = x_4, y_2 = y_3, y_1 = y_4$

$\Rightarrow \quad \dfrac{x_1 + x_2}{2} = \dfrac{x_3 + x_4}{2}$ and $\dfrac{y_1 + y_2}{2} = \dfrac{y_4 + y_3}{2}$

Hence, AB and CD bisect each other.

Therefore, $ACBD$ is a parallelogram

Also

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x_3 - x_4)^2 + (y_4 - y_3)^2$$
$$= CD^2$$

Thus $ACBD$ is a parallelogram and $AB = CD$, hence is a rectangle

55. c. For any point $P(x, y)$ that is equidistant from given line, we have

$$x + y - \sqrt{2} = -(x + y - 2\sqrt{2})$$

$\Rightarrow \quad 2x + 2y - 3\sqrt{2} = 0$

56. a. The distance between $(2/m, 2)$ and $(6/m, 6)$ is less than 5. Hence,

$$\left(\frac{2}{m} - \frac{6}{m}\right)^2 + (2 - 6)^2 < 25$$

$\Rightarrow \quad \dfrac{16}{m^2} < 9$

$\Rightarrow \quad m^2 > \dfrac{16}{9}$

$\Rightarrow \quad m > \dfrac{4}{3}$ or $m < \dfrac{-4}{3}$

57. a.

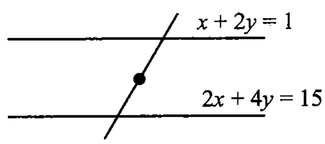


Fig. 1.124

Let P be on $x + 2y = 1$. Then,

$$1 + \frac{t}{\sqrt{2}} + 2\left(2 + \frac{t}{\sqrt{2}}\right) = 1$$

or

$$t = \frac{-4\sqrt{2}}{3}$$

Let P be on $2x + 4y = 15$. Then,

$$2\left(1 + \frac{t}{\sqrt{2}}\right) + 4\left(2 + \frac{t}{\sqrt{2}}\right) = 15$$

or

$$t = \frac{5\sqrt{2}}{6}$$

Since point lies between the lines and $x = t$, then $t \in \left(\frac{-4\sqrt{2}}{3}, \frac{5\sqrt{2}}{3}\right)$.

58. a.

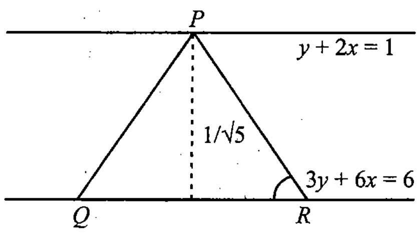


Fig. 1.125

The given lines are $y + 2x = 1$ and $y + 2x = 2$. The distance between the lines is $(2 - 1)/\sqrt{5} = 1/\sqrt{5}$.

The side length of the triangle is $(1/\sqrt{5})$ cosec $60° = 2/\sqrt{5}$.

59. c.

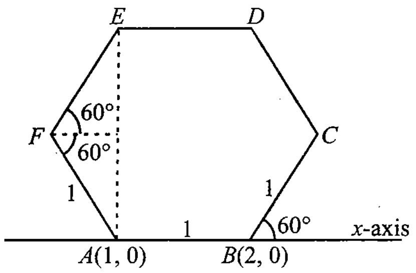


Fig. 1.126

$$AB = 1$$

$$\Rightarrow C \equiv (2 + 1 \times \cos 60°, 1 \times \sin 60°) = \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$$

$$E \equiv (1, 1 \times \sin 60° + 1 \times \sin 60°) = (1, \sqrt{3})$$

Therefore, the equation of CE is

$$y - \sqrt{3} = \frac{\sqrt{3} - \frac{\sqrt{3}}{2}}{1 - \frac{5}{2}}(x - 1)$$

60. d.

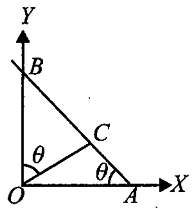


Fig. 1.127

$$\tan(180° - \theta) = \text{slope of } AB = -3$$

$$\therefore \quad \tan\theta = 3$$

$$\therefore \quad \frac{OC}{AC} = \tan\theta, \frac{OC}{BC} = \cot\theta$$

$$\Rightarrow \quad \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$$

61. a. Let $x_1 = a$, $x_2 = ar$ and $x_3 = ar^2$; $y_1 = b$, $y_2 = br$ and $y_3 = br^2$. Now,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{br - b}{ar - a} = \frac{b}{a}$$

and

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{br^2 - br}{ar^2 - ar} = \frac{b}{a}$$

Therefore, slope of PQ is equal to slope of QR. Hence, points P, Q, R are collinear.

62. c. Since the diagonals are perpendicular, so the given quadrilateral is a rhombus. So, the distances between two pairs of parallel sides are equal. Hence,

$$\left|\frac{c' - c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{c' - c}{\sqrt{a'^2 + b'^2}}\right|$$

$$\Rightarrow \quad a^2 + b^2 = a'^2 + b'^2$$

63. c. We have,

$$B \equiv \left(\frac{6}{7}, 2\right), C \equiv \left(-\frac{6}{7}, 2\right)$$

$$\Rightarrow \quad BC = \frac{12}{7}, AD = 3$$

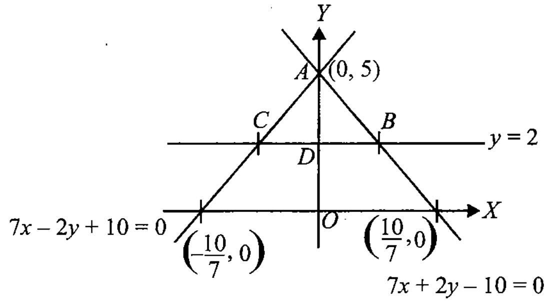


Fig. 1.128

$$\therefore \quad \Delta_{ABC} = \frac{1}{2} \times \frac{12}{7} \times 3 = \frac{18}{7} \text{ sq. units}$$

64. d. Clearly, A will remain as (0, 0); 'f_1' will make B as (0, 4); 'f_2' will make it (12, 4) and 'f_3' will make it (4, 8); 'f_1' will make 'C' as (2, 4); 'f_2' will make it (14, 4); 'f_3' will make it (5, 9). Finally 'f_1' will make 'D' as (2, 0) 'f_2' will make ixt (2, 0) 'f_3' will make it (1, 1). So we finally get $A \equiv (0, 0)$, $B \equiv (4, 8)$, $C \equiv (5, 9)$, $D \equiv (1, 1)$. Hence,

$$m_{AB} = \frac{8}{4}, m_{BC} = \frac{9-8}{5-4} = 1, m_{CD} = \frac{9-1}{5-1} = \frac{8}{4}, m_{AD} = 1, m_{AC} = \frac{9}{5},$$

$$m_{BD} = \frac{8-1}{4-1} = 7/3$$

Hence, the final figure will be a parallelogram.

65. d.

Angle bisector will make the angles $(\theta_1 + \theta_2)/2$ and $(\pi/2 + (\theta_1 + \theta_2)/2)$ with the x-axis. Hence, their equations are

$$\frac{x - x_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}$$

or

$$\frac{x - x_1}{-\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$$

66. c. $ax + by = 1$ will be one of the bisectors of the given line. Equation of bisectors of the given lines are

$$\frac{3x + 4y - 5}{5} = \pm\left(\frac{5x - 12y - 10}{13}\right)$$

$\Rightarrow \quad 64x - 8y = 115$

or $\quad 14x + 112y = 15$

$\Rightarrow \quad a = \frac{64}{115}, b = -\frac{8}{115}$

or $\quad a = \frac{14}{15}, b = \frac{12}{115}$

67. b. Lines $x \cos\alpha + y \sin\alpha = p$ and $x \sin\alpha - y \cos\alpha = 0$ are mutually perpendicular. Thus $ax + by + p = 0$ will be equally inclined to these lines and would be the angle bisector of these lines. Now equations of angle bisectors is

$x \sin\alpha - y \cos\alpha = \pm(x \cos\alpha + y \sin\alpha - p)$

$\Rightarrow x(\cos\alpha - \sin\alpha) + y(\sin\alpha + \cos\alpha) = p$

or

$x(\sin\alpha + \cos\alpha) - y(\cos\alpha - \sin\alpha) = p$

Comparing these lines with $ax + by = p = 0$, we get

$$\frac{a}{\cos\alpha - \sin\alpha} = \frac{b}{\sin\alpha + \cos\alpha} = 1$$

$\Rightarrow \quad a^2 + b^2 = 2$

or

$$\frac{a}{\sin\alpha + \cos\alpha} = \frac{b}{\cos\alpha - \sin\alpha} = 1$$

$\Rightarrow \quad a^2 + b^2 = 2$

68. a. Let the equation of line be $x/a + y/b = 1$.

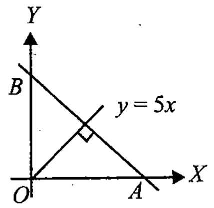


Fig. 1.129

AB is perpendicular to $y = 5x$. Hence,

$$-b/a \times 5 = -1 \Rightarrow 5b = a$$

Area of $\quad \Delta_{OAB} = \frac{1}{2}|ab|$

$\Rightarrow \quad 10 = \frac{1}{2}|5b^2|$

$\Rightarrow \quad b^2 = 4 \Rightarrow b = \pm 2, a = \pm 10$

The line can be $x/10 + y/2 = 1$ or $x/10 + y/2 = -1$.

69. a. Given lines are mutually perpendicular and intersect at (6/5, 13/5).

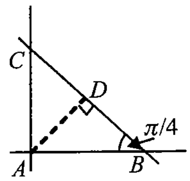


Fig. 1.130

Equations of angle bisectors of the given lines are $x - 2y + 4 = \pm(2x + y - 5)$,

i.e., $x + 3y = 9$ and $3x - y = 1$.

Side BC will be parallel to these bisectors. Let

$AD \quad = a$

$\Rightarrow \quad AB = a\sqrt{2}$ and area of ΔABC is

$$\Delta_{ABC} = \frac{1}{2} \times (a\sqrt{2})^2 = a^2 = 10$$

$\Rightarrow \quad a = \sqrt{10}$

Let equation of BC be $x + 3y = \lambda$. Then,

$$\sqrt{10} = \frac{\frac{6}{5} - \frac{39}{5} - \lambda}{\sqrt{10}} \Rightarrow \lambda = -1, 19$$

Therefore, equation of BC is $x + 3y = -1$ or $x + 3y = 19$.

If equation of BC is $3x - y = \lambda$, then

$$\sqrt{10} = \frac{\frac{18}{5} - \frac{13}{5} - \lambda}{\sqrt{10}} \Rightarrow \lambda = -9, 11$$

Hence, equation of BC is $3x - y = -9$ or $3x - y = 11$.

70. d. All values of 'a'.

71. c.

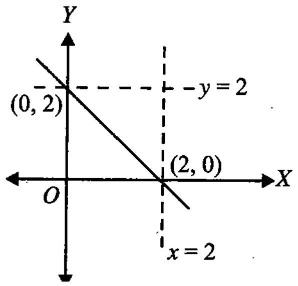


Fig. 1.131

Above figure represents the given isosceles triangle. Clearly the equation of other equal side is $y = 2$.

72. d. Let (h, k) be the point on the locus. Then by the given conditions,

$$(h-a_1)^2+(k-b_1)^2=(h-a_2)^2+(k-b_2)^2$$

$$\Rightarrow 2h(a_1-a_2)+2k(b_1-b_2)+a_2^2+b_2^2-a_1^2-b_1^2=0$$

$$\Rightarrow h(a_1-a_2)+k(b_1-b_2)+\frac{1}{2}(a_2^2+b_2^2-a_1^2-b_1^2)=0 \quad \text{(i)}$$

Also, since (h, k) lies on the given locus, therefore

$$(a_1-a_2)h+(b_1-b_2)k+c=0 \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$c=\frac{1}{2}(a_2^2+b_2^2-a_1^2-b_1^2)$$

73. c. Clearly, the equation of PQ in the new position is $x = 2$.

74. b. If the line cuts off the axes at A and B, then area of triangle is $\frac{1}{2}\times OA\times OB=T$.

$$\Rightarrow \quad \frac{1}{2}\times a\times OB=T\Rightarrow OB=\frac{2T}{a}$$

Hence, the equation of line is

$$\frac{x}{-a}+\frac{y}{2T/a}=1$$

$$\Rightarrow \quad 2Tx-a^2y+2aT=0$$

75. a.

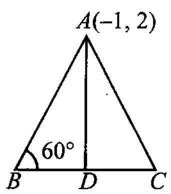


Fig. 1.132

$$AD=\left|\frac{-2-2-1}{\sqrt{(2)^2+(-1)^2}}\right|=\left|\frac{-5}{\sqrt{5}}\right|=\sqrt{5}$$

$$\because \quad \tan 60°=\frac{AD}{BD}\Rightarrow\sqrt{3}=\frac{\sqrt{5}}{BD}$$

$$\Rightarrow \quad BD=\sqrt{\frac{5}{3}}$$

$$\therefore \quad BC=2BD=2\sqrt{\frac{5}{3}}=\sqrt{\frac{20}{3}}$$

76. a. Solving $3x + 4y = 9$, $y = mx + 1$, we get $x=\frac{5}{3+4m}$. Here, x is an integer if $3 + 4m = 1, -1, 5, -5$. Hence, $m = -2/4, -4/4, 2/4, -8/4$. So, m has two integral values.

77. a. Given that slope is $-\sqrt{3}$. Therefore, the line is

$$y=-\sqrt{3}x+c$$

$$\Rightarrow \quad \sqrt{3}x+y=c$$

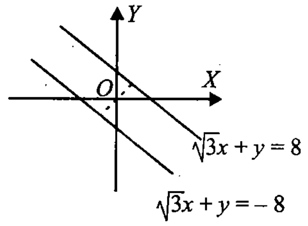


Fig. 1.133

Now,

$$\left|\frac{c}{2}\right|=4\Rightarrow c=\pm 8\Rightarrow x\sqrt{3}+y=\pm 8$$

78. a. Line perpendicular to $x\sec\theta+y\operatorname{cosec}\theta=a$ is

$$x\operatorname{cosec}\theta-y\sec\theta=\lambda.$$

This line passes through the point $(a\cos^3\theta, a\sin^3\theta)$. Then,

$$(a\cos^3\theta)\operatorname{cosec}\theta-(a\sin^3\theta)\sec\theta=\lambda$$

$$\Rightarrow \quad \lambda=a\left(\frac{\cos^3\theta}{\sin\theta}-\frac{\sin^3\theta}{\cos\theta}\right)$$

$$=a\frac{\cos 2\theta}{\cos\theta\sin\theta}$$

Hence, the equation of line is $x\cdot\cos\theta-y\sin\theta=a\cos 2\theta$.

79. c. The line passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is

$$ax+2by+3b+\lambda(bx-2ay-3a)=0$$

$$\Rightarrow (a+b\lambda)x+(2b-2a\lambda)y+3b-3\lambda a=0 \quad \text{(i)}$$

Line Eq. (i) is parallel to x-axis. Therefore,

$$a+b\lambda=0\Rightarrow\lambda=-\frac{a}{b}=0$$

Putting the value of λ in Eq. (i), we get

$$y\left(2b+\frac{2a^2}{b}\right)+3b+\frac{3a^2}{b}=0$$

$$\Rightarrow \quad y=-\frac{3}{2}$$

So, it is 3/2 units below x-axis.

80. a. Midpoint of $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is

$$P\left(\frac{a(\cos\alpha+\cos\beta)}{2},\frac{a(\sin\alpha+\sin\beta)}{2}\right).$$

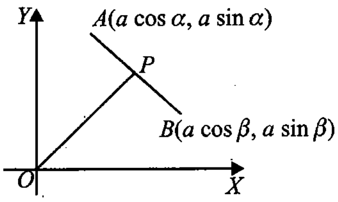


Fig. 1.134

Slope of line AB is

$$\frac{a\sin\beta - a\sin\alpha}{a\cos\beta - a\cos\alpha} = \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = m_1$$

and slope of OP is

$$\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = m_2$$

Now,

$$m_1 \times m_2 = \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\beta - \cos^2\alpha} = -1$$

Hence, the lines are perpendicular.

81. b. The set of lines is $4ax + 3by + c = 0$, where $a + b + c = 0$. Eliminating c, we get

$$4ax + 3by - (a + b) = 0$$

$$\Rightarrow a(4x - 1) + b(3y - 1) = 0$$

This passes through the intersection of the lines $4x - 1 = 0$ and $3y - 1 = 0$, i.e., $x = 1/4, y = 1/3$, i.e., $(1/4, 1/3)$.

82. a. The three lines are concurrent, if

$$\begin{vmatrix} 1 & 2 & -9 \\ 3 & 5 & -5 \\ a & b & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 35a - 22b + 1 = 0$$

which is true if the line $35x - 22y + 1 = 0$ passes through (a, b).

83. b. The line passing through (2, 3) and perpendicular to $-y + 3x + 4 = 0$ is

$$\frac{y-3}{x-2} = -\frac{1}{3}$$

or $\quad 3y + x - 11 = 0$

Therefore, foot is $\quad x = -1/10, y = 37/10$.

84. b. Let the coordinates of the third vertex be $(2a, t)$. Now, $AC = BC$. Hence,

$$t = \sqrt{4a^2 + (a - t)^2} \Rightarrow t = \frac{5a}{2}$$

So the coordinates of third vertex C are $(2a, 5a/2)$. Therefore, area of the triangle is

$$\pm\frac{1}{2}\begin{vmatrix} 2a & \frac{5a}{2} & 1 \\ 2a & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = \begin{vmatrix} a & \frac{5a}{2} & 1 \\ 0 & -\frac{5a}{2} & 0 \\ 0 & a & 1 \end{vmatrix} = \frac{5a^2}{2} \text{ sq. units}$$

85. d. If we reflect $y = |x - 2|$ in y-axis, it will become $y = |-x - 2| = |x + 2|$. The reflected lines are $y = x + 2$, $y = -x - 2$. Their combined equation is

$$(y - x - 2)(y + x + 2) = 0$$

$$\Rightarrow \quad y^2 - (x + 2)^2 = 0$$

$$\Rightarrow \quad y^2 - x^2 - 4x - 4 = 0$$

86. d. $a^3x^2 - 2hxy + b^3y^2 = 0$

Let the slope of lines be m_1 and m_2. Then,

$$m_1 + m_2 = \frac{2h}{b^3}, \; m_1m_2 = \frac{a^3}{b^3}$$

Given $\quad m_2^2 = m_1 \quad \Rightarrow m_2^3 = \frac{a^3}{b^3}$

$$\Rightarrow \quad m_2 = \frac{a}{b}$$

Also $\quad m_2^2 + m_2 = \frac{2h}{b^3}$

$$\Rightarrow \quad \frac{2h}{b^3} = \frac{a}{b} + \frac{a^2}{b^2}$$

$$\Rightarrow \quad ab + a^2 = \frac{2h}{b}$$

$$\Rightarrow \quad 2h = a^2b + ab^2 = ab(a + b)$$

87. d.

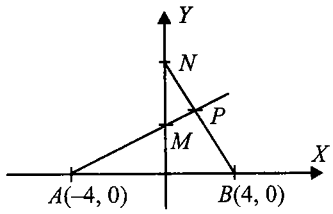


Fig. 1.135

Let $\quad M = (0, h)$

$\Rightarrow \quad N = (0, h + 4)$. Equation of AM is

$$\frac{x}{-4} + \frac{y}{h} = 1$$

$$\Rightarrow \quad \frac{y}{h} = \frac{4 + x}{4} \Rightarrow h = \frac{4y}{4 + x}$$

Equation of BN is

$$\frac{x}{4} + \frac{y}{h + 4} = 1$$

$$\Rightarrow \quad \frac{y}{h + 4} = \frac{4 - x}{4}$$

$$\Rightarrow \quad h + 4 = \frac{4y}{4 - x}$$

$$\Rightarrow \quad h = \frac{4y - 16 + 4x}{4 - x}$$

$$\Rightarrow \quad \frac{4(y - 4 + x)}{4 - x} = \frac{4y}{4 + x} \text{ (eliminating } h\text{)}$$

$\Rightarrow \quad 2xy - 16 + x^2 = 0$, which is a required locus.

88. b.

$$OM_r = OA_r + \frac{OA_{r+1} - OA_r}{2}$$

$$= \frac{OA_r + OA_{r+1}}{2}$$

$$= \frac{1}{2}\{OA_1 \times k^{r-1} + OA_1 \times k^r\}$$

$$= \frac{OA_1}{2}(1+k)\,k^{r-1}$$

$$\therefore \quad \sum_{r=1}^{\infty} OM_r = \frac{OA_1}{2}(1+k)\sum_{r=1}^{\infty} k^{r-1}$$

$$= \frac{OA_1}{2}(1+k) \times \frac{1}{1-k}$$

$$= \frac{OA_1}{2} \times \frac{1 + \frac{OA_2}{OA_1}}{1 - \frac{OA_2}{OA_1}}$$

$$= \frac{OA_1(OA_1 + OA_2)}{2(OA_1 - OA_2)}$$

89. c. A rough sketch of the lines is given below. There are three triangles namely *ABC*, *ABD*, *ACD*.

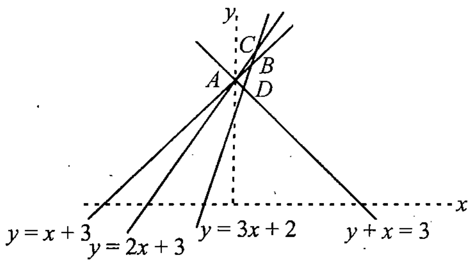


Fig. 1.136

90. c. We know that $PA + PB \geq AB$ (by triangle inequality). So, $PA + PB$ is the minimum if $PA + PB = AB$, i.e., *A*, *P*, *B* are collinear.

$$\therefore \quad \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 1 \\ 2k-1 & 2k+1 & 1 \end{vmatrix} = 0$$

or

$$3(2 - 2k - 1) + 4(1 - 2k + 1) + 1(2k + 1 - 4k + 2) = 0$$

or

$$3 - 6k + 8 - 8k + 3 - 2k = 0$$

or

$$14 - 16k = 0$$

$$\therefore \quad k = \frac{7}{8}$$

91. b.

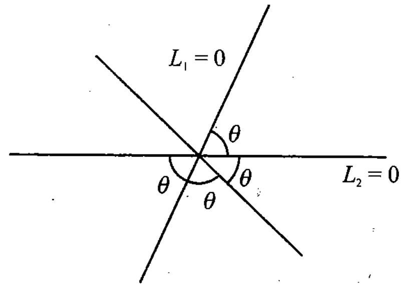


Fig. 1.137

From the figure,

$$3\theta = 180 \Rightarrow \theta = 60°$$

92. c.

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ a & b & -c \end{vmatrix} = 0$$

$$\Rightarrow \quad a + 5b - 3c = 0$$

$$\Rightarrow \quad -\frac{a}{3} - \frac{5}{3}b + c = 0$$

Hence, $2ax + 3by + c = 0$ is concurrent at $2x = -1/3$ and $3y = -5/3$. So, $x = -1/6$, $y = -5/9$.

93. a. Slope of is $AG = -b/(2a)$. Now,

$$\tan 30° = \frac{\frac{3b}{2a}}{1 + \frac{b^2}{a^2}}$$

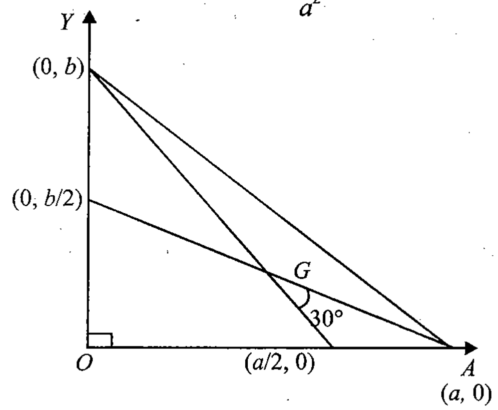


Fig. 1.138

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{3ba}{2(a^2 + b^2)}$$

$$\Rightarrow \quad \frac{1}{2}ab = \left(\frac{a^2 + b^2}{3\sqrt{3}}\right)$$

$$= 9/3\sqrt{3} = \sqrt{3} \text{ (Putting } a^2 + b^2 = 9\text{)}$$

94. c. *a*, 8, *b* are in *H.P*

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} = 1/4$$

$$\Rightarrow \quad b = \frac{4a}{a-4}$$

$\Rightarrow \quad \text{area, } A = \dfrac{4a^2}{2(a-4)}$

A is minimum at $a = 8$. Hence, minimum value of A is 32 sq. units.

95. a. We have,

$$3x + 5y = 2007 \Rightarrow x + \frac{5y}{3} = 669$$

Clearly, 3 must divide $5y$ and so $y = 3k$, for some $k \in N$. Thus,

$$x + 5k = 669$$

$$\Rightarrow \quad 5k \le 668$$

$$\Rightarrow \quad k \le \frac{668}{5} \Rightarrow k \le 133$$

96. d. Three non-collinear points form *a* triangle and the line joining the midpoints of any two sides is equidistant from all the three vertices.

97. a. The point (4, 5) lies on the given line $7x - 3y - 13 = 0$. The locus of the point equidistant from the given point and the line is a line perpendicular to $7x - 3y - 13 = 0$ at (4, 5).

98. b.

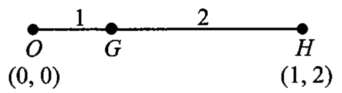


Fig. 1.139

$$G \equiv \left(\frac{1\times 1 + 2\times 0}{3}, \frac{2\times 1 + 2\times 0}{3}\right)$$

$$= \left(\frac{1}{3}, \frac{2}{3}\right)$$

99. c. Family of line through the given lines is

$L \equiv x - 7y + 5 + \lambda(x + 3y - 2) = 0$ (i)

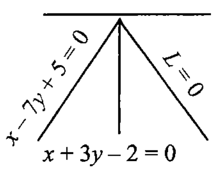


Fig. 1.140

For line $L = 0$ in the diagram, distance of any point say (2, 0) on the line $x + 3y - 2 = 0$ from the line $x - 7y + 5 = 0$ and the line $L = 0$ must be same.

$$\Rightarrow \quad \left|\frac{2+5}{\sqrt{50}}\right| = \left|\frac{2 + 2\lambda + 5 - 2\lambda}{\sqrt{(1+\lambda)^2 + (3\lambda - 7)^2}}\right|$$

$$\Rightarrow \quad 10\lambda^2 - 40\lambda = 0$$

$$\Rightarrow \quad \lambda = 4 \text{ or } 0$$

Hence, $\quad L = 0, \lambda = 4$

$\Rightarrow$ Required line is $5x + 5y - 3 = 0$

100. b. Given $AB = BC$.

$$\tan\theta = \frac{AB}{OA} = m_1$$

$$\tan\alpha = \frac{2\,AB}{OA} = m_2$$

$$\frac{m_2}{m_1} = 2 \Rightarrow \frac{m_2 + m_1}{m_2 - m_1} = \frac{2+1}{2-1} = 3$$

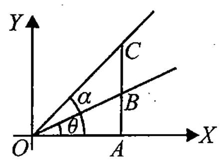


Fig. 1.141

$$\Rightarrow \quad \frac{m_1 + m_2}{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}} = 3$$

$$\Rightarrow \quad -\frac{\frac{2h}{b}}{\sqrt{\frac{4h}{b^2} - \frac{4a}{b}}} = 3$$

$$\Rightarrow \quad \frac{4h^2}{b^2} - \frac{4a}{b} = \frac{4h^2}{9b^2}$$

$$\Rightarrow \quad \frac{4h^2}{b^2} \times \frac{8}{9} = \frac{4a}{b}$$

$$\Rightarrow \quad 8h^2 = 9ab$$

101. d.

$$A = \frac{1}{2}\begin{vmatrix} 1 & p^2 & 1 \\ 0 & 1 & 1 \\ p & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[1(1-0) + p(p^2 - 1)]$$

$$= \frac{1}{2}(p^3 - p + 1)$$

Hence, $A = \frac{1}{2}|p^3 - p + 1|$. Now, minimum value of modulus is zero. Since $A(p)$ is cubic, it must vanish for some p other than given in (a), (b), (c).

102. b.

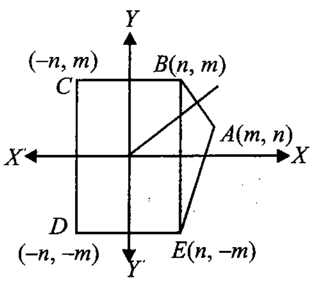


Fig. 1.142

Area of rectangle $BCDE$ is $4mn$. Area of ΔABE is $\frac{2m(m-n)}{2} = m^2 - mn$.

Therefore, the area of pentagon is $4mn + m^2 - mn = m^2 + 3mn$.

103. b.

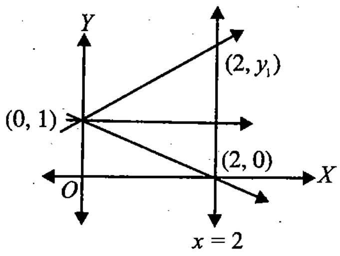


Fig. 1.143

From the figure,

$$2^2 + (y_1 - 1)^2 = y_1^2$$
$$4 + y_1^2 + 1 - 2y_1 = y_1^2$$
$$5 = 2y_1 \text{ or } y_1 = 5/2$$

Equation of the line from (2, 5/2) to the given base is

$$y - 5/2 = 2(x - 2)$$

or

$$2y - 5 = 4(x - 2)$$

at $\quad y = 1,\ -3/4 = x - 2 \quad$ or $\quad x = 5/4$.

104. b.

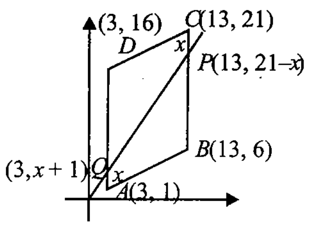


Fig. 1.144

$$m = \frac{21-x}{13} = \frac{x+1}{3}$$

$\Rightarrow \quad 63 - 3x = 13x + 13$

$\Rightarrow \quad 16x = 50$

$\Rightarrow \quad x = \frac{25}{8}$

Hence,

$$m = \left(\frac{25}{8} + 1\right) \times \frac{1}{3} = \frac{33}{24} = \frac{11}{8}$$

105. b.

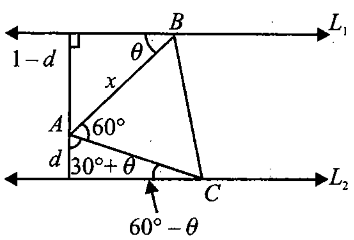


Fig. 1.145

From the figure,

$$x \cos(\theta + 30°) = d \qquad \text{(i)}$$
$$x \sin\theta = 1 - d \qquad \text{(ii)}$$

Dividing (i) by (ii), we have

$$\sqrt{3}\cot\theta = \frac{1+d}{1-d}$$

Squaring Eq. (ii) and putting the value of $\cot\theta$, we have

$$x^2 = \frac{1}{3}(4d^2 - 4d + 4)$$

$$\Rightarrow \qquad x = 2\sqrt{\frac{d^2 - d + 1}{3}}$$

106. d.

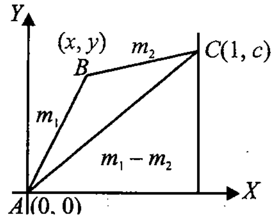


Fig. 1.146

Let the coordinates of C be $(1, c)$. Then,

$$m_2 = \frac{c-y}{1-x}$$

or

$$m_2 = \frac{c - m_1 x}{1-x}$$

$\Rightarrow \quad m_2 - m_2 x = c - m_1 x$

$\Rightarrow \quad (m_1 - m_2)x = c - m_2$

$\Rightarrow \quad c = (m_1 - m_2)x + m_2 \qquad \text{(i)}$

Now area of ΔABC is

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & m_1 x & 1 \\ 1 & c & 1 \end{vmatrix} = \frac{1}{2}[cx - m_1 x]$$
$$= \frac{1}{2}\left|[((m_1 - m_2)x + m_2)x - m_1 x]\right|$$
$$= \frac{1}{2}\left|[(m_1 - m_2)x^2 + m_2 x - m_1 x]\right|$$
$$= \frac{1}{2}(m_1 - m_2)(x - x^2)$$
$$[\because x > x^2 \text{ in } (0, 1)]$$

Hence,

$$f(x) = \frac{1}{2}(x - x^2)$$

$$f(x)_{\max} = \frac{1}{8} \text{ when } x = 1/2.$$

107. d. $R(x, y)$ lies on $9x + 7y + 4 = 0$

Hence, $R\left(a\ \frac{-(4+9a)}{7}\right), a \in R$.

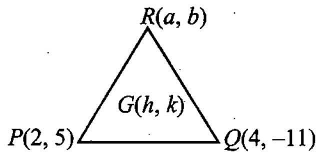


Fig. 1.147

$$h = \left(\frac{2+4+a}{3}\right) = \frac{6+a}{3} \qquad \text{(i)}$$

$$k = \frac{5 - 11 - \frac{(4+9a)}{7}}{3}$$

$$= \frac{-46-9a}{7\times 3} \qquad \text{(ii)}$$

From (i) and (ii), we get

$$3h - 6 = \frac{-(21k-46)}{9}$$

$\Rightarrow 27h + 21k - 54 + 46 = 0$

Hence, the locus is $9x + 7y - 8/3 = 0$. This line is parallel to N.

108. a. Equation of line AB is $y - 1 = m(x - 1)$

$\Rightarrow$ Equation of line AC is $y - 1 = -\frac{1}{m}(x - 1)$

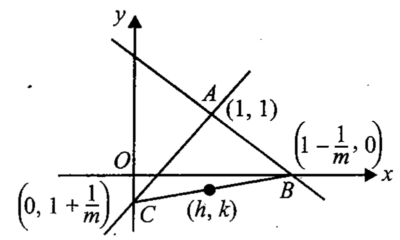


Fig. 1.148

$$2h = 1 - \frac{1}{m}$$

$$2k = 1 + \frac{1}{m}$$

Eliminating m we have locus $x + y = 1$.

109. c. Equation of any line through (2, 3) is

$$y - 3 = m(x - 2)$$

$$\Rightarrow \qquad y = mx - 2m + 3$$

From the figure, area of ΔOAB is $\pm$ 12. That is,

$$\frac{1}{2}\left(\frac{2m-3}{m}\right)(3 - 2m) = \pm 12$$

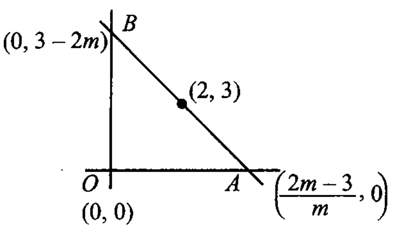


Fig. 1.149

Taking positive sign, we get $(2m + 3)^2 = 0$. This gives one value of m as $-3/2$. Taking negative sign, we get

$$4m^2 - 36m + 9 = 0 \; (D > 0)$$

This is a quadratic in m which gives two values of m. Hence, three straight lines are possible.

110. b. BD and BE intersect at B. Coordinates of B are (–3, –2).

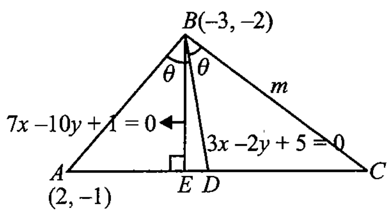


Fig. 1.150

$$m_{AB} = \frac{1}{5}$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{3}{2} - \frac{1}{5}}{1 + \frac{3}{10}}\right| = \left|\frac{\frac{3}{2} - m}{1 + \frac{3m}{2}}\right|$$

$$\Rightarrow \qquad 1 = \left|\frac{3-2m}{2+3m}\right|$$

$$\Rightarrow \qquad \pm 1 = \frac{3-2m}{2+3m}$$

$$\Rightarrow \qquad m = 1/5 \text{ (rejected) or } -5$$

Equation of BC is

$$y + 2 = -5(x + 3)$$

$$\Rightarrow \qquad 5x + y + 17 = 0$$

Alternative Solution:

Take image of (2, –1) in the line BD to get a point on BC.

111. a. Coordinates of circumcentre are $l/(l^2 - m^2)$, $m/(m^2 - l^2)$. Hence,

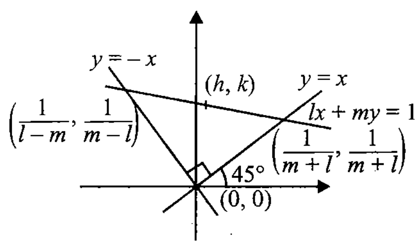


Fig. 1.151

$$h = \frac{l}{l^2 - m^2} \qquad \text{(i)}$$

$$k = -\frac{m}{l^2 - m^2} \qquad \text{(ii)}$$

Square and adding (i) and (ii), we get

$$h^2 + k^2 = \frac{l^2 + m^2}{(l^2 - m^2)^2} = \frac{1}{(l^2 - m^2)^2}$$

(putting $l^2 + m^2 = 1$)

$$\therefore \quad \frac{1}{(l^2 - m^2)^2} = (h^2 - k^2)^2$$

Therefore, the locus is $x^2 + y^2 = (x^2 - y^2)^2$.

112. d. $\frac{\Delta AQP}{\Delta AOB} = \frac{3}{8}$

or $$\frac{\frac{P^2\lambda}{(\lambda+1)^2}}{\frac{1}{2}p^2} = \frac{3}{8}$$

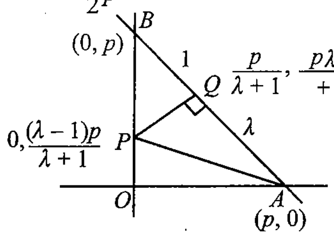


Fig. 1.152

$$\Rightarrow \quad \lambda = 3, \frac{1}{3}$$

$$\Rightarrow \quad \frac{AQ}{BQ} = 3 \text{ or } \frac{1}{3}$$

The value 1/3 is rejected because this gives negative coordinates of P and it is given that P lies on OB.

113. b.

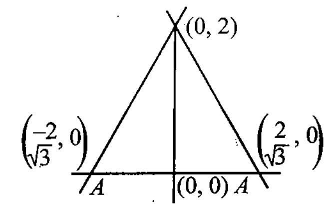


Fig. 1.153

From the figure

$$y + \sqrt{3}x = 2 \text{ for } x > 0$$
$$y - \sqrt{3}x = 2 \text{ for } x < 0$$

114. a. $2x + 3y = 6$

$$\tan 45° = \left| \frac{m - \left(\frac{-2}{3}\right)}{1 + m\left(\frac{-2}{3}\right)} \right|$$

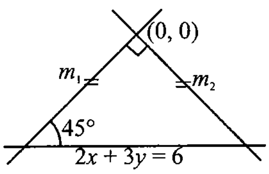


Fig. 1.154

Hence, $\quad m_1 = -5, m_2 = 1/5.$

115. d. $\tan\theta = 7$

$$OA = OB = r$$

$$\sin\theta = \frac{7}{5\sqrt{2}}, \cos\theta = \frac{1}{5\sqrt{2}}$$

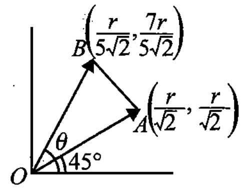


Fig. 1.155

Now, $\quad m_{AB} = -1/2.$

116. b. We can assume that OP and OR are x-axis and y-axis, respectively. Let $OP = a$. Then area of square $OPQR$ is a^2.

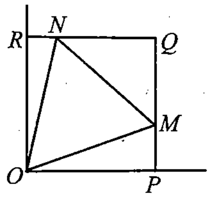


Fig. 1.156

Coordinates of M and N are $(a, a/2)$ and $(a/2, a)$, respectively.

$$\therefore \quad \text{ar}(\Delta OMN) = \frac{1}{2}\begin{vmatrix} a & a/2 \\ a/2 & a \end{vmatrix} = \frac{3a^2}{8}$$

$$\therefore \quad \frac{8}{3} = \frac{\lambda}{6}$$

$$\lambda = 16$$

117. d. Orthocentre of triangle BCH is the vertex $A(-1, 0)$.

118. a. Distance of (0, 0) from the line $2x + 3y - 6 = 0$ is $6/\sqrt{4+9} = 6/\sqrt{13}$. The area of the triangle is $(6/\sqrt{13})^2 = 36/13$.

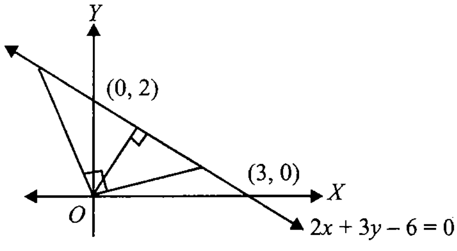


Fig. 1.157

119. d. The point Q is $(-b, -a)$ and the point R is $(-a, -b)$. Therefore, the midpoint of PR is (0, 0).

120. b. The point B is (2, 1). The image of $A(1, 2)$ in the line $x - 2y + 1 = 0$ is given by

$$\frac{x-1}{-1}=\frac{y-2}{-2}=\frac{4}{5}$$

Hence, coordinates of the point are (9/5, 2/5).

Since this point lies on BC, therefore the equation of BC is $3x - y - 5 = 0$. Hence, $a + b = 2$.

121. a.

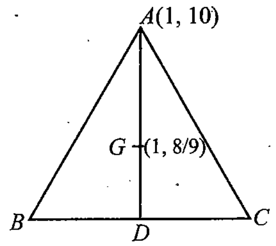


Fig. 1.158

Circumcentre $O \equiv (-1/3, 2/3)$ and orthocentre $H \equiv (11/3, 4/3)$.

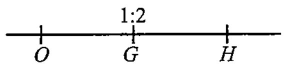


Fig. 1.159

Therefore, the coordinates of G are (1, 8/9). Now, the point A is (1, 10) as G is (1, 8/9). Hence,

$AD{:}DG = 3{:}\ 1$

$$\therefore \quad D_x = \frac{3-1}{2} = 1, D_y = \frac{\frac{8}{3}-10}{2} = -\frac{11}{3}$$

Hence, the coordinates of the midpoint of BC are $(1, -11/3)$.

122. b. (2, 0) is midpoint of $B(0, 0)$ and C, then C has coordinates (4, 0). Also, A has coordinates $(0 + 2\cos\pi/3, 0 + 2\sin\pi/3) \equiv (1, \sqrt{3})$. Then centroid is $(5/3, 1/\sqrt{3})$.

123. d.

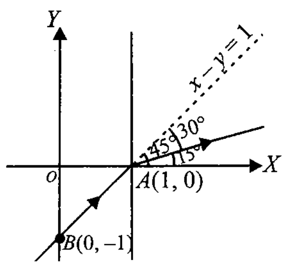


Fig. 1.160

From figure refracted ray makes an angle of 15° with positive direction of x-axis and passes through the point (1, 0). Its equation is

$$(y - 0) = \tan(45° - 30°)(x - 1)$$

or

$$y = (2 - \sqrt{3})(x - 1)$$

124. a. Let the line cut the axis at points $A(a, 0)$ and $B(0, b)$. Now given that $(-4, 3)$ divides AB in ratio 5:3. Then,

$-4 = 3a/8$ and $3 = 5b/8$. Therefore, $a = -32/3$ and $b = 24/5$. Then using the intercept form $x/a + y/b = 1$, the equation of line is

$$-\frac{3x}{32}+\frac{5y}{24}=1$$

or

$$9x - 20y + 96 = 0$$

125. d.

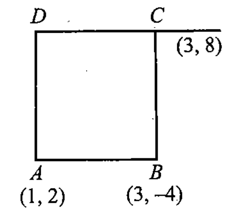


Fig. 1.161

$$m_{AB}=\frac{-4-2}{3-1}=-3$$

Thus equation of CD is $y - 8 = -3(x - 3)$, i.e., $y + 3x = 17$. Equation of right bisector of AB is

$$y+1=\frac{1}{3}(x-2)$$

$$\Rightarrow \quad 3y = x - 5$$

Solving it with line CD, we get

$x = 24/5, y = 1/5$. Thus midpoint of CD is (24/5, 1/5).

126. b. If the given lines represent the same line, then the lengths of the perpendiculars from the origin to the lines are equal, so that

$$\frac{c}{\sqrt{1+m^2}}=\frac{p}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$$

$$\Rightarrow \quad c = p\sqrt{1+m^2}$$

127. b.

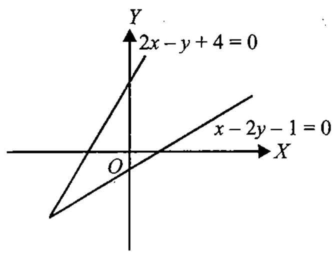


Fig. 1.162

Clearly from the figure, the origin is contained in the acute angle. Writing the equations of the lines as $2x - y + 4 = 0$ and $-x + 2y + 1 = 0$, the required bisector is

$$\frac{2x-y+4}{\sqrt{5}}=\frac{-x+2y+1}{\sqrt{5}}$$

128. c. We have to find locus of the point (h, k) whose image in the line $2x - y - 1 = 0$ lies on the line $y = x$. Now, image of (h, k) in the line $2x - y - 1 = 0$ is given by

$$\frac{x_2-h}{2}=\frac{y_2-k}{-1}=-\frac{2(2h-k-1)}{5}$$

$$\Rightarrow \quad x_2 = \frac{-3h + 4k + 4}{5}$$

and $$y_2 = \frac{4h + 3k - 2}{5}$$

This point lies on $y = x$. Then,

$$\frac{-3h + 4k + 4}{5} = \frac{4h + 3k - 2}{5} \Rightarrow 7h - k = 6$$

129. c.

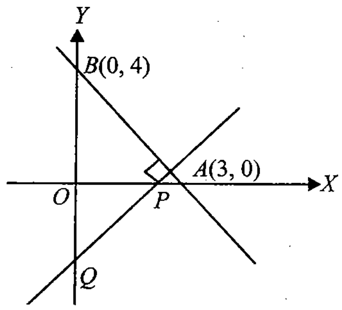


Fig. 1.163

Clearly, circumcentre of triangle ABQ will lie on the perpendicular bisector of line AB. Now equation of perpendicular bisector of line AB is $3x - 4y + 7/2 = 0$. Hence, locus of circumcentre is $6x - 8y + 7 = 0$.

130. d. Arranging the lines in descending order of slope, we have

$$m_1 = 5, m_2 = 3 \text{ and } m_3 = -1$$

$$\therefore \quad \tan A = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{2}{1 + 15} = \frac{1}{8}$$

$$\tan B = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{3 + 1}{1 - 3} = -2$$

$$\tan C = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{-1 - 5}{1 - 5} = \frac{3}{2}$$

$$\sum \tan^2 A = \frac{1}{64} + 4 + \frac{9}{4} = \frac{1 + 256 + 144}{64}$$

$$= \frac{401}{64}$$

$$\Rightarrow \quad p + q = 465$$

131. b. If the given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0 \text{ (Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1)$$

$$\Rightarrow a(b-1)(c-1) - (c-1)(1-a) - (b-1)(1-a) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

[Dividing by $(1 - a)(1 - b)(1 - c)$]

Adding 1 on both sides, we get

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

132. b. Let $y = mx$ be a line common to the given pairs of lines. Then

$$am^2 + 2m + 1 = 0 \text{ and } m^2 + 2m + a = 0$$

$$\Rightarrow \quad \frac{m^2}{2(1-a)} = \frac{m}{a^2 - 1} = \frac{1}{2(1-a)}$$

$$\Rightarrow \quad m^2 = 1 \text{ and } m = -\frac{a+1}{2}$$

$$\Rightarrow \quad (a + 1)^2 = 4$$

$$\Rightarrow \quad a = 1 \text{ or } -3$$

But for $a = 1$, the two pairs have both the lines common. So $a = -3$ and the slope m of the line common to both the pairs is 1. Now,

$$x^2 + 2xy + ay^2 = x^2 + 2xy - 3y^2 = (x - y)(x + 3y)$$

and

$$ax^2 + 2xy + y^2 = -3x^2 + 2xy + y^2 = -(x - y)(3x + y)$$

So the equation of the required lines is

$$(x + 3y)(3x + y) = 0$$

$$\Rightarrow \quad 3x^2 + 10xy + 3y^2 = 0$$

133. d. The given equation is

$$(a - b)(x^2 + y^2) - 2abx = 0 \quad \text{(i)}$$

The origin is shifted to $(ab/(a - b), 0)$. Any point (x, y) on the curve (i) must be replaced with new point (X, Y) with reference to new axes, such that

$$x = X + \frac{ab}{a-b} \text{ and } y = Y + 0$$

Substituting these in (i), we get

$$(a-b)\left[\left(X + \frac{ab}{a-b}\right)^2 + Y^2\right] - 2ab\left[X + \frac{ab}{a-b}\right] = 0$$

$$\Rightarrow (a - b)\left[X^2 + \frac{a^2b^2}{(a-b)^2} + Y^2 + \frac{2abX}{a-b}\right] - 2abX - \frac{2a^2b^2}{a-b} = 0$$

$$\Rightarrow \quad (a - b)(X^2 + Y^2) = \frac{a^2b^2}{a-b}$$

$$\Rightarrow \quad (a - b)^2 (X^2 + Y^2) = a^2b^2$$

134. b. The straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ are

$$y - mx = \pm a\sqrt{1 + m^2}$$

i.e.,

$$y - mx = a\sqrt{1 + m^2} \quad \text{(i)}$$

and

$$y - mx = -a\sqrt{1+m^2} \quad \text{(ii)}$$

Similarly, the straight lines represented by $(y - nx)^2 = a^2(1 + n^2)$ are

$$y - nx = a\sqrt{1+n^2} \quad \text{(iii)}$$

and

$$y - nx = -a\sqrt{1+n^2} \quad \text{(iv)}$$

Since the lines (i) and (ii) are parallel, so the distance between (i) and (ii) is

$$\left|\frac{a\sqrt{1+m^2} + a\sqrt{1+m^2}}{\sqrt{1+m^2}}\right| = |2a|$$

Similarly the lines (iii) and (iv) are parallel lines and the distance between them is $|2a|$. Since the distances between parallel lines are same, hence the four lines form a rhombus.

135. a. Let the common line be $y = mx$. Then it must satisfy both the equations. Therefore, we have

$$bm^2 + 2hm + a = 0 \quad \text{(i)}$$

$$b'm^2 + 2h'm + a' = 0 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\frac{m^2}{2(ha' - h'a)} = \frac{m}{ab' - a'b} = \frac{a}{2(bh' - b'h)}$$

Eliminating m, we get

$$\left[\frac{ab' - a'b}{2(bh' - b'h)}\right]^2 = \frac{ha' - h'a}{bh' - b'h}$$

$$\Rightarrow \quad (ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$$

136. b. Being a pair of lines, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. This gives $m = 4$. Now find angle between lines.

137. d. Here $my(y - mx) + x(y - mx) = 0$, i.e., $(y - mx) \times (my + x) = 0$. So, the lines are $y = mx$ and $y = (-1/m)x$. Bisectors between the lines $xy = 0$ are $y = x$ and $y = -x$. Therefore, $m = 1$ or -1.

138. a. $3a + a^2 - 2 = 0$

$\Rightarrow a^2 + 3a - 2 = 0;$

$$\Rightarrow a = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}$$

$\Rightarrow$ two values of a

139. d. Since the product of the slope of the four lines represented by the given equation is 1 and a pair of lines represents the bisectors of the angles between the other two, the product of the slopes of each pair is -1.

So let the equation of one pair be $ax^2 + 2hxy - ay^2 = 0$. Then the equation of its bisectors is

$$\frac{x^2 - y^2}{2a} = \frac{xy}{h}$$

By hypothesis,

$x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = (ax^2 + 2hxy + ay^2)(hx^2 - 2axy - hy^2)$

$= ah(x^4 + y^4) + 2(h^2 - a^2)(x^3y - xy^3) - 6ahx^2y^2$

140. b. The given equation of pair of straight lines can be rewritten as $(\sqrt{3}\,y - x + \sqrt{3})(\sqrt{3}\,y + x - \sqrt{3}) = 0$

Their separate equations are $\sqrt{3}y - x + \sqrt{3} = 0$ and $\sqrt{3}\,y + x - \sqrt{3} = 0$

or $y = \frac{1}{\sqrt{3}}x - 1$ and $y = -\frac{1}{\sqrt{3}}x + 1$

or $y = (\tan 30°)\,x - 1$ and $y = (\tan 150°)\,x + 1$

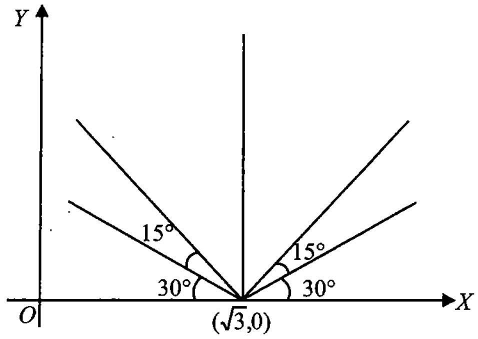


Fig. 1.164

After rotation through an angle of 15°, the lines are $(y - 0) = \tan 45°\,(x - \sqrt{3})$ and $(y - 0) = \tan 135°\,(x - \sqrt{3})$ or $y = x - \sqrt{3}$ and $y = -x + \sqrt{3}$

Their combined equation is

$(y - x + \sqrt{3})(y + x - \sqrt{3}) = 0$ or $y^2 - x^2 + 2\sqrt{3}x - 3 = 0$

141. c. Equations of the given lines are

$$y - 1 = \tan\theta\,(x - 1) \text{ and}$$

$$y - 1 = \cot\theta\,(x - 1)$$

So their joint equation is

$[(y - 1) - \tan\theta\,(x - 1)]\,[(y - 1) - \cot\theta\,(x - 1)] = 0$

$\Rightarrow (y - 1)^2 - (\tan\theta + \cot\theta)(x - 1)(y - 1) + (x - 1)^2 = 0$

$\Rightarrow x^2 - (\tan\theta + \cot\theta)\,xy + y^2 + (\tan\theta + \cot\theta - 2)(x + y - 1) = 0$

Comparing with the given equation, we get

$$\tan\theta + \cot\theta = a + 2$$

$$\Rightarrow \quad \frac{1}{\sin\theta\cos\theta} = a + 2$$

$$\Rightarrow \quad \sin 2\theta = \frac{2}{a+2}$$

142. c. We have, $6x^2 - xy - 12y^2 = 0$

$\Rightarrow (2x - 3y)(3x + 4y) = 0$ (i)

and

$$15x^2 + 14xy - 8y^2 = 0$$

$\Rightarrow (5x - 2y)(3x + 4y) = 0$ (ii)

Equation of the line common to (i) and (ii) is

$$3x + 4y = 0 \quad \text{(iii)}$$

Equation of any line parallel to (iii) is

$$3x + 4y = k$$

Since its distance from (iii) is 7, so

$$\left|\frac{k}{\sqrt{3^2+4^2}}\right| = 7 \Rightarrow k = \pm 35$$

143. c. We have,

$x^2y^2 - 9y^2 - 6x^2y + 54y = 0$

$\Rightarrow y^2(x^2-9) - 6y(x^2-9) = 0$

$\Rightarrow y(y-6)(x-3)(x+3) = 0$

$\Rightarrow \quad y = 0, y = 6, x = 3, x = -3$

So, the given equation represents four straight lines which form a square.

144. d. The combined equation of bisectors of angles between the lines of the first pair is

$$\frac{x^2-y^2}{2-1} = \frac{xy}{9}$$

As these equations are the same, the two pairs are equally inclined to each other.

145. a. Acute angle between the lines $x^2 + 4xy + y^2 = 0$ is $\tan^{-1}(2\sqrt{4-1})/(1+1) = \tan^{-1}\pi/3$. Angle bisectors of $x^2 + 4xy + y^2 = 0$ are given by

$$\frac{x^2-y^2}{1-1} = \frac{xy}{2}$$

$$x^2 - y^2 = 0 \Rightarrow x = \pm y$$

As $x + y = 0$ is perpendicular to $x - y = 4$, the given triangle is isosceles with vertical angle equal to $\pi/3$ and hence it is equilateral.

146. c. The given equations are

$a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ (i)

and

$ax^2 + 2hxy + by^2 = 0$ (ii)

The equation of the bisectors of the angles between the lines represented by (i) is

$$\frac{x^2-y^2}{a^2-b^2} = \frac{xy}{h(a+b)}$$

or

$$\frac{x^2-y^2}{a-b} = \frac{xy}{h}$$

which is same as equation of the bisectors of angles between the line pair (ii). Thus, two line pairs are equally inclined to each other.

147. a. $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$

$\Rightarrow (3x - 4y + 2)(3x - 4y - 6) = 0$

Hence, distance between lines is $\dfrac{|6-(-2)|}{5} = 8/5$.

Multiple Correct Answers Type

1. a., c.

Let $L = 3x - 4y - 8$. Then the value of L at (3, 4) is $3 \times 3 - 4 \times 4 - 8 = -15 < 0$. Hence, for the point $P(x, y)$, we should have $L > 0$

$\Rightarrow \quad 3x - 4y - 8 > 0$

$\Rightarrow \quad 3x - 4(-3x) - 8 > 0$

[$\because P(x, y)$ lies on $y = -3x$]

$\Rightarrow \quad x > 8/15$

and $\quad -y - 4y - 8 > 0$

$\Rightarrow \quad y < -8/5$

2. b., c., d. If the remaining vertex is (h, k), then

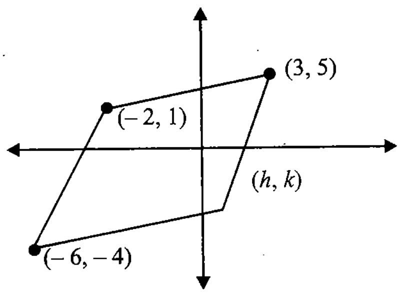


Fig. 1.165

$$h - 2 = -6 + 4, k + 1 = 5 - 4$$

$$\Rightarrow \quad h = -1, k = 0$$

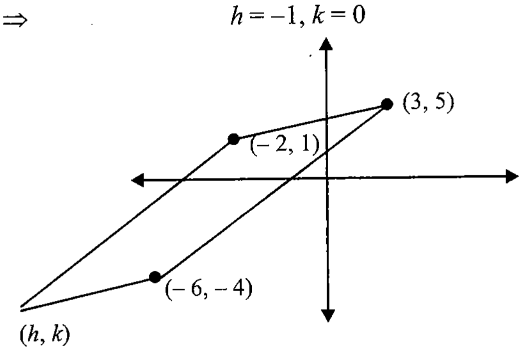
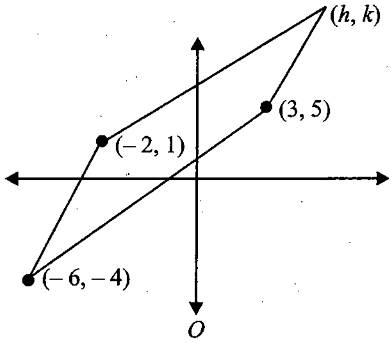


Fig. 1.166

$$h + 3 = -6 - 2, k + 5 = -4 + 1$$

$$\Rightarrow \quad h = -11, k = -8$$



$$h - 6 = 3 - 2,\ k - 4 = 5 + 1$$

$$\Rightarrow \quad h = 7,\ k = 9$$

Fig. 1.167

3. a., d. Distance between $x + 2y + 3 = 0$ and $x + 2y - 7 = 0$ is $10/\sqrt{5}$. Let the remaining side parallel to $2x - y - 4 = 0$ be $2x - y + \lambda = 0$. We have,

$$\frac{|\lambda + 4|}{\sqrt{5}} = \frac{10}{\sqrt{5}} \Rightarrow \lambda = 6, -14$$

Thus the remaining side is $2x - y + 6 = 0$ or $2x - y - 14 = 0$.

4. a., d. Let point P be (x, y)

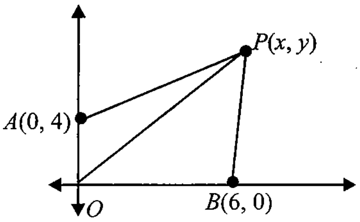


Fig. 1.168

$$\Delta_{POA} = \frac{1}{2}(OA)|x| = 2|x|$$

$$\Delta_{POB} = \frac{1}{2}(OB)|y| = 2|y|$$

where

$$P \equiv (x, y) \Rightarrow 2|x| = 6|y|$$

$$\Rightarrow \quad |x| = 3|y| \Rightarrow 3y - x = 0 \text{ or } 3y + x = 0$$

5. a., c., d.

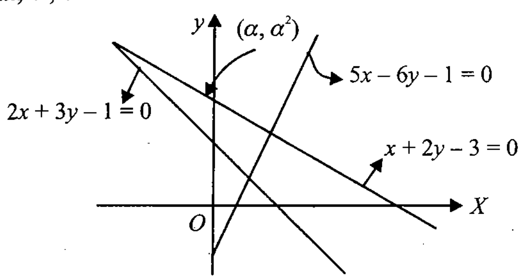


Fig. 1.169

O and the point (α, α^2) lie on the opposite sides w.r.t. $2x + 3y - 1 = 0$. Hence,

$$\Rightarrow \quad 2\alpha + 3\alpha^2 - 1 > 0 \quad \text{(i)}$$

O and the point (α, α^2) lie to the same side w.r.t. $x + 2y - 3 = 0$. Hence,

$$\Rightarrow \quad \alpha + 2\alpha^2 - 3 < 0 \quad \text{(ii)}$$

Again O and the point (α, α^2) lie to the same side w.r.t. $5x - 6y - 1 = 0$. Hence,

$$5\alpha - 6\alpha^2 - 1 < 0$$

$$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0$$

6. b., d. Let the angle be θ. Then, equation of the given line is

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} \quad \text{(i)}$$

The coordinates of a point on (i) at a distance $\sqrt{6}/3$ from (1, 2) are $(1 + \sqrt{6}/3 \cos\theta, 2 + \sqrt{6}/3 \sin\theta)$. This point lies on $x + y = 4$. Therefore,

$$1 + \frac{\sqrt{6}}{3}\cos\theta + 2 + \frac{\sqrt{6}}{3}\sin\theta = 4$$

$$\Rightarrow \quad \cos\theta + \sin\theta = \sqrt{\frac{3}{2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos(\theta - \pi/4) = \cos(\pm\pi/6)$$

$$\Rightarrow \quad \theta - \pi/4 = \pm\pi/6$$

$$\Rightarrow \quad \theta = 75^\circ \text{ or } \theta = 15^\circ$$

7. b., c. For the two lines $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$, the angle bisectors are given by

$$\frac{24x + 7y - 20}{25} = \pm\frac{4x - 3y - 2}{5}$$

Taking positive sign, we get

$$2x + 11y - 5 = 0$$

8. c., d. Let the third vertex be (x, y). Then,

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \pm 4$$

$$\Rightarrow \quad x + 5y + 4 = \pm 8$$

$$\Rightarrow \quad x + 5y + 12 = 0 \text{ or } x + 5y - 4 = 0$$

Hence, the third vertex lies on $x + 5y + 12 = 0$ or $x + 5y - 4 = 0$

9. a., b., c. The equation is

$$x^2(x + y) - y^2(x + y) = 0$$

or $\quad (x + y)^2 (x - y) = 0$

It represents the lines $x + y = 0$, $x + y = 0$, $x - y = 0$.

10. a., c. The equation represents a pair of straight lines. Hence,

$$1 \times (-2)(-1) + 2\left(\frac{3}{2}\right) \times 0 \times \frac{m}{2} - 1 \times \left(\frac{3}{2}\right)^2 - (-2) \times 0^2 - (-1) \times \left(\frac{m}{2}\right)^2 = 0$$

$\Rightarrow \quad m = 1, -1$

The points of intersection of the pair of lines are obtained by solving

$$\frac{\partial S}{\partial x} \equiv 2x + my = 0$$

and
$$\frac{\partial S}{\partial y} \equiv mx - 4y + 3 = 0$$

When $m = 1$, the required point is the intersection of $2x + y = 0$, $x - 4y + 3 = 0$. When $m = -1$, the required point is the intersection of $2x - y = 0$, $-x - 4y + 3 = 0$.

11. **a, c., d.** Vertices of the given triangle are (0, 0), $(a/m_1, a)$, and $(a/m_2, a)$. So the area of the triangle is equal to $a^2(m_2 - m_1)/(2m_1m_2)$. Since m_1, m_2 are the roots of $x^2 - ax - a - 1 = 0$, so

$$m_1 + m_2 = a;\ m_1 m_2 = -(a+1)$$

$$\Rightarrow \quad (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$
$$= a^2 + 4(a+1)$$
$$= (a+2)^2$$
$$\Rightarrow \quad m_1 - m_2 = \pm(a+2)$$

So the required area is

$$\Delta = \pm\frac{a^2(a+2)}{-2(a+1)} = \pm\frac{a^2(a+2)}{2(a+1)}$$

Since area is a positive quantity, so

$$\Delta = \frac{a^2(a+2)}{2(a+1)} \text{ if } a > -1 \text{ or } a < -2$$

and
$$\Delta = -\frac{a^2(a+2)}{2(a+1)} \text{ if } -2 < a < -1$$

12. **a., b., d.** Since the given point lies on the line $lx + my + n = 0$, so a, b, c are the roots of the equation

$$l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$$

or
$$lt^3 + mt^2 + nt - (3m+n) = 0 \quad \text{(i)}$$

Hence,
$$a + b + c = -\frac{m}{l}$$
$$ab + bc + ca = \frac{n}{l} \quad \text{(ii)}$$
$$abc = \frac{3m+n}{l} \quad \text{(iii)}$$

So, from Eqs. (i), (ii) and (iii), we get

$$abc - (bc + ca + ab) + 3(a + b + c) = 0$$

13. **a,d.** Here,

$$my(y - mx) + x(y - mx) = 0$$
$$\Rightarrow \quad (y - mx)(my + x) = 0$$

So, the lines are $y = mx$ or $y = (-1/m)x$. Bisectors between the lines $xy = 0$ are $y = x$ and $y = -x$. Therefore, $m = 1, -1$.

14. **b., c.** The chord subtends 90° at the centre (0, 0). Making $x^2 + y^2 = 1$ homogeneous in the second degree with the help of $y = mx + 1$, we get

$$x^2 + y^2 = (y - mx)^2$$
or $\quad (1 - m^2)x^2 + 2mxy = 0$

The angle between these lines is 90° if $1 - m^2 + 0 = 0$, i.e., $m = \pm 1$.

15. **a., b.**

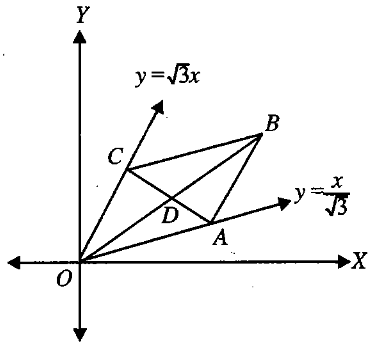


Fig. 1.170

Here, $\quad \angle COA = 30°$.

Area of rhombus $= 2 \times \frac{1}{2} \times OA \times OC \sin 30°$

$$\Rightarrow \quad 2 = \frac{1}{2}x^2$$
$$\Rightarrow \quad OA = OC = 2$$

Also, $\quad \angle OAB = 150°$

$$\Rightarrow \quad \cos 150° = \frac{OA^2 + AB^2 - OB^2}{2\, OA \times AB}$$

$$OB^2 = 8 + 4\sqrt{3} \Rightarrow OB = \sqrt{2}(\sqrt{3} + 1)$$

Hence, the coordinates of B are $(\pm\sqrt{2}(\sqrt{3}+1)\cos 45°, \pm\sqrt{2}(\sqrt{3}+1)\sin 45°)$.

16. **a., b., c., d.**

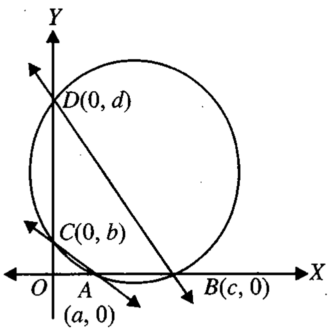


Fig. 1.171

If points A, B, C, D are concyclic, then $ac = bd$. The coordinates of the points of intersection of lines are

$$\left(\frac{ac(b-d)}{bc-ad}, \frac{bd(c-a)}{bc-ad}\right)$$

Let coordinates of the point of intersection be (h, k). Then

$$h = \frac{ac(b-d)}{bc-ad},\ k = \frac{bd(c-a)}{bc-ad}$$

Given $\quad c^2 + a^2 = b^2 + d^2$. Since $ac = bd$, so

$$(c-a)^2 = (b-d)^2$$

or $\quad (c-a) = \pm(b-d)$

Then the locus of the points of intersection is $y = \pm x$

17. a., d.

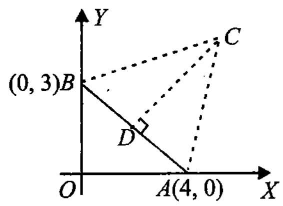


Fig. 1.172

$$AB = 5,\ D \equiv \left(2, \frac{3}{2}\right)$$

$CD = 5\times\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$, slope of AB is $-3/4$, slope of CD is $4/3$.

If $\quad C \equiv (h, k)$, then

$$\frac{h-2}{3/5} = \frac{k-3/2}{4/5} = \pm\frac{5\sqrt{3}}{2}$$

$$\Rightarrow \quad h = 2\left(1 - \frac{3\sqrt{3}}{4}\right),\ k = \frac{3}{2}\left(1 - \frac{4}{\sqrt{3}}\right)$$

or

$$h = 2\left(1 + \frac{3\sqrt{3}}{4}\right),\ k = \frac{3}{2}\left(1 + \frac{4}{\sqrt{3}}\right)$$

18. a., c. The equations of lines passing through (1, 0) are given by $y = m(x-1)$. Its distance from origin is $\sqrt{3}/2$. Hence,

$$\Rightarrow \quad \left|\frac{-m}{\sqrt{1+m^2}}\right| = \sqrt{3}/2 \Rightarrow m = \pm\sqrt{3}$$

Hence, the lines are $\sqrt{3}\,x + y - \sqrt{3} = 0$ and $\sqrt{3}\,x - y - \sqrt{3} = 0$.

19. b., c.

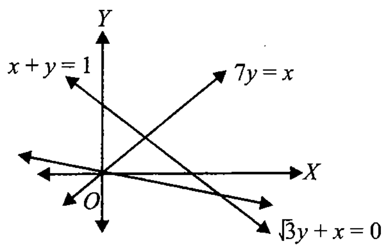


Fig. 1.173

Clearly from the figure above, the triangle is obtuse angled. Hence, centroid and incentre lie inside the triangle. Orthocentre and circumcentre lie outside the triangle.

Therefore, it is an obtuse angled triangle.

20. b., d.

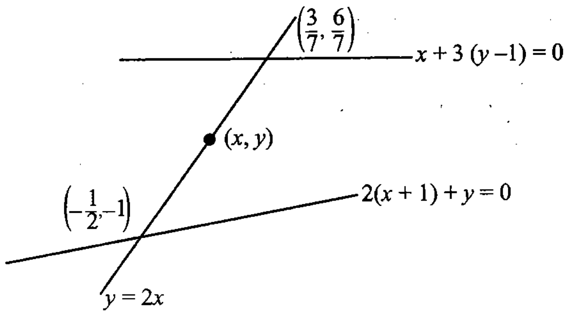


Fig. 1.174

Solving $y = 2x$, $2(x+1) + y = 0$, we get $x = -1/2$, $y = -1$. Solving $y = 2x$, $x + 3(y-1) = 0$, we get $x = 3/7$, $y = 6/7$.

21. a., b.

$$x^2 - 3|x| + 2 = 0$$

$$\Rightarrow \quad (|x|-1)(|x|-2) = 0 \Rightarrow x = \pm1, \pm2$$

$$y^2 - 3x + 2 = 0$$

$$\Rightarrow \quad (y-1)(y-2) = 0 \Rightarrow y = 1, 2$$

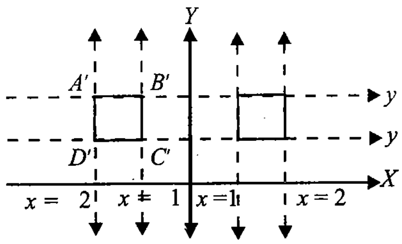


Fig. 1.175

From the figure two such squares are possible whose coordinates are $A(1, 2)$, $B(2, 2)$, $C(2, 1)$, $D(1, 1)$ and $A'(-2, 2)$, $B'(-1, 2)$, $C'(-1, 1)$, $D'(-2, 1)$.

22. b., d. Let any point on the line $x - y = 2$ be $C(h, h-2)$. Given area of ΔABC is

$$\left|\frac{1}{2}\begin{vmatrix} h & h-2 & 1 \\ -5 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix}\right| = 20$$

$$\Rightarrow \quad |8(h-2)| = 40$$

$$\Rightarrow \quad h - 2 = \pm 5$$

$$\Rightarrow \quad h = 7, -3$$

Hence, the points are (7, 5) and (–3, –5).

23. a., b. The area of the triangle is given by

$$= \frac{1}{2} \times \frac{2b}{a} \times \frac{2b}{c} = \frac{2b^2}{ac} = 2$$

$$\Rightarrow \quad b^2 = ac$$

$\Rightarrow$ a, b, c are in G.P. So, $a, -b, c$ are in G.P.

24. a., b. Verteces $(a \cos \theta_1, a \sin \theta_1)$, $(a \cos \theta_2, a \sin \theta_2)$ and $(a \cos \theta_3, a \sin \theta_3)$ are equidistant from origin (0, 0). Hence, the origin is circumcentre (centroid) of circumcircle. Therefore, the coordinates of centroid are

$$\left(\frac{a(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)}{3}, \frac{a(\sin \theta_1 + \sin \theta_2 + \sin \theta_3)}{3}\right)$$

But as the centroid is the origin (0, 0), therefore $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$ and $\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$.

25. b., c. Draw the diagram and verify.

26. a., c. Since points $AB = AC = 1$, Δ is right angled at point A. We have,

$$\tan \alpha \tan \beta = -1$$

$$\Rightarrow \quad \cos(\alpha - \beta) = 0 \Rightarrow \alpha - \beta = \frac{\pi}{2}$$

27. a., b.

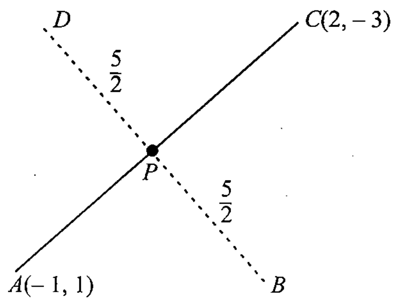


Fig. 1.176

$$AC = \sqrt{3^2 + 4^2} = 5$$

The midpoint P of $AC = \left(\frac{1}{2}, -1\right)$.

Slope of AC is $\frac{-4}{3}$.

Therefore, slope of BD is $\frac{3}{4} = \tan \theta$

Therefore, coordinates of B and D are

$$\equiv \left(1/2 \pm \frac{5}{2} \cos \theta, -1 \pm \frac{5}{2} \sin \theta\right)$$

28. a., c.

29. a., d. Equation of line passing though two given points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Now given expression is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 + x_3 & y_2 + y_3 & 1 \end{vmatrix} = 0$$

or
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2 + x_3}{2} & \frac{y_2 + y_3}{2} & 1 \end{vmatrix} = 0$$

This is the equation of the line passing through the points (x_1, y_1) and $((x_2 + x_3)/2, (y_2 + y_3)/2)$. This is a equation of median through vertex A.

30. a., b., c., d. The point $A(\alpha, \beta)$ lies on $y = 2x + 3$. Hence,

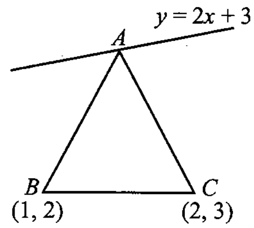


Fig. 1.177

$$\beta = 2\alpha + 3$$

$$A \equiv (\alpha, 2\alpha + 3)$$

Area of ΔABC is

$$\left|\frac{1}{2}\begin{vmatrix} \alpha & 2\alpha + 3 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}[\alpha(2 - 3) + (2\alpha + 3)(2 - 1) + 1(3 - 4)]\right|$$

$$= \frac{1}{2}|-\alpha + 2\alpha + 3 - 1| = \frac{1}{2}|\alpha + 2| = S$$

$$[S] = 2 \Rightarrow 2 \le S < 3$$

$$\therefore \quad 2 \le \frac{1}{2}|\alpha + 2| < 3$$

$$4 \le |\alpha + 2| < 6$$

$$|\alpha + 2| < 6 \Rightarrow -6 < \alpha + 2 < 6$$

$$\Rightarrow \quad -8 < \alpha < 4 \quad \text{(i)}$$

and

$$|\alpha + 2| \ge 4 \Rightarrow \alpha + 2 \ge 4 \text{ or } \alpha + 2 \le -4$$

$$\Rightarrow \quad \alpha \ge 2 \text{ or } \alpha \le -6 \quad \text{(ii)}$$

From Eqs. (i) and (ii),

$$-8 < \alpha \le -6 \text{ or } 2 \le \alpha < 4$$

$$\Rightarrow \quad \alpha = -7, -6, 2, 3$$

Possible coordinates of A are (–7, –11), (–6, –9), (2, 7), (3, 9).

31. a., c.

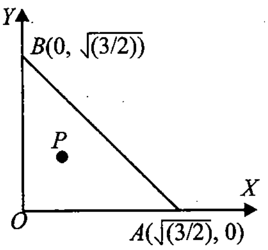


Fig. 1.178

Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{3/2}$, respectively.

Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$. Similarly, P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{3/2}$ if $\sin\theta + \cos\theta < \sqrt{3/2}$. Hence, P will lie inside the ΔABC if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{3/2}$. Now,

$$\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$$

$$\Rightarrow \quad \sin(\theta + \pi/4) < \sqrt{\frac{3}{4}}$$

Since $\sin\theta > 0$ and $\cos\theta > 0$, so $0 < \theta < \pi/12$ or $5\pi/12 < \theta < \pi/2$.

32. b., d.

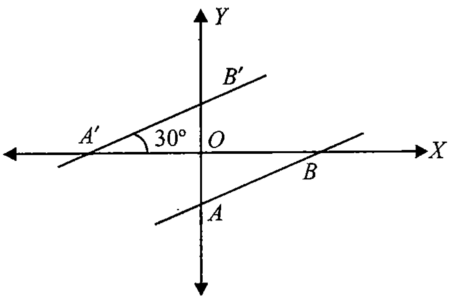


Fig. 1.179

According to question $AB = 10$. So, $OA = 10 \sin 30° = 5$. Then equation of line is

$$y = \frac{1}{\sqrt{3}}x \pm 5$$

or

$$x - \sqrt{3}\,y \pm 5\sqrt{3} = 0$$

33. c., d. If lines $x + y - 1 = 0$, $(m - 1)x + (m^2 - 7)y - 5 = 0$ and $(m - 2)x + (2m - 5)y = 0$ are concurrent, then

$$\Delta = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -1 \\ m-1 & m^2-7 & -5 \\ m-2 & 2m-5 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (m - 2)(-5 + m^2 - 7) - (2m - 5)(-5 + m - 1) + 0 = 0$$

$$\Rightarrow (m - 2)(m^2 - 12) - (2m - 5)(m - 6) = 0$$

$$\Rightarrow m^3 - 4m^2 + 5m - 6 = 0$$

$$\Rightarrow (m - 3)(m^2 - m + 2) = 0$$

$\Rightarrow m = 3$ but $m^2 - m + 2 = 0$ has no real roots. If $m = 3$, then two lines are parallel.

34. b., c. Let slope of line is m. Then

$$\frac{1}{2} = \left|\frac{m - (-2)}{1 + (-2)m}\right|$$

$$\Rightarrow \quad m = -3/4 \text{ and } \infty$$

Hence, equation of line is $y - 3 = -3/4\,(x - 2)$ and $x = 2$.

35. a., b. Let p be the length of the perpendicular from the origin on the given line. Then its equation in normal form is

$$x \cos 30° + y \sin 30° = p$$

or $\quad \sqrt{3}\,x + y = 2p$

This meets the coordinate axes at $A\left(\frac{2p}{\sqrt{3}}, 0\right)$ and $B(0, 2p)$. Therefore, area of ΔAOB is

$$\frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)2p = \frac{2p^2}{\sqrt{3}}$$

By hypothesis, $\quad \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p = \pm 5$

Hence, the lines are $\sqrt{3}x + y \pm 10 = 0$.

36. a, c. It is clear that diagonals of the rhombus will be parallel to the bisectors of the given lines and will pass through (1, 3). Equations of bisectors of the given lines are

$$\frac{x + y - 1}{\sqrt{2}} = \pm\left(\frac{7x - y - 5}{5\sqrt{2}}\right)$$

or $\quad 2x - 6y = 0,\ 6x + 2y = 5$

Therefore, the equations of diagonals are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$. Thus the required vertex will be the point where these lines meet the line $y = 2x$. Solving these lines we get possible coordinates as (8/5, 16/5) and (6/5, 12/5).

37. a., b., c., d.

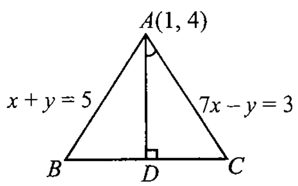


Fig. 1.180

Side BC will be perpendicular to the bisector of the angle BAC. Now equations of the bisectors of lines AB and AC are

$$\frac{(x + y - 5)}{\sqrt{2}} = \pm\frac{(7x - y - 3)}{5\sqrt{2}}$$

$$\Rightarrow \quad x - 3y + 11 = 0 \text{ or } 3x + y - 7 = 0$$

Let equation of side BC be $x - 3y + \lambda = 0$ and altitude through vertex A be AD. Then equation of AD is $3x + y - 7 = 0$. If $AD = \lambda$, then $\Delta ABC = \frac{1}{2} \times |BC| = \frac{1}{2} \times \lambda \times 2\lambda$ $|\tan\theta| = l^2|\tan\theta|$. Hence,

$$\lambda^2 \times \frac{1}{2} = 5 \Rightarrow \lambda^2 = 10$$

$\Rightarrow$ $$(11 - \lambda)^2 = 100$$

$\Rightarrow$ $$11 - \lambda = \pm 10 \Rightarrow \lambda = 1, -21$$

Hence, equation of BC is $x - 3y + 1 = 0$ or $x - 3y - 21 = 0$. Similarly, if equation of BC is $3x + y + \lambda = 0$, then equation of AD will be $x - 3y + 11 = 0$. Therefore,

$$|\tan\theta| = \left|\frac{7 - \frac{1}{3}}{1 + \frac{7}{3}}\right| = 2$$

$\Rightarrow$ $$\lambda^2 |\tan\theta| = 2\lambda^2 = 5 \Rightarrow \lambda^2 = \frac{5}{2}$$

Also,

$$\frac{5}{2} = \frac{(3 + 4 + \lambda)^2}{10}$$

$\Rightarrow$ $$7 + \lambda = \pm 5 \Rightarrow \lambda = 2, -12$$

Hence, equation of BC is $3x + y + 2 = 0$ or $3x + y - 12 = 0$. Finally, there are four possible equations of side BC, viz., $x - 3y + 1 = 0$, $x - 3y - 21 = 0$, $3x + y + 2 = 0$ or $3x + y - 12 = 0$.

38. a., b., c., d.

Let the slope of $u = 0$ be m. Then slope of $v = 0$ is $\frac{9m}{2}$.

Therefore, $$\frac{7}{9} = \left|\frac{m - \frac{9m}{2}}{1 + m \times \frac{9m}{2}}\right| = \left|\frac{-7m}{2 + 9m^2}\right|$$

$\Rightarrow$ $$9m^2 - 9m + 2 = 0 \text{ or } 9m^2 + 9m + 2 = 0$$

$$m = \frac{9 \pm \sqrt{81 - 72}}{18} = \frac{9 \pm 3}{18} = \frac{2}{3}, \frac{1}{3}$$

or $$m = \frac{-9 \pm 3}{18} = -\frac{2}{3}, -\frac{1}{3}$$

Therefore, equations of the lines are

i. $3y = x$ and $2y = 3x$

ii. $3y = 2x$ and $y = 3x$

iii. $x + 3y = 0$ and $3x + 2y = 0$

iv. $2x + 3y = 0$ and $3x + y = 0$

39. a, d. Note that the lines are perpendicular. Assume the coordinate axes to be directed along $u = 0$ and $v = 0$. Now the lines $k_1 u - k_2 v = 0$ and $k_1 u + k_2 v = 0$ are equally inclined with uv axes. Hence, the bisectors are $u = 0$ and $v = 0$.

40. a., b., d. Equation of the lines given by $x^2 + 2hxy + y^2 = 0$ be $y = m_1 x$ and $y = m_2 x$. Since these make an angle α with $y + x = 0$ whose slope is -1, so

$$\frac{m_1 + 1}{1 - m_1} = \tan\alpha = \frac{-1 - m_2}{1 - m_2}$$

$\Rightarrow$ $$m_1 + m_2 = \frac{(\tan\alpha - 1)^2 + (\tan\alpha + 1)^2}{\tan^2\alpha - 1}$$

$$= \frac{-2\sec^2\alpha \times \cos^2\alpha}{\cos 2\alpha}$$

$\therefore$ $$-2\sec 2\alpha = -2h$$

$\Rightarrow$ $$\sec 2\alpha = h$$

$\Rightarrow$ $$\cos 2\alpha = \frac{1}{h} \Rightarrow 2\cos^2\alpha - 1 = \frac{1}{h}$$

$\Rightarrow$ $$\cos\alpha = \sqrt{\frac{1 + h}{2h}} \text{ and } \cot\alpha = \sqrt{\frac{h + 1}{h - 1}}$$

41. a., d.

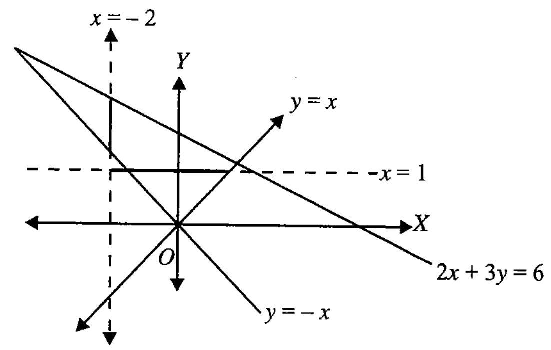


Fig. 1.181

The separate equations of the sides are $x - y = 0$, $x + y = 0$ and $2x + 3y - 6 = 0$. The point $(-2, a)$ moves on the line $x = -2$ and $(b, 1)$ moves on the line $y = 1$. From the figure, the y-coordinates of points of intersection of $x = -2$ with $y = -x$ and $2x + 3y = 6$ give the range of values of a. The x-coordinates of the points of intersection of $y = 1$ with $y = -x$ and $y = x$ give the range of values of b.

42. a., b., c., d. Let,

$$\frac{2}{3}x^2 + \frac{p}{3}xy + y^2 = (y - mx)(y - m'x)$$

and

$$\frac{2}{-3}x^2 + \frac{q}{-3}xy + y^2 = \left(y + \frac{1}{m}x\right)(y - m'x)$$

Then,

$$m + m' = -\frac{p}{3}, mm' = \frac{2}{3} \quad \text{(i)}$$

$$\frac{1}{m} - m' = \frac{-q}{3}, -\frac{m'}{m} = -\frac{2}{3} \quad \text{(ii)}$$

$\Rightarrow$ $$m^2 = 1 \Rightarrow m = \pm 1$$

If $m = 1$, $m' = 2/3$ and so $p = -5$, $q = -1$. If $m = -1$, $m' = -2/3$ and so $p = 5$, $q = 1$.

43. a., b., c., d. Equation of the lines joining the origin to the points of intersection of the given lines is

$3x^2 + mxy - 4x(2x + y) + 1(2x + y)^2 = 0$

(by homogenization)

$\Rightarrow$ $$x^2 - mxy - y^2 = 0$$

which are perpendiculars for all values of m.

Reasoning Type

1. **a.** Statement 1 is true and follows from statement 2 as the family of lines can be written as $a(x + y - 1) + b(x - 2y) = 0$.
2. **c.** Centroid $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ is a rational point. Orthocentre is intersection point of two altitudes which will bear rational coefficients when expressed as a straight line. So, orthocentre is also rational. Circumcentre is intersection point of two perpendicular bisectors which will bear rational coefficient when expressed as a straight line. So, circumcentre is also rational. But statement 2 is not true as in equilateral triangle all the centres coincide.
3. **d.** We know that if sum of algebraic distances from three points on the variable line is zero, then the line always passes through the mean of the given point, which is centroid of triangle formed by given three points. But centroid of triangle is (1, 2). Hence, the line must pass through it, for which $a + 2b + c = 0$. Therefore, statement 1 is false and statement 2 is true.
4. **a.** Bisectors of the given lines are $(3x + 4y - 12)/5 = \pm (4x + 3y - 24)/5$, of which one the bisectors is $y - x + 12 = 0$. Also any point on the bisector is always equidistant from the given lines.
5. **d.** According to given data $2a - 5 + a^2 - 3 < 0$ or $a^2 + 2a - 8 < 0$ or $(a - 2)(a + 4) < 0$ or $a \in (-4, 2)$.
6. **a.** Any line equally inclined to given lines is always parallel to angle bisectors.
7. **a.** Area of triangles is unaltered by shifting origin to any point. If origin is shifted to (1000, 1002), A, B, C become P (0, 0) Q (1, 2), R (2, 1). Both are true.
8. **a.** The quadrilateral is obviously a parallelogram and if the diagonals are at right angles, it must be a rhombus. Hence, the distance between the pairs of opposite sides must be the same, i.e.,

$$\frac{|r - r'|}{\sqrt{p^2 + q^2}} = \frac{|r - r'|}{\sqrt{p'^2 + q'^2}}$$

$$\Rightarrow \quad p^2 + q^2 = p'^2 + q'^2$$

9. **d.** The joint equation of $y = x$ and $y = -x$ is $(x - y)(x + y) = 0$, i.e., $x^2 - y^2 = 0$.
10. **b.** Bisectors of angle C are

$$\frac{3x + 2y}{\sqrt{13}} = \pm \frac{2x + 3y + 6}{\sqrt{13}}$$

or

$$x - y - 6 = 0 \text{ and } 5x + 5y + 6 = 0$$

According to given equations of sides, internal angle bisector at C will have negative slope. Also, image of A will lie on BC respect to both bisectors, from which we can conclude that $5x + 5y + 6 = 0$ is internal angle bisector. Hence, statement 2 is not correct explanation of statement 1.

11. **a.** From the figure, both the statements are true and statement 2 correctly explains statement 1.

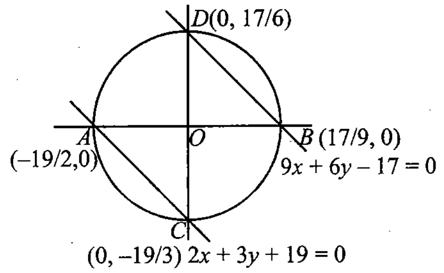


Fig. 1.182

12. **a.** We have,

$$(m_{AC})(m_{BC}) = \left(\frac{-4+2}{5+5}\right)\left(\frac{-4-6}{5-7}\right) = -1$$

Therefore, ABC is right-angled triangle with C as the right angle. Hence, circumcentre is midpoint of AB, i.e., (1, 2).

13. **c.** Statement 2 is false as point satisfying such property can be ex-centre of the triangle. However, statement 1 is true as (0, 0) is at distance π from all the lines and it lies inside the triangle.
14. **d.** Statement 1 is false since $(x - 2) + (2x - 4) + (6 - 3x) = 0$ but the lines $x - 2 = 0$, $2x - 4 = 0$ and $6 - 3x = 0$ are parallel. Statement 2 is a standard result whose more general form as follows. Let $L_1 = 0, L_2 = 0, L_3 = 0$ be three lines. Now, if we can find λ, μ, ν (not all zero) such that $\lambda L_1 + \mu L_2 + \nu L_3 = 0$, then the three lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are either concurrent or parallel.
15. **a.** The given lines are

$$(a + b)x + (a - b)y - 2ab = 0 \quad \text{(i)}$$

$$(a - b)x + (a + b)y - 2ab = 0 \quad \text{(ii)}$$

$$x + y = 0 \quad \text{(iii)}$$

The triangle formed by the lines (i), (ii) and (iii) is an isosceles triangle if the internal bisector of the vertical angle is perpendicular to the third side. Now equations of bisectors of the angle between lines (i) and (ii) are

$$\frac{(a + b)x + (a - b)y - 2ab}{\sqrt{[(a + b)^2 + (a - b)^2]}} = \pm \frac{(a - b)x + (a + b)y - 2ab}{\sqrt{[(a - b)^2 + (a + b)^2]}}$$

or

$$x - y = 0 \quad \text{(iv)}$$

and

$$x + y = 2b \quad \text{(v)}$$

Obviously the bisector (iv) is perpendicular to the third side of the triangle. Hence, the given lines form an isosceles triangle.

16. b. Put $2h = -(a + b)$ in $ax^2 + 2hxy + by^2 = 0$. Then,

$$ax^2 - (a + b)xy + by^2 = 0$$

$$\Rightarrow \quad (x - y)(ax - by) = 0$$

Therefore, one of the lines bisects the angle between coordinates axes in positive quadrant. Also putting $b = -2h - 1$ in $ax - by$, we have $ax - by = ax - (-2h - a)y = ax + (2h + a)y$. Hence, $ax + (2h + a)$ is a factor of $ax^2 + (2h + a)$. However, statement 2 is not correct explanation of statement 1.

Linked Comprehension Type

For Problems 1–3

1. a., 2. c., 3. d.

Sol. 1. a.

Given lines $(x + y + 1) + b(2x - 3y - 8) = 0$ are concurrent at point of intersection of the lines $x + y + 1 = 0$ and $2x - 3y - 8 = 0$, which is $(1, -2)$. Now line through $A(1, -2)$ which is farthest from the point $B(2, 2)$, is perpendicular to AB. Now, slope of AB is 4. Then the required line is $y + 2 = -(1/4)(x - 1)$ or $x + 4y + 7 = 0$.

2. c.

Also this line $x + 4y + 7 = 0$ meets axis at $C(-7, 0)$ and $D(0, -7/4)$. Then area of triangle OCD is $1/2\,|(-7)(-7/4)|$ or 49/8.

3. d. Lines $x + 4y + 7 = 0$ and $x - 2y + 1 = 0$ intersect at $(-3, -1)$ which must satisfy the line $3x - 4y + \lambda =$. Then $-9 + 4 + \lambda = 0$ or $\lambda = 5$.

For Problems 4–6

4. b., 5. d., 6. a.

Sol. 4. b.

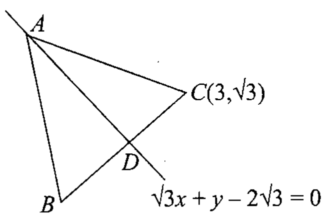


Fig. 1.183

Let the triangle be ABC with $C \equiv (3, \sqrt{3})$ and altitude drawn through vertex (meeting BC at D) be $\sqrt{3}x + y - 2\sqrt{3} = 0$. If B is (x_b, y_b), then we have

$$\frac{2(x_b - 3)}{\sqrt{3}} = \frac{y_b - \sqrt{3}}{2} = -\frac{2(3\sqrt{3} + \sqrt{3} - 2\sqrt{3})}{2} = -2\sqrt{3}$$

$$\Rightarrow \quad x_b = 0, y_b = 0$$

and coordinates of D is $(3/2, \sqrt{3}/2)$. Let coordinates of vertex A be (x_a, y_a). Then,

$$\frac{x_a - \frac{3}{2}}{-1/2} = \frac{y_a - \sqrt{3}/2}{\sqrt{3}/2} = \pm 3$$

$$\Rightarrow \quad (x_a, y_a) \equiv (0, 2\sqrt{3}) \text{ or } (3, -\sqrt{3})$$

Hence, the remaining vertices are $(0, 0)$ and $(0, 2\sqrt{3})$ or $(0, 0)$ and $(3, -\sqrt{3})$. Also, the orthocentre is $(1, \sqrt{3})$ or $(2, 0)$.

For Problems 7–9

7. d., 8. b., 9. d.

Sol.

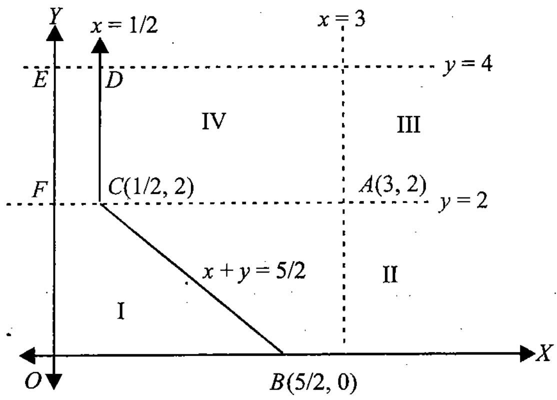


Fig. 1.184

Let $P = (h, k)$ be a general point in the first quadrant such that

$$d(P, A) = d(P, O)$$

$$|h - 3| + |k - 2| = |h| + |k| = h + k \qquad \text{(i)}$$

[h and k are +ve, piont $P(h, k)$ being in first quadrant]

If $h < 3, k < 2$, then (h, k) lies in region I. Then,

$$3 - h + 2 - k = h + k \Rightarrow h + k = 5/2$$

If $h > 3, k < 2$, then (h, k) lies in region II. Then,

$$h - 3 + 2 - k = h + k$$

$\Rightarrow \quad k = -1/2$ (not possible)

If $h > 3$, $k > 2$ then (h, k) lies in region III. Then,

$$h - 3 + k - 2 = h + k \Rightarrow -5 = 0 \text{ (not possible)}$$

If $h < 3$, $k > 2$, then (h, k) lies in region IV. Then,

$$3 - h + k - 2 = h + k \Rightarrow h = 1/2$$

Hence, the required set consists of line segment $x + y = 5/2$ of finite length as shown in the first region and the ray

$$x = 1/2 \text{ in the fourth region.}$$

7. **d.** Obviously locus of P is union of line segment and one infinite ray.

8. **b.** Area of region $OBCDEFO$

A = area of trapezium $OBCF$ + area of rectangle $FCDE$

$$= \frac{1}{2} \times \left(\frac{5}{2} + \frac{1}{2}\right) \times 2 + \frac{1}{2} \times 2$$

$$= 4$$

9. **d.** Obviously locus of P is a relation but not a function.

For Problems 10–12

10. d., 11. c., 12. a.

Sol. 10. d.

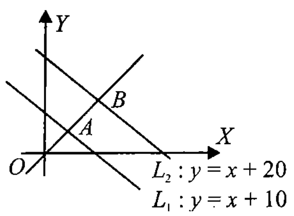


Fig. 1.185

Let the parametric equation of drawn line be

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r$$

$$\Rightarrow \quad x = r\cos\theta, y = r\sin\theta$$

Putting it in 'L_1', we get

$$r\sin\theta = r\cos\theta + 10$$

$$\Rightarrow \quad \frac{1}{OA} = \frac{\sin\theta - \cos\theta}{10}$$

Similarly, putting the general point of drawn line in the equation of L_2, we get

$$\frac{1}{OB} = \frac{\sin\theta - \cos\theta}{20}$$

Let $P = (h, k)$ and $OP = r \Rightarrow r\cos\theta = h$, $r\sin\theta = k$, we have

$$\frac{2}{r} = \frac{\sin\theta - \cos\theta}{10} + \frac{\sin\theta - \cos\theta}{20}$$

$$\Rightarrow \quad 40 = 3r\sin\theta - 3r\cos\theta$$

$$\Rightarrow \quad 3y - 3x = 40$$

11. c.

$$r^2 = \frac{10 \times 20}{(\sin\theta - \cos\theta)^2}$$

$$\Rightarrow \quad (r\sin\theta - r\cos\theta)^2 = 200$$

Hence, locus is $(y - x)^2 = 200$.

12. a. $\frac{1}{r^2} = \frac{(\sin\theta - \cos\theta)^2}{100} + \frac{(\sin\theta - \cos\theta)^2}{400}$

$$\Rightarrow \quad 400 = 5(r\sin\theta - r\cos\theta)^2$$

Hence, the locus is $400 = 5(x - y)^2$, i.e., $(x - y)^2 = 80$.

For Problems 13–15

13. a., 14. b., 15. c.

Sol.

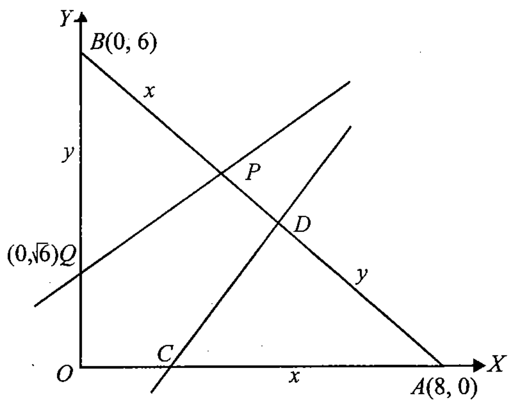


Fig. 1.186

Case I: Let the line L cut AO and AB at distances x and y from A. Then, the area of the triangle with sides x and y is

$$\frac{1}{2}xy\sin(\angle CAD) = \frac{1}{2}xy\frac{3}{5} = \frac{3xy}{10} = 12$$

$$\Rightarrow \quad xy = 40$$

Also, $x + y = 12$ (from perimeter bisection). Then x and y are roots of $r^2 - 12x + 40 = 0$ which has imaginary roots.

Case II: If the line L cuts OB and BA at distances y and x from B, then we have $xy = 30$ and $x + y = 12$.

Solving, we get $x = 6 + \sqrt{6}$ and $y = 6 - \sqrt{6}$.

Case III: If the line L cuts the sides OA and OB at distances x and y from O, then

$$x + y = 12 \text{ and } xy = 24$$

$$\therefore \quad x, y = 6 \pm 2\sqrt{3} \text{ (not possible)}$$

So there is a unique line possible. Let point P be (α, β).

Using parametric equation of AB, we have

$$\beta = 6 - \frac{3}{5}(6 + \sqrt{6})$$

and
$$\alpha = \frac{4}{5}(6 + \sqrt{6})$$

Hence, slope of PQ is

$$\frac{\beta - \sqrt{6}}{\alpha - 0} = \frac{10 - 5\sqrt{6}}{10}$$

For Problems 16–18

16. b., 17. c., 18. a.

Sol.

Image of $A(1, 3)$ in line $x + y = 2$ is $(1 - 2(2)/2, 3 - 2(2)/2) \equiv (-1, 1)$.

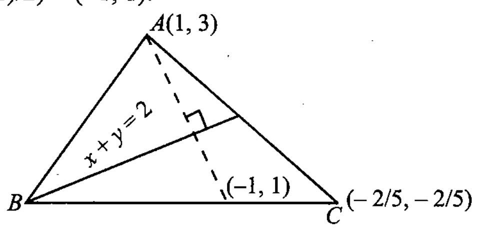


Fig. 1.187

So line BC passes through $(-1, 1)$ and $(-2/5, -2/5)$. The equation of line BC is

$$y - 1 = \frac{-2/5 - 1}{-2/5 + 1}(x + 1)$$

$$\Rightarrow \quad 7x + 3y + 4 = 0$$

17. c. Vertex B is point of intersection of $7x + 3y + 4 = 0$ and $x + y = 2$, i.e., $B \equiv (-5/2, 9/2)$.

18. a. Line AB is $y - 3 = \frac{3 - 9/2}{1 + 5/2}(x - 1)$

$$\Rightarrow \quad 3x + 7y = 24$$

For Problems 19–21

19. c., 20. a., 21. a.

Sol. 19. c.

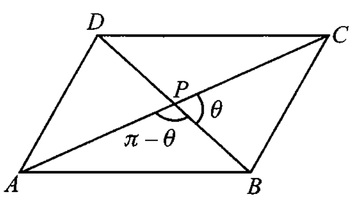


Fig. 1.188

Angle between the diagonals is given by

$$\tan\theta = \left|\frac{-\frac{1}{2} + 2}{1 + 1}\right| = \frac{3}{4}$$

$$\Rightarrow \quad \sin\theta = \frac{3}{5}$$

Area of ΔCPB is

$$\frac{1}{2} \times PC \times PB \sin\theta = 2 \Rightarrow PB = \frac{10}{3}$$

$$\Rightarrow \quad BD = \frac{20}{3}$$

20. a. $\cos(\pi - \theta) = \dfrac{AP^2 + PB^2 - AB^2}{2AP \times PB}$

$$\Rightarrow \quad -\frac{4}{5} = \frac{4 + \frac{100}{9} - AB^2}{2 \times 2 \times \frac{10}{3}}$$

$$\Rightarrow \quad AB = \frac{2\sqrt{58}}{3}$$

21. a. In ΔCPB,

$$\cos\theta = \frac{PC^2 + PB^2 - BC^2}{2PC \times PB}$$

$$\Rightarrow \quad BC = \frac{2\sqrt{10}}{3}$$

For Problems 22–24

22. a., 23. b., 24. c.

Sol.

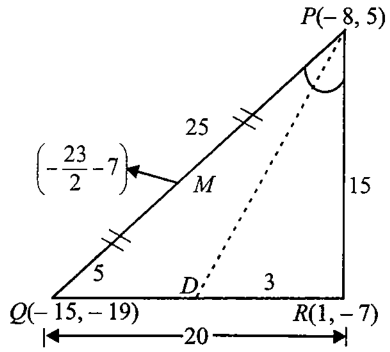


Fig. 1.189

Since triangle is right angled, circumcentre is the mid-point of PQ and orthocentre is $R(1, -7)$. Hence,

$$RM = \left|\sqrt{\left(\frac{23}{2} + 1\right)^2}\right| = 12\frac{1}{2}$$

23. b. The coordinates of the incentre are given as follows:

$$\bar{x} = \frac{20(-8) + 15(-15) + 25(1)}{20 + 15 + 25}$$
$$= -6$$

$$\bar{y} = \frac{20(5) + 15(-19) + 25(-7)}{60}$$
$$= -6$$

Hence, incentre I is $(-6, -6)$.

Now, equation of side PR is

$$y+7=\frac{12}{-9}(x-1)$$

or

$$4x-4=-3y-21$$

or

$$4x+3y+17=0$$

Inradius is given as the distance of I from side PR, i.e.,

$$\frac{|-24-18+17|}{5}=5.$$

24. c. Coordinates of D using section formula are $(-5,-23/2)$ and $m_{PD}=-11/2$. Equation PD is

$$11x+2y+78=0$$

$$\Rightarrow \quad a+c=89$$

For Problems 25–27

25. b., 26. d., 27.

Sol.

$$\theta=60°,\ m=2$$

$$\tan\theta=\frac{m\sin\omega}{1+m\cos\omega}=\frac{2\sin 60°}{1+2\cos 60°}$$

$$=\frac{2\times\sqrt{3}/2}{1+2\times 1/2}=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

26. d. $\omega=60°,\ m_1=2,\ m_2=-\frac{1}{2}$

$$\tan\theta_1=\frac{m_1\sin\omega}{1+m_1\cos\omega}=\frac{2\times\sqrt{3}/2}{1+2\times 1/2}$$

$$=\frac{\sqrt{3}}{2}$$

$$\tan\theta_2=\frac{-1/2\times\sqrt{3}/2}{1-1/2\times 1/2}=\frac{-\sqrt{3}}{4}\times\frac{4}{3}=\frac{-1}{\sqrt{3}}$$

Let angle between the lines be ϕ. Then,

$$\tan\phi=\left|\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}\right|$$

$$=\left|\frac{\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}}{1-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{3}}}\right|$$

$$\Rightarrow \quad \phi=\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$

27. c. $m=\frac{\sin 60°}{\sin(30°-60°)}=-\sqrt{3}$

Therefore, the equation of the line is $y-0=-\sqrt{3}(x-2)$, i.e. $\sqrt{3}x+y=2\sqrt{3}$.

For Problems 28–29

28. d., 29. b.

Sol. 28. d.

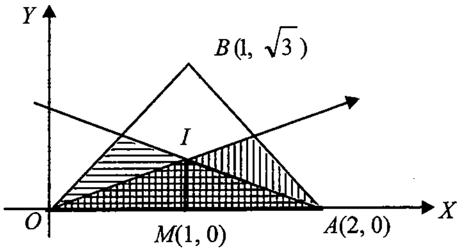


Fig. 1.190

$$d(P, OA)\le \min[d(P, OB), d(P, AB)]$$

$$\Rightarrow \quad d(P, OA)\le d(P, OB)$$

and $\quad d(P, OA)\le d(P, AB)$

When $d(P, OA)=d(P, OB)$, P is equidistant from OA and OB, or P lies on angle bisector of lines OA and OB. Hence, when $d(P, OA)\le d(P, OB)$, point P is nearer to OA than OB or lies below bisector of OA and OB. Similarly, when $d(P, OA)\le d(P, AB)$, P is nearer to OA than AB, or lies below bisector of OA and AB. Therefore, the required area is equal to the area of ΔOIA.

Now,

$$\tan\angle BOA=\frac{\sqrt{3}}{1}=\sqrt{3}$$

$$\Rightarrow \quad \angle BOA=60°$$

Hence, triangle is equilateral. Then I coincides with centroid, which is $(1, 1/\sqrt{3})$.

Therefore, area of ΔOIA is $\frac{1}{2}\,OA\times IM=(1/2)\times 2\times(1/\sqrt{3})=1/\sqrt{3}$ sq. units.

29. b.

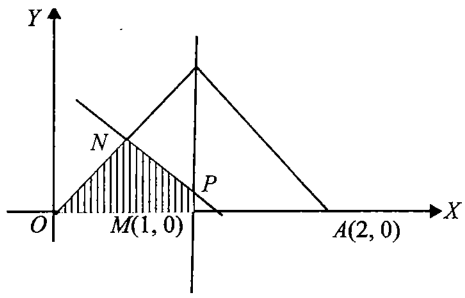


Fig. 1.191

$$OP\le \min[BP, AP]$$

$$\Rightarrow \quad OP\le BP \text{ (when } BP<AP)$$

$$OP\le AP \text{ (when } AP<BP)$$

Let $OP=BP$. Then P lies on the perpendicular bisector of OB. For $OP=AP$, P lies on the perpendicular bisector of OA. Then for required condition, P lies in the region as shown in the diagram. The area of the region $OMPN$ is

$$\frac{1}{2} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{2\sqrt{3}}$$

For Problems 30–32

30. c., 31. d., 32. b.

Sol. 30. c.

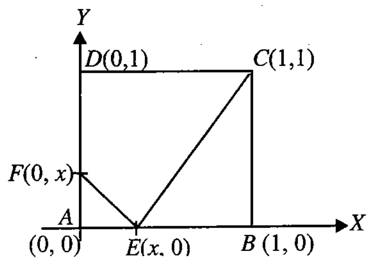


Fig. 1.192

Area of $CDFE$

$$A = 1 - \frac{1}{2}x^2 - \frac{1}{2}(1 - x)$$

$$= \frac{2 - x^2 - 1 + x}{2} = \frac{1 + x - x^2}{2}$$

$$A_{max} = \frac{1 + \frac{1}{2} - \frac{1}{4}}{2} = \frac{5}{8} \text{ at } x = \frac{1}{2}$$

31. d.

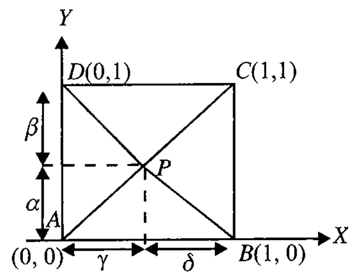


Fig. 1.193

$$(PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$$

$$= (\alpha^2 + \gamma^2) - (\alpha^2 + \delta^2) + (\delta^2 + \beta^2) - (\gamma^2 + \beta^2)$$

32. b.

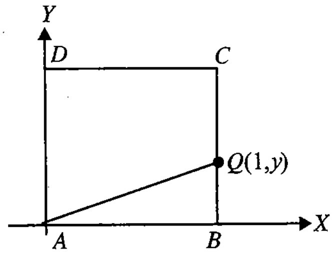


Fig. 1.194

$$\frac{1}{2}\, y(1) = \frac{1}{3}(1)$$

$$y = \frac{2}{3}$$

$$LAQ = \sqrt{(1)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{13}}{3}$$

Matrix-Match Type

1. a → s; b → r, c → p; d → q.

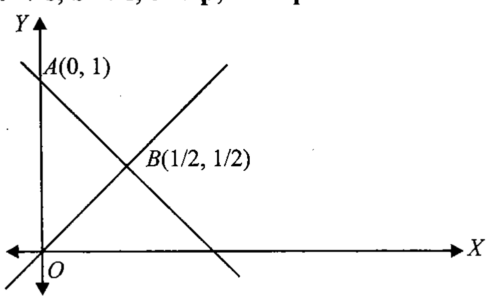


Fig. 1.195

The given lines are $x(x + y - 1)(x - y) = 0$. In other words, lines $x = 0$, $x + y - 1 = 0$ and $x - y = 0$ form triangle OAB as shown in the above diagram.

The triangle is right angled at point B, hence orthocentre is (1/2, 1/2). Also, circumcentre is midpoint of OA which is (0, 1/2). The centroid is

$$\left(\frac{0 + \frac{1}{2} + 0}{3}, \frac{0 + \frac{1}{2} + 1}{3}\right) \text{ or } \left(\frac{1}{6}, \frac{1}{2}\right)$$

Also, $OA = 1$, $OB = OC = 1/\sqrt{2}$. Hence, the incentre is

$$\left(\frac{0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(1) + 0\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}}, \frac{0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(1) + 1\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}}\right)$$

$$\equiv \left(\frac{1}{2 + 2\sqrt{2}}, \frac{1}{2}\right)$$

2. a → s; b → r; c → q; d → p.

a.

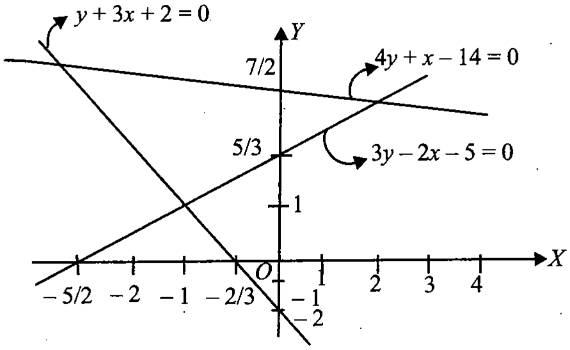


Fig. 1.196

b. Clearly, point (α, 0) lies on the x-axis, which is not intersecting any side of triangle, hence no such α exists.

c.

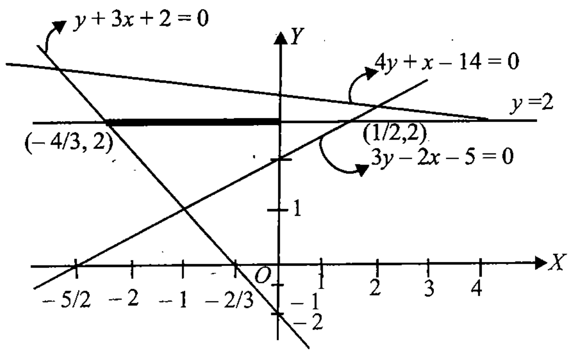


Fig. 1.197

d.

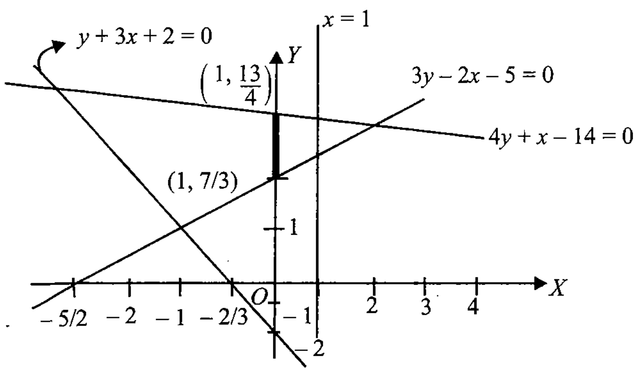


Fig. 1.198

3. a → s; b → p; c → q; d → r.

a.

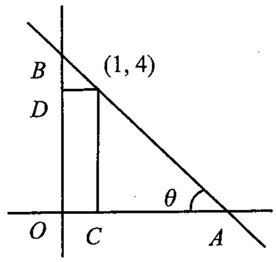


Fig. 1.199

$$OA = 1 + 4\cot\theta$$

$$OB = 4 + \tan\theta$$

$$OA + OB = 5 + 4\cot\theta + \tan\theta$$

$$\geq 5 + 2\sqrt{4\cot\theta\tan\theta}$$

$$= 5 + (2 \times 2) = 9$$

b.

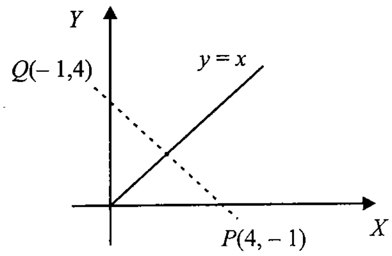


Fig. 1.200

Reflection of P (4, –1) in $y = x$ is Q (–1, 4). Hence,

$$PQ = \sqrt{(4+1)^2 + (-1-4)^2}$$

$$= \sqrt{50} = 5\sqrt{2}$$

c.

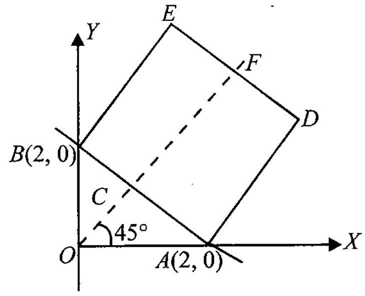


Fig. 1.201

$$AB = 2\sqrt{2}$$

$$OC = \sqrt{2}$$

Maximum value of d is

$$OF = \sqrt{2} + 2\sqrt{2}$$
$$= 3\sqrt{2}$$

d. The given line is

$$x = 4 + \frac{1}{\sqrt{2}}\left(\frac{y+1}{\sqrt{2}}\right) \Rightarrow y = 2x - 9$$

Hence, the intercept made by x-axis is 9/2.

4. a → p, s; b → q, s; c → p, r; d → s.

a. Given lines are concurrent. So,

$$\begin{vmatrix} 3 & 1 & -4 \\ 1 & -2 & -6 \\ \lambda & 4 & \lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda - 8 = 0$$
$$\Rightarrow \quad \lambda = 2, -4$$

b. Points are collinear. Hence,

$$\begin{vmatrix} \lambda+1 & 1 & 1 \\ 2\lambda+1 & 3 & 1 \\ 2\lambda+2 & 2\lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2\lambda^2 - 3\lambda - 2 = 0 \Rightarrow \lambda = 2, -1/2$$

c. The point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is (1, 2). It lies on the line $x + y - 1 - |\lambda/2| = 0$ $\Rightarrow \lambda = \pm 4$.

d. The midpoint of (1, –2) and (3, 4) will satisfy

$$y - x - 1 + \lambda = 0.$$
$$\Rightarrow \lambda = 2$$

5. a → p; b → q; c → s; d → r.

a. $AH \perp BC \Rightarrow \left(\frac{k}{h}\right)\left(\frac{3+1}{-2-5}\right) = -1$

$\therefore \quad 4k = 7h \quad$ (i)

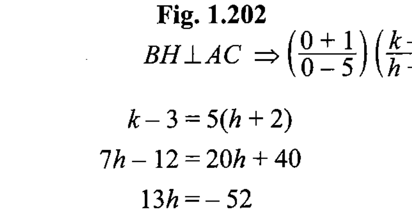


Fig. 1.202

$$BH \perp AC \Rightarrow \left(\frac{0+1}{0-5}\right)\left(\frac{k-3}{h+2}\right) = -1$$

$\therefore \quad k - 3 = 5(h+2) \quad$ (ii)

$\Rightarrow \quad 7h - 12 = 20h + 40$

$\Rightarrow \quad 13h = -52$

$\Rightarrow \quad h = -4$

$\therefore \quad k = -7$

Hence, point, A is (–4, –7)

b. $\quad x + y - 4 = 0 \quad$ (i)

$\quad 4x + 3y - 10 = 0 \quad$ (ii)

Let $(h, 4 - h)$ be the point on (i). Then,

$$\left|\frac{4h + 3(4-h) - 10}{5}\right| = 1$$

$\Rightarrow \quad h + 2 = \pm 5.$

$\Rightarrow \quad h = 3, h = -7$

Hence, the required point is either (3, 1) or (–7, 11).

c. Since lines $x + y - 1 = 0$ and $x - y + 3 = 0$ are perpendicular, orthocentre of the triangle is the point of intersection of these lines, i.e., (–1, 2).

d. Since $2a, b, c$ are in A.P., so

$$b = \frac{2a+c}{2}$$

$\Rightarrow \quad 2a - 2b + c = 0$

Comparing with the line $ax + by + c = 0$, we have $x = 2$ and $y = -2$. Hence, lines are concurrent at (2, – 2).

6. a → q, r, s; b → p; c → q, s; d → q.

a.

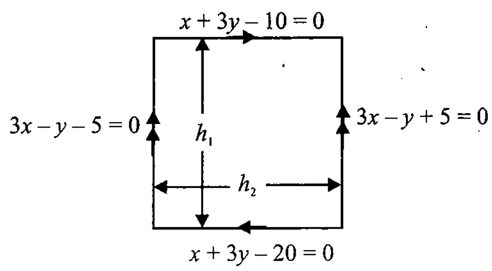


Fig. 1.203

$$h_1 = \left|\frac{10}{\sqrt{10}}\right| = \sqrt{10}$$

$$h_2 = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Hence, the given lines form a square of side $\sqrt{10}$. Therefore, the area 10 sq. units.

b.

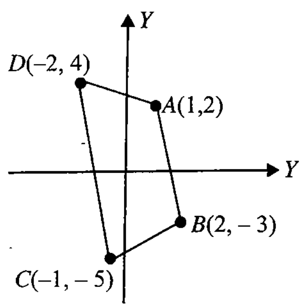


Fig. 1.204

$$m_{AB} = \frac{5}{-1} = -5$$

$$m_{DC} = \frac{9}{-1} = -9$$

Hence, the figure is not a parallelogram.

c. Lines $7x + 3y - 33 = 0$, $7x + 3y - 4 = 0$ are parallel and distance between them is $|29/\sqrt{58}|$. Lines $3x - 7y + 19 = 0$, $3x - 7y - 10$ are parallel and distance between them $|29/\sqrt{58}|$. Also, lines $7x + 3y - 33 = 0$ and $3x - 7y + 19 = 0$ are perpendicular. Hence, given lines form a square.

d. Lines $4y - 3x - 7 = 0$ and $4y - 3x - 21 = 0$ are parallel. Lines $3y - 4x + 7 = 0$, $3y - 4x + 14 = 0$ are parallel. Also, lines $4y - 3x - 7 = 0$ and $3y - 4x + 14 = 0$ are not perpendicular. Hence, given lines form parallelogram.

7. a → p; b → s; c → q; d → r.

a. $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$

$$\Rightarrow (x + 7y)^2 + 7\sqrt{2}(x + y) - 3\sqrt{2}(x + y) - 42 = 0$$

$$\Rightarrow (x + y)\,[x + 7y + 7\sqrt{2}] - 3\sqrt{2}(x - 7y + 7\sqrt{2}) = 0$$

$$\Rightarrow (x + 7y + 7\sqrt{2})(x + 7y - 3\sqrt{2}) = 0$$

$$\Rightarrow x + 7y + 7\sqrt{2} = 0 \text{ and } x + 7y - 3\sqrt{2} = 0$$

$$\Rightarrow d = \left|\frac{7\sqrt{2} + 3\sqrt{2}}{\sqrt{1 + 49}}\right| = \frac{10\sqrt{2}}{\sqrt{50}} = 2$$

b.

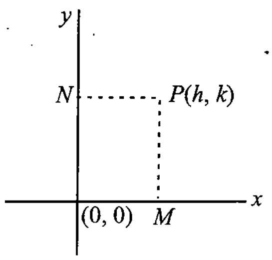


Fig. 1.205

Let two perpendicular lines are coordinate axes. Then,

$$PM + PN = 1$$

$$\Rightarrow \quad h + k = 1$$

Hence, the locus is $x + y = 1$.

But if the point lies in other quadrants also, then $|x| + |y| = 1$. Hence, value of k is 1.

c. Angle bisector between the lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$ is

$$\frac{x + 2y + 4}{\sqrt{1 + 4}} = \pm \frac{(-4x + 2y + 1)}{\sqrt{16 + 4}}$$

$$\Rightarrow \quad x + 2y + 4 = \pm \frac{(-4x - 2y + 1)}{2}$$

$$\Rightarrow \quad 2(x + 2y + 4) = \pm(-4x - 2y + 1)$$

Since $AA' + BB' < 0$, so +ve sign gives acute angle bisector. Hence,

$$2x + 4y + 8 = -4x - 2y + 1$$

$$\Rightarrow \quad 6x + 6y + 7 = 0$$

$$\Rightarrow \quad m = 7$$

d. We have,

$$y^2 - 9xy + 18x^2 = 0$$

or

$$y^2 - 6xy - 3xy + 18x^2 = 0$$

$$\Rightarrow y(y - 6x) - 3x(y - 6x) = 0$$

$$\Rightarrow \quad (y - 3x) = 0 \text{ and } y - 6x = 0$$

The third line is $y = 6$. Therefore, area of the triangle formed by these lines,

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 6 & 1 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|6 - 12|$$

$$= 3 \text{ units}^2$$

8.

a → s; b → r; c → p, q; d → p, q, r, s.

a. The equation $4x^2 + 8xy + ky^2 - 9 = 0$ represents a pair of straight lines if $(4)(k)(-a) - (-a)(4)^2 = 0$ $\Rightarrow k = 4$.

b.

$$m_1 + m_2 = 4m_1m_2$$

$$\Rightarrow \quad -\frac{2h}{b} = \frac{4a}{b}$$

$$\Rightarrow \quad -\frac{2(-c)}{-7} = \frac{4 \times 1}{-7}$$

$$\Rightarrow \quad 2c = 4 \Rightarrow c = 2$$

c. Let m_1, m_2 be the slopes of the lines $x^2 + hxy + 2y^2 = 0$. Then,

$$m_1 + m_2 = -\frac{h}{2},\ m_1m_2 = \frac{1}{2}$$

But $\quad m_1 = 2m_2$ (given). Therefore,

$$3m_2 = -h/2 \text{ and } 2m_2^2 = 1/2,$$

i.e., $m_2^2 = \frac{1}{4}$. Also, $m_2 = -h/6$.

$\therefore \quad \frac{h^2}{36} = \frac{1}{4} \Rightarrow h^2 = 9 \Rightarrow h = \pm 3$

d. Equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is

$$\frac{x^2 - y^2}{(a+\lambda) - (b+\lambda)} = \frac{xy}{h}$$

or

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

which is same as the equation of the bisectors of angles between the lines $ax^2 + 2hxy + by^2 = 0$.

Thus, the two line pairs are equally inclined to each other for any value of λ.

9. a → p; b → p, s; c → p, r; d → q, s.

a. Given that points are $O(0, 0)$ and $B(2, 0)$.

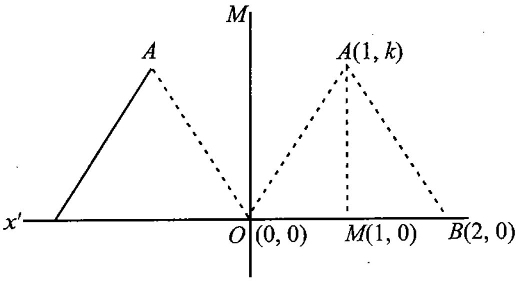


Fig. 1.206

From figure, ΔABC is equilateral.

Hence, $\tan 60° = k$

or $k = \sqrt{3}$(for first quadrant) or $k = -\sqrt{3}$ (for fourth quadrant). Then possible coordinates are $(1, \pm\sqrt{3})$.

Similarly, for second quadrant, the point is $(-1, \sqrt{3})$ and for third quadrant, the point is $(-1, -\sqrt{3})$.

b. Case (i)

If $\quad OA = AB$, then $\angle A = 30°$.

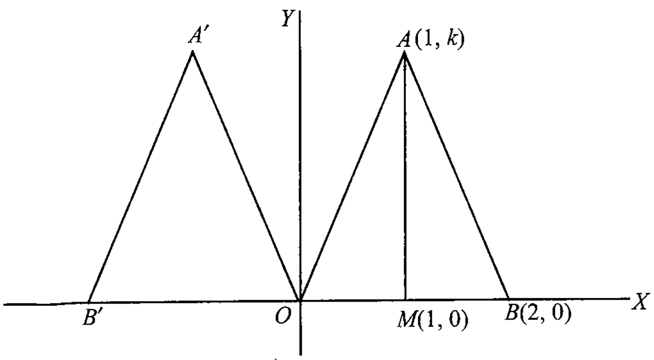


Fig. 1.207

$\therefore \quad \angle AOB = 75°$

$\therefore \quad \frac{AM}{OM} = \tan 75°$

$AM = OM \tan 75°$

$k = 1 \times (2 + \sqrt{3})$

$\therefore \quad k = 2 + \sqrt{3}$

Hence, point A is $(1, 2 + \sqrt{3})$. By symmetry, all possible points are $(\pm 1, \pm(2 + \sqrt{3}))$.

Case (ii)

$AO = OB$

$\therefore \quad \angle AOB = 120°$

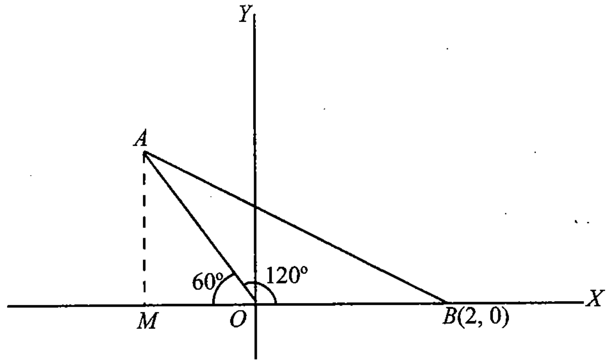


Fig. 1.208

$AM = 2 \sin 60° = \sqrt{3}$

and $\quad OM = 2 \cos 60° = 1$

Hence, point A is $(1, -\sqrt{3})$. By symmetry, all possible points are $(\pm 1, \pm\sqrt{3})$.

c.

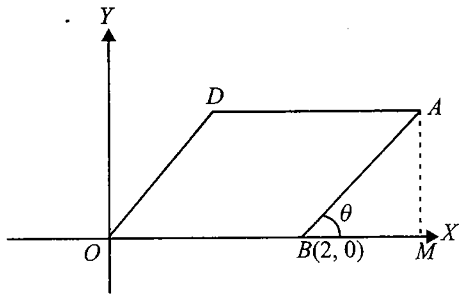


Fig. 1.209

Let $\angle DOB = \angle ABM = \theta$. Area of ΔOAB is $\frac{1}{2} \times OB \times AM = \frac{1}{2} \times \sqrt{3}$

$\Rightarrow \quad 2 \times 2 \sin \theta = \sqrt{3}$

$\Rightarrow \quad \sin \theta = \frac{\sqrt{3}}{4} \Rightarrow AM = \sqrt{3}$ and $BM = 1$

Hence, A has coordinates $(3, \sqrt{3})$. By symmetry, all possible coordinates are $(\pm 3, \pm\sqrt{3})$.

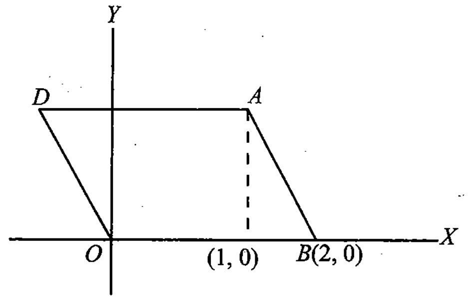


Fig. 1.210

From the above figure A has coordinates $(1, \sqrt{3})$.

By symmetry, all possible coordinates are $(\pm 1, \pm\sqrt{3})$.

d.

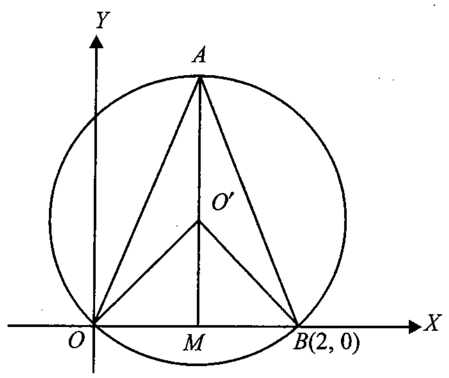


Fig. 1.211

$$OB = 2 \text{ units} = OO' = \text{radius}$$

$$\Rightarrow \quad OM = \frac{2}{2} = 1 \text{ unit}$$

In $\Delta\, OO'M$,

$$O'M = \sqrt{4-1} = \sqrt{3}$$

Since ΔOAB is isosceles hence point A lies on perpendicular bisector of OB.

$$\therefore \quad AM = \sqrt{3} + 2 = O'M + OA$$

Hence, the coordinate A will be $(1, 2+\sqrt{3})$ in first quadrant. By symmetry, all possible coordinates of A are $(\pm 1, \pm(2+\sqrt{3}))$.

10. $a \to p, s$; $b \to p$; $c \to q$; $d \to p, q, r$

a.

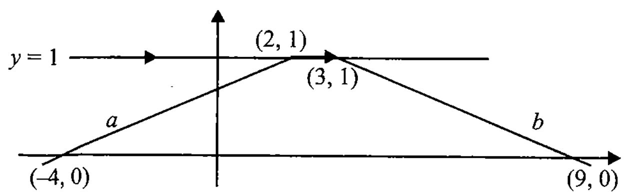


Fig. 1.212

Obviously, trapezium

$$\left.\begin{aligned} a &= \sqrt{37} \\ b &= \sqrt{37} \end{aligned}\right\} \Rightarrow a = b$$

Hence, isosceles trapezium

$\Rightarrow$ a cyclic quadrilateral

b.

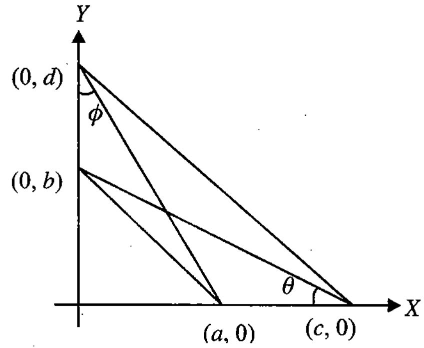


Fig. 1.213

$$ac = bd$$

$$\Rightarrow \quad \frac{b}{c} = \frac{a}{d}$$

$$\left.\begin{aligned} \tan\theta &= \frac{b}{c} \\ \tan\phi &= \frac{a}{d} \end{aligned}\right\} \Rightarrow \theta = \phi$$

Hence, cyclic quadrilateral

c.

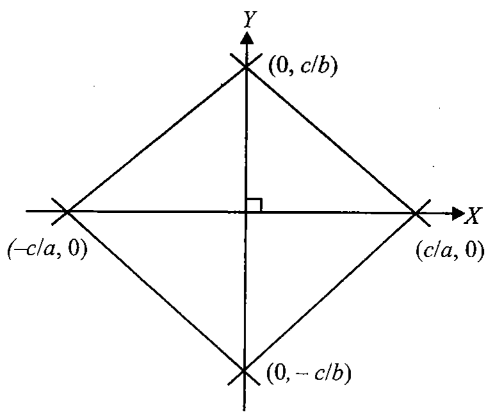


Fig. 1.214

$$ax \pm by \pm c = 0$$

if $\quad y = 0, y = \pm\frac{c}{a}$

if $\quad x = 0, y = \pm\frac{c}{d}$

$\Rightarrow$ rhombus

d.

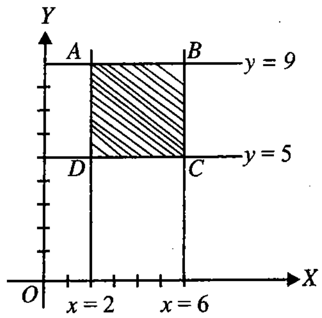


Fig. 1.215

$(x-6)(x-2)=0$

$x=6$ and $x=2$

$y^2-14y+45=0$

$(y-9)(y-5)=0$

$\Rightarrow$ a square

Integer Type

1. (5) The given lines $7x+4y=168$ and $5x+3y=121$ intersect at $P(20, 7)$

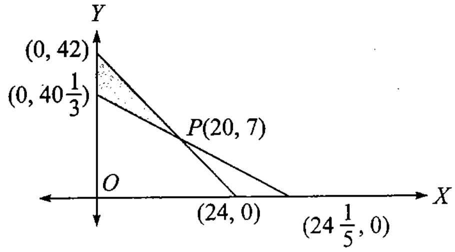


Fig. 1.216

$\therefore$ Area of shaded region

$$A=\frac{1}{2}\left(42-40\frac{1}{3}\right)20$$

$$=\frac{1}{2}\left(\frac{5}{3}\right)20=\frac{50}{3} \text{ (square units)}$$

2. (6) $x^2y^2-9x^2-25y^2+225=0$

$\Rightarrow x^2(y^2-9)-25(y^2-9)=0$

$\Rightarrow (y^2-9)(x^2-25)=0$

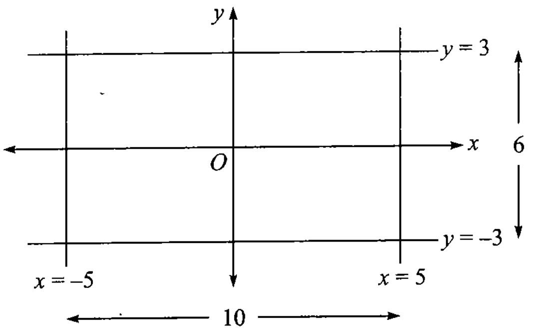


Fig. 1.217

$\therefore$ Area $A=10\times 6=60$ sq. units.

3. (1) Lines $(k+1)x+8y=4k$ and $kx+(k+3)y=3k-1$ are coincident then we can compare ratio of coefficients

$\Rightarrow \dfrac{k+1}{k}=\dfrac{8}{k+3}=\dfrac{4k}{3k-1}$

$\Rightarrow k^2+4k+3=8k$ and $24k-8=4k^2+12k$

$\Rightarrow (k-3)(k-1)=0$ and $(k-2)(k-1)=0$

$\Rightarrow k=1$

4. (0) Equation of angle bisector of angle A

$\dfrac{3x+4y}{5}=\pm\dfrac{4x+3y}{5} \Rightarrow x=\pm y$

equation of internal bisector is $x=-y$

since h and k lie on the line $x=-y$

$\Rightarrow h+k=0$

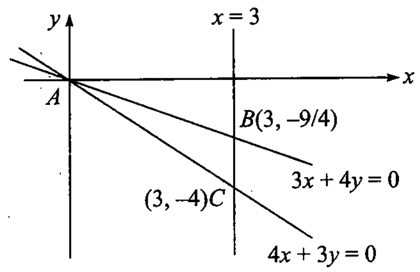


Fig. 1.218

5. (0) As H, G and S are collinear

$$\therefore \begin{vmatrix} 4 & b & 1 \\ b & 2b-8 & 1 \\ -4 & 8 & 1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 4 & b & 1 \\ b-4 & b-8 & 0 \\ -(b+4) & 16-2b & 0 \end{vmatrix}=0$$

$\Rightarrow (b-4)(16-2b)+(b+4)(b-8)=0$

$\Rightarrow 2(b-4)(8-b)+(b+4)(b-8)=0$

$\Rightarrow (8-b)[(2b-8)-(b+4)]=0$

$\Rightarrow (8-b)(b-12)=0$

Also

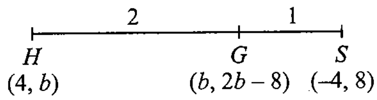


Fig. 1.219

$\therefore \dfrac{-8+4}{3}=b \quad \Rightarrow b=\dfrac{-4}{3}$

And $\dfrac{16+b}{3}=2b-8 \Rightarrow b=8$

But no common value of 'b' is possible

6. **(7)** Using section formula $A\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of triangle ABC is 2 sq. units

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ 7 & -2 & 1 \\ \frac{3k-5}{k+1} & \frac{5k+1}{k+1} & 1 \end{vmatrix} = \pm 2$$

Operating $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & 5 & 1 \\ 6 & -7 & 0 \\ \frac{3k-5}{k+1}-1 & \frac{5k+1}{k+1}-5 & 0 \end{vmatrix} = \pm 4$$

$$\Rightarrow 6\left(\frac{5k+1-5k-5}{k+1}\right)+7\left(\frac{3k-5-k-1}{k+1}\right) = \pm 4$$

$\Rightarrow$ $-24 + 7(2k - 6) = \pm 4(k + 1)$

$\Rightarrow$ $k = 7$ or $k = \frac{31}{9}$

7. **(4)** $x^2 - 3y^2 - 2xy + 8y - 4 \equiv (x - 3y + 2)(x + y - 2)$

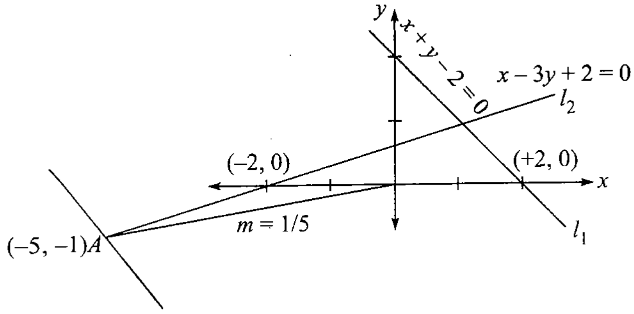


Fig. 1.220

Now $(-5, -1)$ lies on $x - 3y + 2 = 0$

In limiting case line passing through $(-5, -1)$ can be parallel to $x + y - 2 = 0$

i.e. $m > -1$

and maximum slope can occur if it passes through (0, 0)

i.e. $m < \frac{1}{5} \Rightarrow m \in \left(-1, \frac{1}{5}\right)$

$\Rightarrow$ $a = -1$ and $b = \frac{1}{5}$

$\Rightarrow$ $\left(a + \frac{1}{b}\right) = -1 + 5 = 4$

8. **(8)** Given pair of lines $x^2 - (y^2 - 4y + 4) = 0$

$\Rightarrow$ $x^2 - (y - 2) = 0$

$\Rightarrow$ $(x + y - 2)(x - y + 2) = 0$

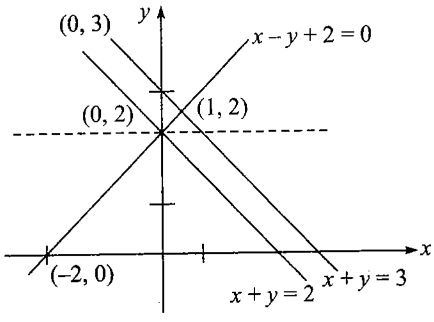


Fig. 1.221

Required area is $A = \frac{1 \cdot 1}{2} = \frac{1}{2}$

9. **(6)** Let $x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow$ $2r\cos\theta + 3r\sin\theta = 6$

$\Rightarrow$ $r = \frac{6}{2\cos\theta + 3\sin\theta}$; and $r = \sqrt{x^2 + y^2}$

for r to be minimum $2\cos\theta + 3\sin\theta$ must be maximum i.e. $\sqrt{13}$

$\therefore$ $r_{\text{min}} = \frac{6}{\sqrt{13}}$

10. **(5)** Given vertices of triangle are $O(0, 0)$, $B(6, 8)$ and $C(-4, 3)$

Slope of $OB = \frac{8}{6}$

Slope of $OC = -\frac{3}{4}$

$\therefore$ $\angle BOC = \frac{\pi}{2}$

Δ OBC is right angled at O

Circumcentre = midpoint of hypotenuse $BC = \left(1, \frac{11}{2}\right)$

Orthocentre = vertex $O(0, 0)$

Required distance $= \sqrt{\left(1 + \frac{121}{4}\right)} = \frac{5\sqrt{5}}{2}$ unit

11. **(4)** Any point on the line $x + y = 4$ is $(t, 4 - 4)$

where $t \in R$

Now distance of this point from the line $4x + 3y - 10 = 0$ is 1

$\Rightarrow$ $\frac{|4t + 3(4 - t) - 10|}{5} = 1$

$\Rightarrow$ $|t + 2| = 5$

$\Rightarrow$ $t = 3$ or $t = -7$

$\Rightarrow$ sum of values is -4

12. **(3)** Area of $\Delta OAB = \frac{1}{2}(1)(8) = 4$ sq. units

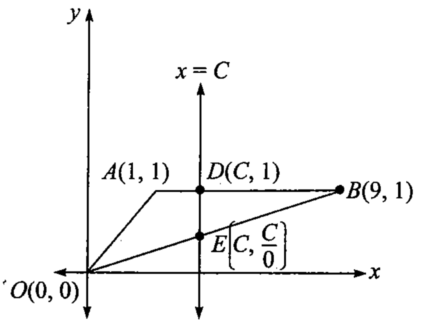


Fig. 1.222

Equation of OB is $y = \frac{1}{9}x$

Hence point E is $\left(C, \frac{C}{9}\right)$

Now area of ΔBDE is 2 square units.

$\Rightarrow \frac{1}{2}\left(1-\frac{C}{9}\right)(9-C)=2$

$\Rightarrow (9-C)^2 = 36$

$\Rightarrow 9 - C = \pm 6$

$\Rightarrow C = 3$

13. (3) For $PR = RQ$ to be minimum it should be the path of light

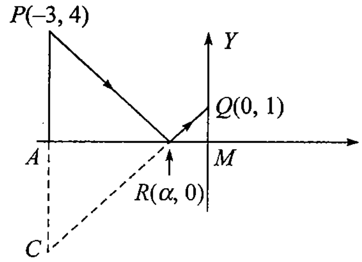


Fig. 1.223

$\therefore \angle PRA = \angle QRM$

From similar Δ PAR and Δ QMR

$\frac{AR}{RM} = \frac{PA}{QM}$

$\Rightarrow \frac{\alpha+3}{0-\alpha} = \frac{4}{1} \Rightarrow \alpha = -\frac{3}{5}$

14. (8) We know that the area of the triangle formed by joining the mid points of any triangle is one fourth of that triangle. Therefore required area is 8.

15. (2) Lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ or $a(2x + y - 3) + b(x + 3y + 1) = 0$ are concurrent at point of intersection of lines $2x + y - 3 = 0$ and $x + 3y + 1 = 0$ which is $(2, -1)$.

Now line $mx + 2y + 6 = 0$ must pass through this point $\Rightarrow 2m - 2 + 6 = 0$ or $m = -2$

16. (7) Line $3x + 2y = 24$ meets the axis at $B(8, 0)$ and $A(0, 12)$. Midpoint of AB is $D(4, 6)$

Equation of perpendicular bisector of AB is

$$2x - 3y + 10 = 0 \quad (1)$$

Now line through $(0, -1)$ and parallel to x – axis is $y = -1$

Co-ordinates of C where line (1) meets $y = -1$ is $C\left(-\frac{13}{2}, -1\right)$

Now the area of triangle ABC

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 12 & 1 \\ 8 & 0 & 1 \\ -\frac{13}{2} & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[0 - 12\left(8 + \frac{13}{2}\right) + 1(-8)\right]$$

$$= \frac{1}{2}[-6(29) - 8] = 91$$

17. (5)

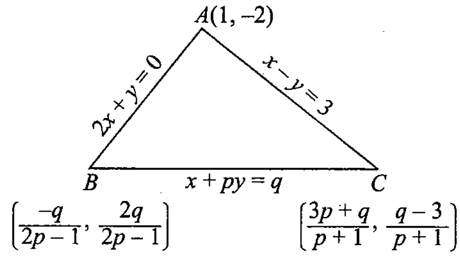


Fig. 1.224

P is orthocenter

$\Rightarrow AP \perp BC$

$\Rightarrow \left(-\frac{1}{p}\right)\left(\frac{3+2}{2-1}\right) = -1$

$\Rightarrow \frac{5}{p} \Rightarrow p = 5$

$\because BP \perp AC$

$\Rightarrow \frac{27-2q}{18+q} = -1 \Rightarrow q = 27 + 18$

$\Rightarrow q = 45$

$\therefore p + q = 5 + 45 = 50$

18. (5)

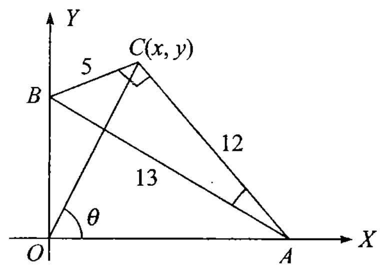


Fig. 1.225

Since $\angle BCA = 90°$

Points A, O, B, C are concyclic

Let $\angle AOC = \theta$

$\angle BOC = \angle BAC$

$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{5}{12}$

$\frac{x}{y} = \frac{5}{12} \Rightarrow 12x - 5y = 0$

Archives

Subjective Type

1. Let $P(x, y)$ divides line segment AB in the ratio 1:2, so that $AP = l/3$ and $BP = 2l/3$ where $AB = l$.

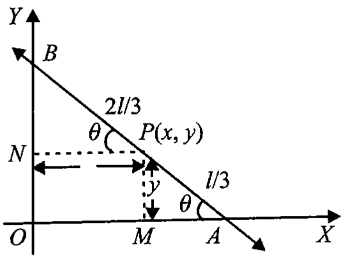


Fig. 1.226

Then $PN = x$ and $PM = y$. Let $\angle PAM = \theta = \angle BPN$. In, ΔPMA,

$$\sin\theta = \frac{y}{l/3} = \frac{3y}{l}$$

In ΔPNB,

$$\cos\theta = \frac{x}{2l/3} = \frac{3x}{2l}$$

Now,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \quad \frac{9y^2}{l^2} + \frac{9x^2}{4l^2} = 1$$

$$\Rightarrow \quad 9x^2 + 36y^2 = 4l^2$$

which is the required locus.

2. As C lies on the line $y = x + 3$, let the coordinates of C be $(\lambda, \lambda + 3)$. Also $A \equiv (2, 1)$, $B \equiv (3, -2)$. Then area of ΔABC is given by

$$\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ \lambda & \lambda+3 & 1 \end{vmatrix} = 5$$

$$\Rightarrow \quad |4\lambda - 4| = 10$$

$$\Rightarrow \quad 4\lambda - 4 = 10 \text{ or } 4\lambda - 4 = -10$$

$$\Rightarrow \quad 4\lambda = 14 \text{ or } 4\lambda = -6$$

$$\Rightarrow \quad \lambda = 7/2 \text{ or } \lambda = -3/2$$

Hence, Coordinate of C are (7/2, 13/2) or (– 3/2, 3/2).

3. Let side AB of rectangle $ABCD$ lie along $4x + 7y + 5 = 0$. As (–3, 1) lies on this line, let it be vertex A. Now (1, 1) is either vertex C or D.

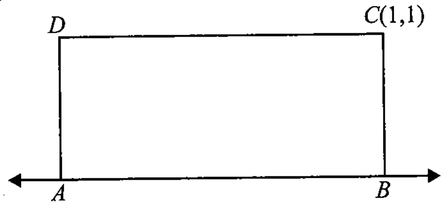


Fig. 1.227

If (1, 1) is vertex D then slope of AD is 0. Hence, AD is not perpendicular to AB. But it is a contradiction as $ABCD$ is a rectangle. Therefore, (1, 1) are the coordinates of vertex C.

CD is a line parallel to AB and passing through C, therefore equation of CD is

$$y - 1 = -\frac{4}{7}(x - 1)$$

$$\Rightarrow \quad 4x + 7y - 11 = 0$$

Also, BC is a line perpendicular to AB and passing through C, therefore equation of BC is

$$y - 1 = \frac{7}{4}(x - 1)$$

$$\Rightarrow \quad 7x - 4y - 3 = 0$$

Similarly, AD is a line perpendicular to AB and passing through A (–3, 1), therefore equation of line AD is

$$y - 1 = 7/4\,(x + 3)$$

$$\Rightarrow \quad 7x - 4y + 25 = 0$$

4. a. We have,

$AH \perp BC$. Therefore, $m_{AH} \times m_{BC} = -1$

$$\Rightarrow \quad \frac{k}{h} \times \frac{3+1}{-2-5} = -1$$

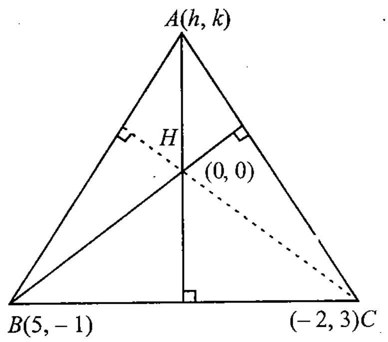


Fig. 1.228

$$\Rightarrow \quad 4k - 7h = 0 \qquad \text{(i)}$$

also $BH \perp AC$. Therefore,

$$\Rightarrow \quad \frac{-1}{5} \times \frac{3-k}{-2-h} = -1$$

$$\Rightarrow \quad 3 - k = -10 - 5h$$

$$\Rightarrow \quad 5h - k + 13 = 0 \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get $h = -4$, $k = -7$. Hence, the third vertex is (–4, –7).

b. The given lines are

$$x - 2y + 4 = 0 \quad \text{(i)}$$

and

$$4x - 3y + 2 = 0 \quad \text{(ii)}$$

Both the lines have constant terms of same sign. Therefore, the equations of bisectors of the angles between the given lines are

$$\frac{x-2y+4}{\sqrt{1+4}} = \pm \frac{4x-3y+2}{\sqrt{16+9}}$$

Here $a_1a_2 + b_1b_2 > 0$. Therefore, taking +ve sign on RHS, we get obtuse angle bisector as

$$(4-\sqrt{5})x + (2\sqrt{5}-3)y - (4\sqrt{5}-2) = 0 \quad \text{(iii)}$$

5. The given line is $5x - y = 1$. Therefore, the equation of line L which is perpendicular to the given line is $x + 5y = \lambda$. This line meets co-ordinates axes at A (λ, 0) and B (0, $\lambda/5$).

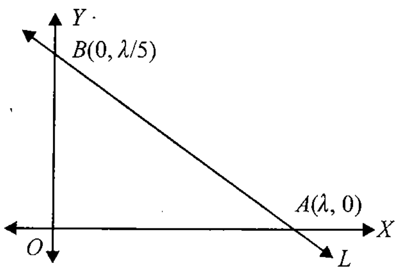


Fig. 1.229

$$\therefore \quad \text{Area of } \Delta OAB = \frac{1}{2} \times OA \times OB$$

$$\Rightarrow \quad 5 = \frac{1}{2} \times \lambda \times \frac{\lambda}{5}$$

$$\Rightarrow \quad \lambda^2 = 5^2 \times 2 \Rightarrow \lambda = \pm 5\sqrt{2}$$

Hence, the equation of line L is $x + 5y - 5\sqrt{2} = 0$ or $x + 5y + 5\sqrt{2} = 0$.

6. Let $ABCD$ be a rectangle and coordinates of its opposite vertices A and C are (1, 3) and (5, 1) respectively.

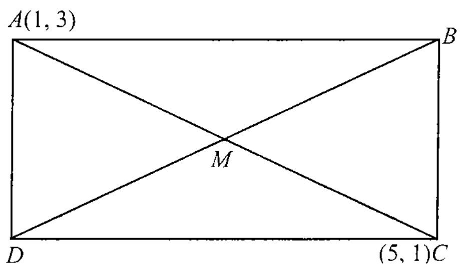


Fig. 1.230

Given that other two vertices B and D lie on $y = 2x + c$. Hence, equation of diagonal BD is $2x - y + c = 0$.

Now we know that diagonals of a rectangle bisect each other. Therefore, the mid-point of AC will lie on BD, which is given by

$$M \equiv \left(\frac{1+5}{2}, \frac{3+1}{2}\right) = (3, 2)$$

As (3, 2) lies on $2x - y + c = 0$, so

$$\therefore \quad 2 \times 3 - 2 + x = 0$$

$$\Rightarrow \quad c = -4$$

Hence, Equation of BD is $2x - y - 4 = 0$. Now slope of BD is

$$m = 2 \Rightarrow \tan\theta = 2$$

$$\Rightarrow \quad \sin\theta = 1/\sqrt{5};\ \sin\theta = 2/\sqrt{5}$$

Therefore, equation of BD in symmetric form is

$$\frac{x-3}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r \quad \text{(i)}$$

Now length of diagonal AC is $\sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$. As diagonals of a rectangle are equal in length, so $BD = 2\sqrt{5}$.

Also, M is the mid-point of BD Therefore, $BM = MD = \sqrt{5}$. In order to find B, we use symmetrical form of line BD with $r = \sqrt{5}$.

$$\therefore \quad \frac{x-3}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = \sqrt{5}$$

$$\Rightarrow \quad x = 1 + 3, y = 2 + 2 \Rightarrow B\ (4, 4)$$

Similarly for point D (being in opposite direction of BD we use $r = -\sqrt{5}$), we get

$$\frac{x-3}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = -\sqrt{5}$$

$$\Rightarrow \quad x = -1 + 3, y = -2 + 2$$

$$\Rightarrow \quad x = 2, y = 0$$

$\therefore D$ (2, 0)

Hence, $c = -4$ and remaining vertices are (4, 4) and (2, 0)

7. Here equation of AB is

$$\frac{x}{c\cos\alpha} + \frac{y}{c\sin\alpha} = 1$$

$$\Rightarrow \quad x\sin\alpha + y\cos\alpha = c\sin\alpha\cos\alpha \quad \text{(i)}$$

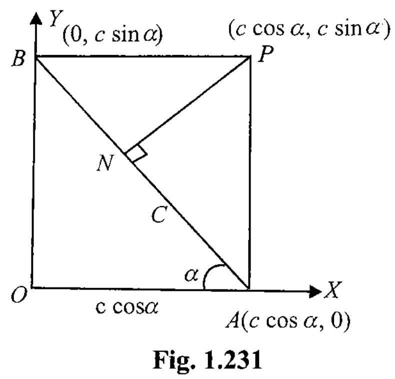


Fig. 1.231

The equation of PN (perpendicular to AB and through P) is

$$y - c\sin\alpha = \cot\alpha\,(x - c\cos\alpha)$$

$$\Rightarrow \quad x\cos\alpha - y\sin\alpha = c(\cos^2\alpha - \sin^2\alpha) \qquad \text{(ii)}$$

N is intersection point of Eqs. (i) and (ii). Multiplying Eq. (i) by $\sin\alpha$ and Eq. (ii) by $\cos\alpha$ and subtracting, we get,

$$x = c\cos^3\alpha,\ y = c\sin^3\alpha$$

$$\therefore \qquad \cos\alpha = \left(\frac{x}{c}\right)^{1/3},\ \sin\alpha = \left(\frac{y}{c}\right)^{1/3}$$

$\therefore$ locus of (x, y) is,

$$\left(\frac{x}{c}\right)^{2/3} + \left(\frac{y}{c}\right)^{2/3} = 1$$

$$\Rightarrow \qquad x^{2/3} + y^{2/3} = c^{2/3}$$

8.

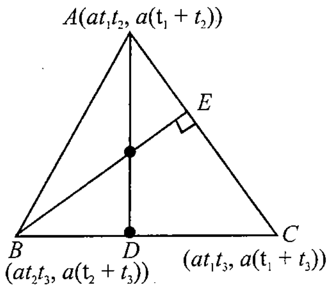


Fig. 1.232

We know that orthocentre of triangle is the meeting point of altitudes. Now, slope of BC is given by

$$\frac{a(t_1 + t_3) - a(t_2 + t_3)}{at_1 - at_2t_3} = \frac{1}{t_3}$$

$\therefore$ Slope of AD is $-t_3$. Hence, equation of AD is

$$y - a(t_1 + t_2) = -t_3(x - at_1t_2)$$

or $$xt_3 + y = at_1t_2t_3 + a(t_1 + t_2) \qquad \text{(i)}$$

Similarly by symmetry equation of BE is

$$xt_1 + y = at_1t_2t_3 + a(t_2 + t_3) \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get

Hence, $$x = -a,\ y = a(t_1 + t_2 + t_3) + at_1t_2t_3$$

Therefore, orthocentre is $H \equiv (-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$

9. Area of $$\Delta ABC = \frac{1}{2}\begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times [6(7) + 3(5)] + 4(-2)] = \frac{49}{2}$$

$$\text{Area of } \Delta PBC = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(7x + 7y - 14) = \frac{7}{2}|x + y - 2|$$

Hence, the ratio of the area of triangles is given by

$$\frac{Ar(\Delta PBC)}{Ar(\Delta ABC)} = \frac{\frac{7}{2}|x + y - 2|}{\frac{49}{2}} = \left|\frac{x + y - 2}{7}\right|$$

10. The given equations of equal sides are

$$7x - y + 3 = 0 \qquad \text{(i)}$$

$$x + y - 3 = 0 \qquad \text{(ii)}$$

Let m be the slope of line which passes through the point $(1, -10)$.

Thus, equation of line will be

$$y + 10 = m(x - 1) \qquad \text{(iii)}$$

Now given lines make same angle with the line in Eq. (iii),

$$\Rightarrow \qquad \left|\frac{m - 7}{1 + 7m}\right| = \left|\frac{m + 1}{1 - m}\right|$$

$$\Rightarrow \qquad m = -3 \text{ or } 1/3$$

Hence, required line is

$$y + 1 = -3(x - 1) \text{ or } y + 1 = \frac{1}{3}(x - 1)$$

$$\Rightarrow \qquad 3x + y + 7 = 0 \text{ or } x + 3y + 31 = 0$$

11.

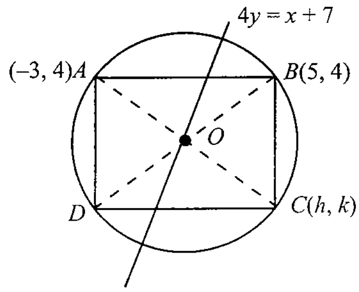


Fig. 1.233

From the diagram, $BC \perp AB$

Slope of AB is 0, then $h - 5 = 0$ (i)

Also midpoint of AC lies on the given diameter

$$\Rightarrow 4\left(\frac{k + 4}{2}\right) = \frac{h - 3}{2} + 7 \qquad \text{(ii)}$$

From (i) and (ii), $h = 5$ and $k = 0$

Hence, C is $(5, 0)$

$\Rightarrow BC = 4$ and $AB = 8$

$\Rightarrow$ Area of rectangle is $AB \times BC = 32$ sq. units

12. A being on y-axis may be chosen as $(0, a)$. The diagonals intersect at $P(1, 2)$.

Again we know that diagonals will be parallel to the bisectors of the two sides $y = x + 2$ and $y = 7x + 3$.

i.e.,

$$\frac{x - y + 2}{\sqrt{2}} = \pm\frac{7x - y + 3}{5\sqrt{2}}$$

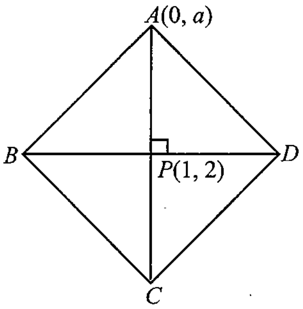


Fig. 1.234

$\Rightarrow \quad 5x - 5y + 10 = \pm(7x - y + 3)$

$\Rightarrow \quad 2x + 4y - 7 = 0$ and $12x - 6y + 13 = 0$

Here, $m_1 = -1/2$ and $m_2 = 2$. Let diagonal d_1 be parallel to $2x + 4y - 7 = 0$ and diagonals d_2 be parallel to $12x - 6y + 13 = 0$. The vertex A could be on any of the two diagonals. Hence, slope of AP is either $-1/2$ or 2. Therefore,

$$\Rightarrow \quad \frac{2-a}{1-0} = 2 \text{ or } -\frac{1}{2}$$

$\Rightarrow \quad a = 0$ or $5/2$

Hence, A is (0, 0) or (0, 5/2).

13.

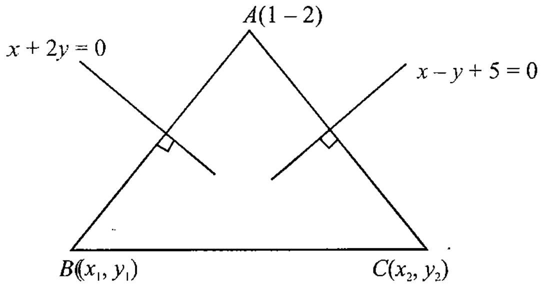


Fig. 1.235

Since given lines are perpendicular bisectors of the sides as shown in the figure, points B and C are the images of the point A in these lines.

$$\Rightarrow \frac{x_1 - 1}{1} = \frac{y_1 + 2}{2} = -\frac{2(1-4)}{1+4}$$

$$\text{and } \frac{x_2 - 1}{1} = \frac{y_2 + 2}{2} = \frac{2(1+2+5)}{1+1}$$

$\Rightarrow B(x_1, y_1) \equiv (-7, 6)$ and $C(x_2, y_2) \equiv (11/5, 2/5)$

Hence, line passing through the points B and C is $14x + 23y - 40 = 0$

14.

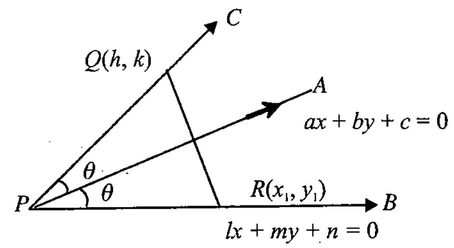


Fig. 1.236

Clearly AP is bisector of $\angle CPB$

Hence, image of Q in line PA must lie on the line PB

$$\Rightarrow \frac{x_1 - h}{a} = \frac{y_1 - h}{b} = -\frac{2(ah + bk + c)}{a^2 + b^2}$$

$$\Rightarrow x_1 = -\frac{2a(ah + bk + c)}{a^2 + b^2} + h$$

$$\text{and } y_1 = -\frac{2b(ah + bk + c)}{a^2 + b^2} + k$$

Now $R(x_1, y_1)$ lies on the line $lx + my + n = 0$

$$\Rightarrow l\left[-\frac{2a(ah + bk + c)}{a^2 + b^2} + h\right] + m\left[-\frac{2b(ah + bk + c)}{a^2 + b^2} + k\right] + n = 0$$

Hence, locus of Q is $2(al + mb)(ax + by + c) - (a^2 + b^2)(lm + my + n) = 0$.

15. Let BC be taken as x-axis with origin at D, the midpoint of BC, and DA will be y-axis. Then,

$AB = AC$. Let $BC = 2a$, then the coordinates of B and C are $(-a, 0)$ and $(a, 0)$. Let $DA = h$. Then, coordinates of A are $(0, h)$.

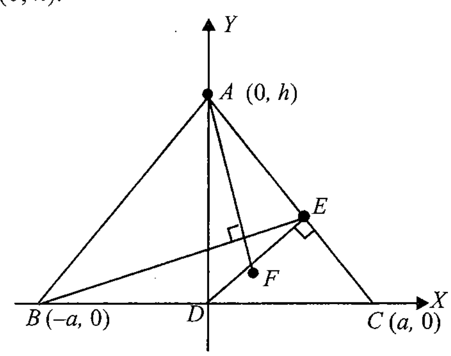


Fig. 1.237

Hence, equation of AC is

$$\frac{x}{a}+\frac{y}{h}=1 \quad \text{(i)}$$

and equation of $DE \perp AC$ and passing through origin is

$$\frac{x}{h}-\frac{y}{a}=0 \quad \Rightarrow \quad x=\frac{hy}{a} \quad \text{(ii)}$$

Solving (i) and (ii) we get the coordinates of E as follows:

$$\frac{hy}{a^2}+\frac{y}{h}=1$$

$$\Rightarrow \quad h^2y+a^2y=a^2h$$

$$\Rightarrow \quad y=\frac{a^2h}{a^2+h^2}$$

$$\Rightarrow \quad x=\frac{ah^2}{a^2+h^2}$$

$$\therefore \quad E \equiv \left(\frac{ah^2}{a^2+h^2}, \frac{a^2h}{a^2+h^2}\right)$$

Since F is midpoint of DE, its coordinates are

$$\left(\frac{ah^2}{2(a^2+h^2)}, \frac{a^2h}{2(a^2+h^2)}\right)$$

Slope of AF is

$$m_1=\frac{h-\dfrac{a^2h}{2(a^2+h^2)}}{0-\dfrac{ah^2}{2(a^2+h^2)}}$$

$$=\frac{2h(a^2+h^2)-a^2h}{-ah^2}$$

$$=-\frac{a^2+2h^2}{ah} \quad \text{(i)}$$

And slope of BE is

$$m_2=\frac{\dfrac{a^2h}{a^2+h^2}-0}{\dfrac{ah^2}{a^2+h^2}+a}$$

$$=\frac{a^2h}{ah^2+a^3+ah^2}$$

$$=\frac{ah}{a^2+2h^2} \quad \text{(ii)}$$

From (i) and (ii), we observe that

$$m_1m_2=-1 \Rightarrow AF \perp BE$$

16. The given straight lines are $3x + 4y = 5$ and $4x - 3y = 15$. Clearly these straight lines are perpendicular to each other ($m_1m_2 = -1$), and intersect at A. Now B and C are points on these lines such that $AB = AC$ and BC passes through (1, 2). From figure it is clear that $\angle B = \angle C = 45°$.

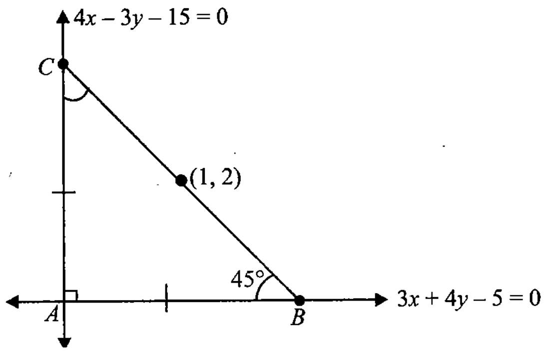


Fig. 1.238

Let slope of BC be m. Then

$$\tan 45° = \left|\frac{m+3/4}{1-\frac{3}{4}m}\right|$$

$$\Rightarrow \quad 4m+3=\pm(4-3m)$$

$$\Rightarrow \quad 4m+3=4-3m \text{ or } 4m+3=-4+3m$$

$$\Rightarrow \quad m=1/7 \text{ or } m=-7$$

Hence, equation of BC is

$$y-2=\frac{1}{7}(x-1) \text{ or } y-2=-7(x-1)$$

$$\Rightarrow \quad 7y-14=x-1 \text{ or } y-2=-7x+7$$

$$\Rightarrow \quad x-7y+13=0 \text{ or } 7x+y-9=0$$

17.

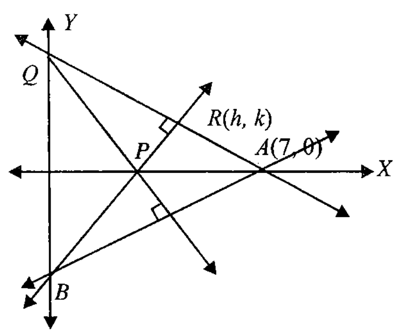


Fig. 1.239

From the figure in ΔABQ,

$AP \perp BQ$, $PQ \perp AB$,

then we have $BP \perp AQ$ (as altitudes of triangle are concurrent)

Hence, $BR \perp AR$

$\Rightarrow \quad \frac{k-0}{h-7} \times \frac{k+5}{h-0} = -1$

$\Rightarrow$ Locus of R is $x^2 + y^2 - 7x + 5y = 0$.

18. See the solution of similar question in examples.

19. The given curve is

$$3x^2 - y^2 - 2x + 4y = 0 \quad \text{(i)}$$

Let $y = mx + c$ be the chord of curve (i) which subtends an angle of 90° at origin. Then the combined equations of lines joining points of intersection of curve (i) and chord $y = mx + c$ to the origin, can be obtained by making the equation of curve homogeneous with the help of equation of chord as follows.

$$3x^2 - y^2 - 2x\left(\frac{y-mx}{c}\right) + 4y\left(\frac{y-mx}{c}\right) = 0$$

$\Rightarrow \quad 3cx^2 - cy^2 - 2xy + 2mx^2 + 4y^2 - 4mxy = 0$

$\Rightarrow \quad (3c + 2m)x^2 - 2(1 + 2m)xy + (4 - c)y^2 = 0.$

As the lines represented by this pair are perpendicular to each other, therefore we must have coefficient of x^2 + coefficient of $y^2 = 0$. Hence,

$3c + 2m + 4 - c = 0$

$\Rightarrow \quad m + c + 2 = 0$

Comparing this result with $y = mx + c$, we can see that $y = mx + c$ passes though $(1, -2)$.

20. See the solution to problem 5 in multiple correct answers type problems.

21. Let θ be the inclination of line through $A(-5, -4)$. Then equation of the line is

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r$$

$\Rightarrow \quad B \equiv (r_1\cos\theta - 5, r_1\sin\theta - 4)$

$C \equiv (r_2\cos\theta - 5, r_2\sin\theta - 4)$

$D \equiv (r_3\cos\theta - 5, r_3\sin\theta - 4)$

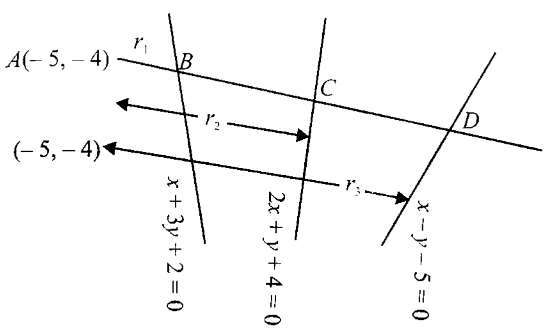


Fig. 1.240

But B lies on $x + 3y + 2 = 0$, therefore

$r_1\cos\theta - 5 + 3r_1\sin\theta - 12 + 2 = 0$

$\Rightarrow \quad \frac{15}{\cos\theta + 3\sin\theta} = r_1$

$\Rightarrow \quad \frac{15}{AB} = \cos\theta + 3\sin\theta \quad \text{(i)}$

As C lies on $2x + y + 4 = 0$, therefore

$2(r_2\cos\theta - 5) + (r_2\sin\theta - 4) + 4 = 0.$

$\Rightarrow \quad r_2 = \frac{10}{2\cos\theta + \sin\theta} = AC$

$\Rightarrow \quad \frac{10}{AC} = 2\cos\theta + \sin\theta \quad \text{(ii)}$

Similarly D lies on $x - y - 5 = 0$, therefore

$r_3\cos\theta - 5 - r_3\sin\theta + 4 - 5 = 0$

$\Rightarrow \quad r_3 = \frac{6}{\cos\theta - \sin\theta} = AD$

$\Rightarrow \quad \frac{6}{AD} = \cos\theta - \sin\theta \quad \text{(iii)}$

Now, given that

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$\Rightarrow (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

[Using (i), (ii) and (iii)]

$\Rightarrow \quad 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$

$\Rightarrow \quad 2\cos\theta + 3\sin\theta = 0$

$\Rightarrow \quad \tan\theta = -2/3$

Hence, equation of required line is $y + 4 = -\frac{2}{3}(x + 5)$

$\Rightarrow \quad 3y + 12 = -2x - 10$

$\Rightarrow \quad 2x + 3y + 22 = 0$

22. See the solution of problems 7–9 in linked comprehension type problems.

23. The line $y = mx$ meets the given lines in $P\left(\frac{1}{m+1}, \frac{m}{m+1}\right)$ and $Q\left(\frac{3}{m+1}, \frac{3m}{m+1}\right)$. Hence, equation of L_1 is

$$\left(y - \frac{m}{m+1}\right) = 2\left(x - \frac{1}{m+1}\right)$$

$\Rightarrow \quad y - 2x - 1 = -\frac{3}{m+1} \quad \text{(i)}$

and that of L_2 is

$$\left(y - \frac{3m}{m+1}\right) = -3\left(x - \frac{3}{m+1}\right)$$

$\Rightarrow \quad y + 3x - 3 = \frac{6}{m+1} \quad \text{(ii)}$

From (i) and (ii), eliminating 'm' we get

$$\frac{y-2x-1}{y+3x-3} = -1/2$$

$\Rightarrow \quad x - 3y + 5 = 0$

which is a straight line.

24. Let the equation of the line be

$$(y-2) = m(x-8) \text{ where } m < 0$$

$\Rightarrow \quad P \equiv \left(8 - \frac{2}{m}, 0\right)$ and $Q \equiv (0, 2 - 8m)$

Now,

$$OP + OQ = \left|8 - \frac{2}{m}\right| + |2 - 8m|$$

$$= 10 + \frac{2}{-m} + (-8m)$$

$$\geq 10 + 2\sqrt{\frac{2}{-m} \times (-8m)} \geq 18$$

25. A line passing through $P(h, k)$ and parallel to x-axis is

$$y = k \qquad \text{(i)}$$

The other two lines given are

$$y = x \qquad \text{(ii)}$$

and $\quad x + y = 2 \qquad \text{(iii)}$

Let ABC be the Δ formed by the points of intersection of the lines (i), (ii) and (iii), as shown in the figure.

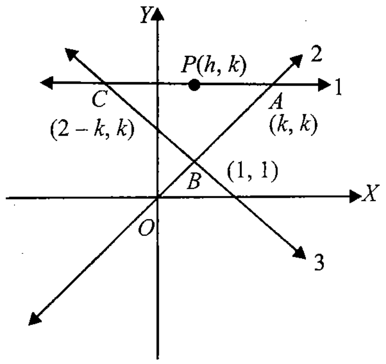


Fig. 1.241

Then $\quad A \equiv (k, k), B \equiv (1, 1), C \equiv (2-k, k)$.

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} k & k & 1 \\ 1 & 1 & 1 \\ 2-k & k & 1 \end{vmatrix} = 4h^2$

Operating $C_1 \to C_1 - C_2$ we get

$$\frac{1}{2}\begin{vmatrix} 0 & k & 1 \\ 0 & 1 & 1 \\ 2-2k & k & 1 \end{vmatrix} = 4h^2$$

$\Rightarrow \frac{1}{2}|(2-2k)(k-1)| = 4h^2$

$\Rightarrow \quad (k-1)^2 = 4h^2$

$\Rightarrow \quad k - 1 = 2h$ or $k - 1 = -2h$

$\Rightarrow \quad k = 2h + 1$ or $k = -2h + 1$

Hence, locus of (h, k) is

$$y = 2x + 1 \text{ or } y = -2x + 1.$$

26. See the solution to similar problem 20 in Objective Type. The answer is

$(m^2 - 1)x - my + b(m^2 + 1) + am = 0$.

Objective Type

Fill in the blanks

1. $|x| + |y| = 1$

The curve represents four lines : $x + y = 1, x - y = 1, -x + y = 1, -x - y = 1$

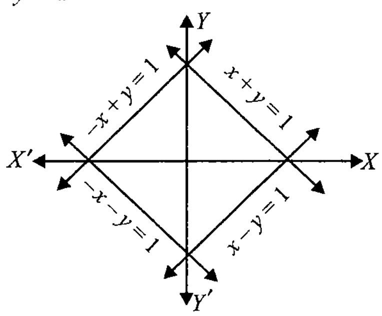


Fig. 1.242

They enclose a square of side equal to the distance between opposite sides $x + y = 1$ and $x + y = -1$, which is given by $(1 + 1)/\sqrt{1+1} = \sqrt{2}$. Therefore, the required area is 2 sq. units.

2. Given that $3a + 2b + 4c = 0$. Hence,

$$\frac{3}{4}a + \frac{1}{2}b + c = 0$$

Hence, the set of lines $ax + by + c = 0$ pass through the points (3/4, 1/2). Therefore, the given lines are concurrent at the point (3/4, 1/2).

3. If a, b, c are in $A.P.$, then

$$a + c = 2b$$

$\Rightarrow \quad a - 2b + c = 0$

Hence, $ax + by + c = 0$ passes through $(1, -2)$.

4. The equations of sides of triangle ABC are as follows:

$$AB: x + y = 1$$

$$BC: 2x + 3y = 6$$

$$CA: 4x - y = -4$$

Solving these pairwise, we get the vertices of the triangle as

$A(-3/5, 8/5), B(-3, 4), C(-3/7, 16/7)$.

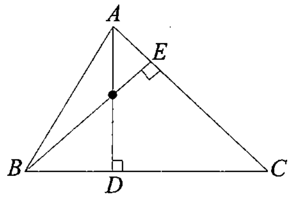


Fig. 1.243

Now AD is perpendicular to BC and passes through A. Any line perpendicular to BC is $3x - 2y + \lambda = 0$. As it passes through $(-3/5, 8/5)$, we have

$$\frac{-9}{5} - \frac{16}{5} + \lambda = 0$$

$$\Rightarrow \quad \lambda = 5$$

Hence, the equation of altitude AD is

$$3x - 2y + 5 = 0 \quad \text{(i)}$$

Any line perpendicular to side AC is $x + 4x + \mu = 0$. As it passes through point $B(-3, 4)$, we have

Therefore, $-3 + 16 + \mu = 0 \Rightarrow \mu = -13$

Therefore, equation of altitude BE is

$$x + 4x - 13 = 0 \quad \text{(ii)}$$

Now orthocentre is the intersection point of Eqs. (i) and (ii) (AD and BE).

Solving (i) and (ii), we get $x = 3/7$, $y = 22/7$. Hence, orthocentre lies in first quadrant.

5. Let the variable line be

$$ax + by + c = 0 \quad \text{(i)}$$

Then perpendicular distance of line from (2, 0) is

$$p_1 = \frac{2a + c}{\sqrt{a^2 + b^2}}$$

The perpendicular distance of line from (0, 2) is

$$p_2 = \frac{2b + c}{\sqrt{a^2 + b^2}}$$

The perpendicular distance of line from (1, 1) is

$$p_3 = \frac{a + b + c}{\sqrt{a^2 + b^2}}$$

According to question

$$p_1 + p_2 + p_3 = 0$$

$$\Rightarrow \frac{2a + c + 2b + c + a + b + c}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \quad 3a + 3b + 3c = 0$$

$$\Rightarrow \quad a + b + c = 0 \quad \text{(ii)}$$

From (i) and (ii), we can say that variable line (i) passes through the fixed point (1, 1).

6. Let BD be the bisector of $\angle ABC$. Then,

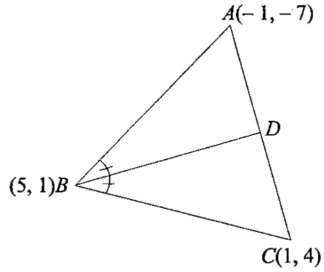


Fig. 1.244

$$AD:DC = AB:BC$$

and

$$AB = \sqrt{(5+1)^2 + (1+7)^2} = 10$$

$$BC = \sqrt{(5-1)^2 + (1-4)^2} = 5$$

$$\therefore \quad AD:DC = 2:1$$

Therefore, by section formula, $D \equiv (1/3, 1/3)$. Therefore, equation of BD is

$$y - 1 = \frac{1/3 - 1}{1/3 - 5}(x - 5)$$

$$\Rightarrow \quad y - 1 = \frac{-2/3}{-14/3}(x - 5)$$

$$\Rightarrow \quad 7y - 7 = x - 5$$

$$\Rightarrow \quad x - 7y + 2 = 0$$

True or false

1. For the given lines, we have $\begin{vmatrix} 5 & 4 & 0 \\ 1 & 2 & -10 \\ 2 & 1 & 5 \end{vmatrix} = 5(10 + 10) - 4(5 + 20)$

$$= 100 - 100 = 0$$

Hence, the statement is true.

2. The given lines cut x-axis at A (17/9, 0) and C (– 19/2, 0).

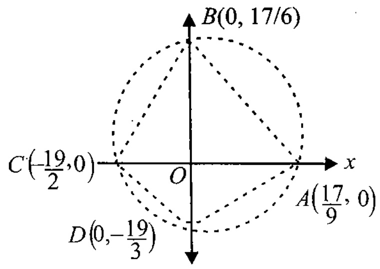


Fig. 1.245

The given lines cut y-axis at $B\left(0, \frac{17}{6}\right)$ and $D\left(0, -\frac{19}{3}\right)$. Now A, B, C, D are concyclic, If $AO \times OC = BO \times OD$, which holds. Hence, the given statement is true.

Multiple choice questions with one correct answer

1. c. Reflection about the line $y = x$, changes the point (4, 1) to (1, 4). On translation of (1, 4) through a distance of

2 units along +*ve* direction of x-axis the point becomes (1 + 2, 4), i.e., (3, 4).

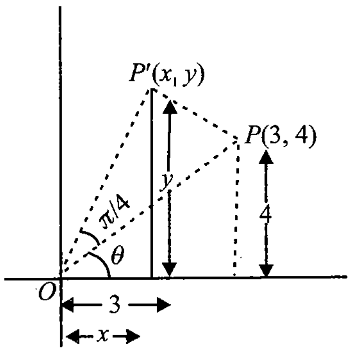


Fig. 1.246

On rotation about origin through an angle $\pi/4$, the point P takes the position P' such that $OP = OP'$. Also $OP = 5 = OP'$ and $\cos\theta = 3/5$, $\sin\theta = 4/5$. Now,

$$x = OP' \cos\left(\frac{\pi}{4} + \theta\right)$$

$$= 5\left(\cos\frac{\pi}{4}\cos\theta - \sin\theta\right)$$

$$= 5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}$$

$$y = OP' \sin\left(\frac{\pi}{4} + \theta\right)$$

$$= 5\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)$$

$$= 5\,\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$\therefore \quad P' \equiv \left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

2. **a.** Solving the given equations of lines pairwise, we get the vertices of triangle as $A(-2, 2)$, $B(2, -2)$, $C(1, 1)$. Then,

$$AB = \sqrt{16 + 16} = 4\sqrt{2}$$

$$BC = \sqrt{1 + 9} = \sqrt{10}$$

$$CA = \sqrt{9 + 1} = \sqrt{10}$$

Hence, the triangle is isosceles.

3. **d.** We have $P = (1, 0)$, $Q = (-1, 0)$, $R = (2, 0)$. Let $S = (x, y)$. Now given that $SQ^2 + SR^2 = 2SP^2$. Hence,

$$(x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2\,[(x - 1)^2 + y^2]$$

$$\Rightarrow \quad 2x^2 + 2y^2 - 2x + 5 = 2x^2 + 2y^2 - 4x + 2$$

$$\Rightarrow \quad 2x + 3 = 0$$

$$\Rightarrow \quad x = -3/2$$

which is a straight line parallel to y-axis.

4. **b.** As L has intercepts a and b on the axes, equation of L is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(i)}$$

Let x-and y-axes be rotated through an angle θ in anticlockwise direction. In new system, intercepts are p and q, therefore equation of L becomes

$$\frac{x}{p} + \frac{y}{q} = 1 \quad \text{(ii)}$$

As the origin is fixed in rotation, the distance of line from origin in both the cases should be same. Hence, we get

$$d = \left|\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = \left|\frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}}\right|$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$$

5. **a.** Let the two perpendicular lines be the coordinate axes. Let (x, y) be the point, sum of whose distances from two axes is 1. Then we must have

$$|x| + |y| = 1$$

or

$$\pm x \pm y = 1$$

These are the four lines $x + y = 1$, $x - y = 1$, $-x + y = 1$, $-x - y = 1$. Any two adjacent sides are perpendicular to each other. Also, each line is equidistant from origin. Therefore, figure formed is a square.

6. **c.** The lines by which triangle is formed are $x = 0$, $y = 0$ and $x + y = 1$. Clearly, it is right triangle and we know that in a right angled triangle orthocentetre coincides with the vertex at which right angle is formed. Therefore, orthocentre is (0, 0).

7. **b.**

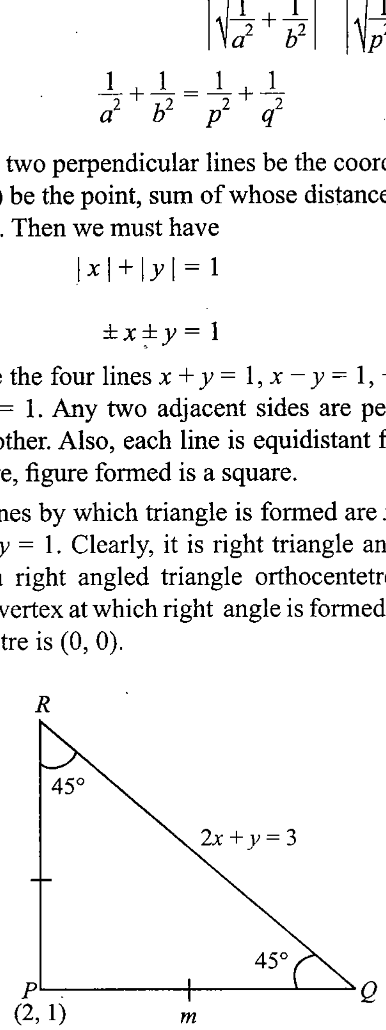


Fig. 1.247

Let m be the slope of PQ. Then.

$$\tan 45° = \left|\frac{m - (-2)}{1 + m(-2)}\right| = \left|\frac{m + 2}{1 - 2m}\right|$$

$$\Rightarrow \quad m + 2 = 1 - 2m \text{ or } -1 + 2m = m + 2$$

$\Rightarrow \quad m = -1/3$ or $m = 3$

Hence, equation of PQ is

$$y - 1 = -\frac{1}{3}(x - 2)$$

or

$$x + 3y - 5 = 0$$

and equation of PR is

$$3x - y - 5 = 0$$

Hence, combined equation of PQ and PR is

$$(x + 3y - 5)(3x - y - 5) = 0$$

$$\Rightarrow 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$$

8. a. Let $x_2 = x_1 r, x_3 = x_1 r^2$ and so is $y_2 = y_1 r, y_3 = y_1 r^2$ where r is common ratio.

$$\therefore \quad \Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

$$= r \times r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}$$

$$= 0$$

Hence, the points are collinear.

9. d. S is the midpoint of Q and R. Therefore, $S = ((7 + 6)/2, (3 - 1)/2) = (13/2, 1)$.

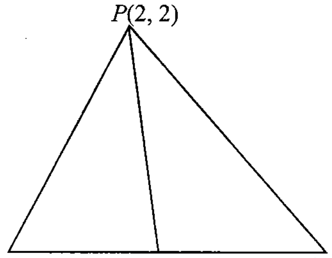


Fig. 1.248

Now slope of PS is $m = \frac{2 - 1}{2 - 13/2} = -\frac{2}{9}$

Then equation of the line passing through $(1, -1)$ and parallel to PS is

$$y + 1 = -\frac{2}{9}(x - 1)$$

or

$$2x + 9y + 7 = 0$$

10. d. Here $AB = BC = CA = 2$. So, it is an equilateral triangle and the incentre coincides with centroid. Therefore, centroid is

$$\left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) = \left(1, \frac{1}{\sqrt{3}}\right)$$

11. a. x-coordinate of the point of intersection is

$$3x + 4(mx + 1) = 9$$

$$\Rightarrow \quad (3 + 4m)x = 5 \Rightarrow x = \frac{5}{3 + 4m}$$

For x to be an integer $3 + 4m$ should be a divisor of 5, i.e., $1, -1, 5$ or -5. Hence,

$3 + 4m = 1 \Rightarrow m = -1/2$ (not integer)

$3 + 4m = -1 \Rightarrow m = -1$ (integer)

$3 + 4m = 5 \Rightarrow m = 1/2$ (not an integer)

$3 + 4m = -5 \Rightarrow m = -2$ (integer)

Hence, there are two integral values of m.

12. d.

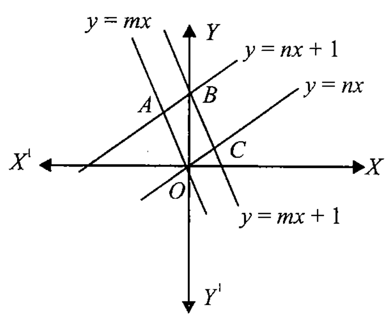


Fig. 1.249

$y = mx$ is a line through $(0, 0)$, $y = mx + 1$ is a line parallel to above line having y-intercept 1.

The vertices are $O(0, 0)$, $A(1/(m - n), m/(m - n))$. Area of parallelogram is given by

$$2 \times \text{Ar}(\Delta OAB)$$

$$= 2 \times \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{1}{m-n} & \frac{m}{m-n} & 1 \end{vmatrix}$$

$$= \frac{1}{|m - n|}$$

13. d. Clearly $OP = OQ = 1$ and $\angle QOP = \alpha - \theta - \theta = \alpha - 2\theta$.

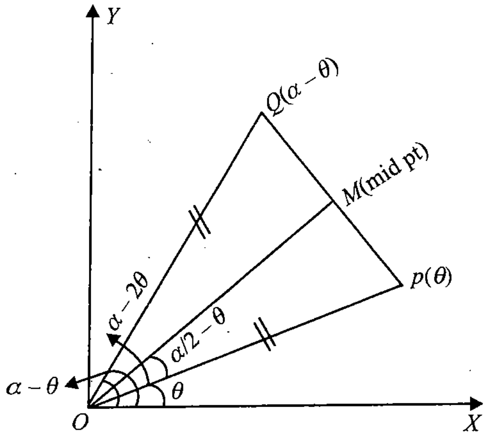


Fig. 1.250

The bisector of $\angle QOP$ will be perpendicular to PQ and also bisect it. Hence, Q is reflection of P in the line OM which makes an angle $\angle MOP + \angle POX$ with x-axis, i.e.,

$\frac{1}{2}(\alpha - 2\theta) + \theta = \alpha/2$, so that slope of OM is $\tan(\alpha/2)$.

14. c. Slope of QR is $(3\sqrt{3} - 0)/(3 - 0) = \sqrt{3}$, i.e., $\theta = 60°$.

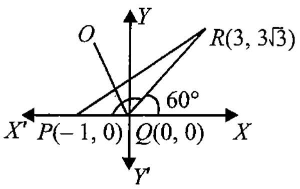


Fig. 1.251

Clearly, $\angle PQR = 120°$. OQ is the angle bisector of the angle, so line OQ makes 120° with the positive direction of x-axis. Therefore, equation of the bisector of $\angle PQR$ is $y = \tan 120°\, x$ or $y = -\sqrt{3}x$, i.e., $\sqrt{3}x + y = 0$.

15. b. The given lines are

$$2x + y = 9/2 \quad \text{(i)}$$
$$2x + y = -6 \quad \text{(ii)}$$

Signs of constants on R.H.S. show that two lines lie on opposite sides of origin. Let any line through origin meet these lines at P and Q, respectively. Then required ratio is OP:OQ.

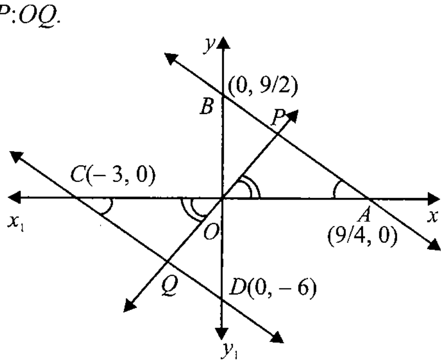


Fig. 1.252

In particular, OP can be taken perpendicular to the two parallel lines. Then required ratio is equal to the ratio of perpendicular distances of lines from origin,

i.e., $\frac{9}{\sqrt{20}} : \frac{6}{\sqrt{5}} \equiv 3:4$

16. b. Total number of points within the square $OABC$ is $20 \times 20 = 400$.

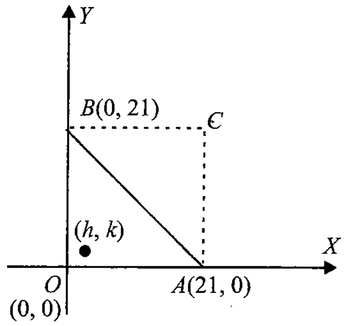


Fig. 1.253

Total number of points on line AB is 20, the points being (1, 1), (2, 2), ..., (20, 20). Therefore, total number of points within ΔOBC and ΔABC is $400 - 20 = 380$. By symmetry, number of points within ΔOAB is $380/2 = 190$.

17. c. We know that orthocentre is the meeting point of altitudes of a triangle.

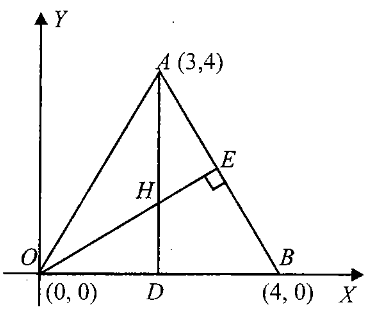


Fig. 1.254

Equation of altitude AD is

$$x = 3 \quad \text{(i)}$$

Similarly, equation of altitude OE is

$$y = -\frac{3-4}{4-0}\, x \Rightarrow y = x/4 \quad \text{(ii)}$$

Solving (i) and (ii), we get orthocentre as (3, 3/4).

18. a. $x^2 - y^2 + 2y = 1 \Rightarrow x = \pm(y - 1)$

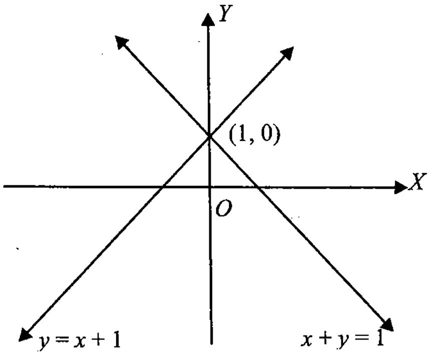


Fig. 1.255

Bisectors of above lines are $x = 0$ and $y = 1$.

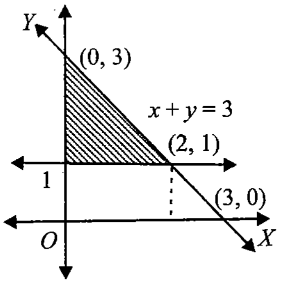


Fig. 1.256

So area between $x = 0$, $y = 1$ and $x + y = 3$ is the shaded region shown in figure. The area is given by $(1/2) \times 2 \times 2 = 2$ sq. units.

19. c. The sides of parallelogram are $x = 2$, $x = 3$, $y = 1$, $y = 5$.

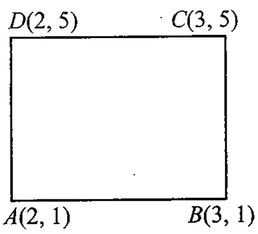


Fig. 1.257

The equation of diagonal AC is

$$= \frac{y-1}{5-1} = \frac{x-2}{3-5}$$

or

$$= 4x - 7$$

and equation of diagonal BD is

$$\frac{x-2}{3-2} = \frac{y-5}{1-5}$$

or

$$4x + y = 13$$

20. c.

$$Ar(\Delta OPR) = Ar(\Delta PQR) = Ar(\Delta OQR)$$

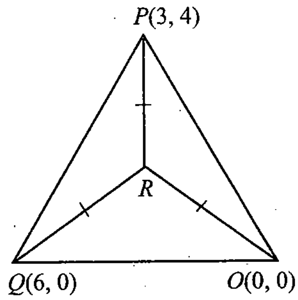


Fig. 1.258

R should be the centroid of ΔPQO whose coordinates are

$$\left(\frac{3+6+0}{3}, \frac{4+0+0}{3}\right) = \left(3, \frac{4}{3}\right)$$

21. d. $P \equiv (-\sin(\beta - \alpha), -\cos\beta) \equiv (x_1, y_1)$

$Q \equiv (\cos(\beta - \alpha), \sin\beta) \equiv (x_2, y_2)$

$R \equiv (x_2 \cos\theta + x_1 \sin\theta, y_2 \cos\theta + y_1 \sin\theta)$

If

$$T \equiv \left(\frac{x_2 \cos\theta + x_1 \sin\theta}{\cos\theta + \sin\theta}, \frac{y_2 \cos\theta + y_1 \sin\theta}{\cos\theta + \sin\theta}\right)$$

then P, Q, T are collinear. Hence, P, Q, R are noncollinear.

22. d.

Intersection point of $y = 0$ with first line is $B(-p, 0)$

Intersection point of $y = 0$ with second line is $A(-q, 0)$

Intersection point of the two lines is $C(pq, (p + 1)(q + 1))$

Altitude from C to AB is $x = pq$

Altitude from B to AC is $y = -\frac{q}{1+q}(x + p)$

Solving these two, we get $x = pq$ and $y = -pg$

$\Rightarrow$ locus of orthocentre is $x + y = 0$.

23. b.

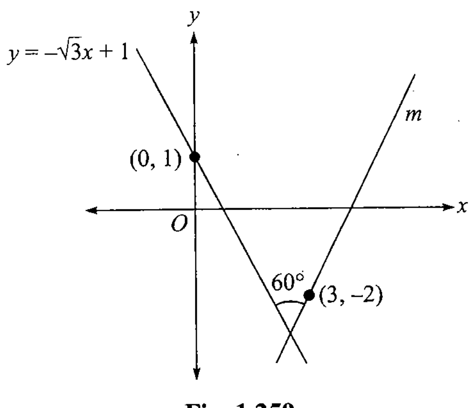


Fig. 1.259

Let slope of required line is m

$$\Rightarrow \left|\frac{m+\sqrt{3}}{1-\sqrt{3}m}\right| = \sqrt{3}$$

$$\Rightarrow m + \sqrt{3} = \pm(\sqrt{3} - 3m)$$

$$\Rightarrow 4m = 0 \Rightarrow m = 0$$

or $2m = 2\sqrt{3} \Rightarrow m = \sqrt{3}$

$\therefore$ Equation is $y + 2 = \sqrt{3}\,(x - 3)$ ($m \neq 0$ as given that line cuts x-axis.)

$$\Rightarrow \sqrt{3}x - y - (2 + 3\sqrt{3}) = 0$$

Multiple choice questions with one or more than one correct answer

1. a., b., c.

For concurrency, of three lines $px + qy + r = 0$, $qx + ry + p = 0$, $rx + py + q = 0$, we must have,

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

$$\Rightarrow 3pqr - p^3 - q^3 - r^3 = 0$$

$$\Rightarrow (p+q+r)(p^2+q^2+r^2-pq-pr-rq) = 0$$

2. d. Let $A \equiv (0, 8/3)$, $B \equiv (1, 3)$ and $C \equiv (82, 30)$

Now, slope of line AB is $(3 - 8/3)/(1 - 0) = 1/3$. Slope of line BC is $(30 - 3)/(82 - 1) = 27/81 = 1/3$. Therefore, $AB \parallel BC$ and B is common point. Hence, A, B, C are collinear.

3. a., c. Substituting the coordinates of the points (1, 3) (5, 0) and (–1, 2) in $3x + 2y$, we obtain the values 9, 15 and 1 which are all +ve. Therefore, all the points lying inside the triangle formed by given points satisfy $3x + 2y \geq 0$.

Substituting the co-ordinates of the given points in $2x + 3y - 13$, we find the values –2, –3, and –9 which are all –ve. So, (b) is not correct

Again substituting the given points in $2x - 3y - 12$ we get, –19, –2, –20 which are all –ve. It follows that all points lying inside the Δ formed by given point satisfy $2x - 3y - 12 \leq 0$. So (c) is the correct answer.

Finally substituting the coordinates of the given point in $-2x + y$, we get 1, – 10 and 4 which are not all +ve. So (d) is not correct.

4. c. $PQRS$ will represent a parallelogram if and only if the mid-point of PR is same as that of the mid-point of QS. Hence,

$$\frac{1+5}{2} = \frac{4+a}{2} \text{ and } \frac{2+7}{2} = \frac{6+b}{2}$$

$$\Rightarrow \quad a = 2 \text{ and } b = 3.$$

5. d. Slope of $x + 3y = 4$ is $-1/3$ and slope of $6x - 2y = 7$ is 3. Therefore, these two lines are perpendicular which shows that both diagonals are perpendicular. Hence, $PQRS$ must be a rhombus.

6. a., c., d. See solutions in examples.

Assertion and reasoning

1. c. Point of intersection of L_1 and L_2 is A (0, 0). Also, P (–2, –2), Q (1, – 2).

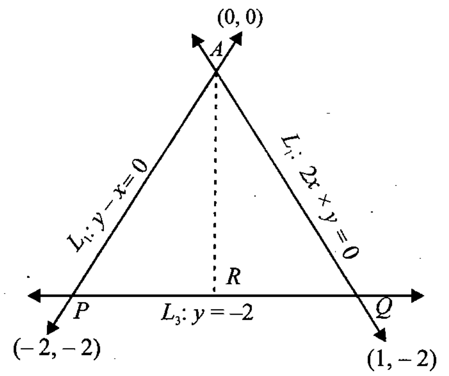


Fig. 1.260

Since AR is the bisector of $\angle PAQ$, therefore R divides PQ in the same ratio as $AP{:}AQ$. Thus $PR : RQ = AP{:}\,AQ = 2\sqrt{2} : \sqrt{5}$. Hence, statement 1 is true. Statement 2 is clearly false

Matrix-match

1. a. → s.; b → p., q; c → r; d → p., q., s.

$x + 3y - 5 = 0$ and $5x + 2y - 12 = 0$ intersect at (2, 1). Hence,

$$6 - k - 1 = 0 \Rightarrow k = 5$$

For L_1, L_2 to be parallel,

$$\frac{1}{3} = \frac{3}{-k} \quad \Rightarrow k = -9$$

For L_2, L_3 to be parallel,

$$\frac{3}{5} = -\frac{k}{2} \Rightarrow k = -\frac{6}{5}$$

For $\quad k \neq 5, -9, -\frac{6}{5}$, these form triangle

For $k = 5$, $k = -9$, $\frac{-6}{5}$, these will not form triangle.

CHAPTER

2

Circle

- Definition
- Different Forms of the Equations of a Circle
- Position of a Point with Respect to a Circle
- Intersection of a Line and a Circle
- Tangent to a Circle at a Given Poin
- Normal to a Circle at a Given Point
- Chord of Contact
- Equation of the Chord Bisected at a Given Point
- Director Circle and Its Equation
- Intersection of Two Circles
- Angle of Intersection of Two Circles
- Radical Axis
- Common Chord of Two Circles
- Family of Circles
- Problems Based on Locus

DEFINITION

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in plane is always a constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

Equation of Circle with Centre (*h*, *k*) and Radius *r*

The equation of circle is

$$(x-h)^2+(y-k)^2=r^2 \quad \text{(i)}$$

In particular: If the centre is at the origin, the equation of circle is $x^2+y^2=r^2$.

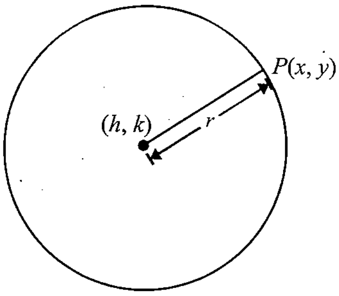


Fig. 2.1

General Equation of a Circle

The general equation of circle is

$$x^2+y^2+2gx+2fy+c=0, \quad \text{(ii)}$$

where g, f and c are constants.

To find the centre and radius: Eq. (ii) can be written as

$$(x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$$

Comparing with the equation of circle given in Eq. (i),

$\therefore \quad h=-g,\ k=-f$

and $\quad r=\sqrt{g^2+f^2-c}$

$\therefore$ Co-ordinates of the centre are $(-g, -f)$ and radius $=\sqrt{g^2+f^2-c}$, $(g^2+f^2\ge c)$

Note:

1. *A general equation of second degree* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *in x, y represent a circle if*
   i. *coefficient of* x^2 = *coefficient of* y^2, *i.e.* $a=b$.
   ii. *coefficient of xy is zero, i.e.* $h=0$.
   iii. $\Delta=abc+2hgf-af^2-bg^2-ch^2\neq 0$
   However for a point circle (whose radius is zero), $\Delta=0$
2. *Rule to find the centre and radius of a circle whose equation is given*:
   i. *Make the coefficients of* x^2 *and* y^2 *equal to 1 and right hand side equal to zero.*
   ii. *Then co-ordinates of centre will be* (α,β), *where* $\alpha=-\frac{1}{2}$ *(coefficient of x) and* $\beta=-\frac{1}{2}$ *(coefficient of y).*
   iii. *Radius* $=\sqrt{(\alpha^2+\beta^2-\text{constant term})}$
3. *Nature of the circle: radius of the circle* $x^2+y^2+2gx+2fy+c=0$ *is* $\sqrt{(g^2+f^2-c)}$.
   Now the following cases are possible:
   i. *If* $g^2+f^2-c>0$, *then the radius of circle will be real. Hence in this case, real circle is possible.*
   ii. *If* $g^2+f^2-c=0$, *then the radius of circle will be zero. Hence, in this case circle is called a point circle.*
   iii. *If* $g^2+f^2-c<0$, *then the radius of circle will be imaginary number. Hence, in this case, circle is called a virtual circle or imaginary circle.*
4. *Concentric circles: Two circles having the same centre C (h, k) but different radii* r_1 *and* r_2, *respectively, are called concentric circles. Thus, the circles* $(x-h)^2+(y-k)^2=r_1^2$ *and* $(x-h)^2+(y-k)^2=r_2^2$, $r_1\neq r_2$, *are concentric circles. Therefore, the equations of concentric circles differ only in constant terms.*

Example 2.1 **If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then find the radius of the circle.**

Sol. The diameter of circle is perpendicular distance between the parallel lines (tangents) $3x-4y+4=0$ and $3x-4y-\frac{7}{2}=0$ and so it is equal to $\frac{|4-(-7/2)|}{\sqrt{9+16}}=\frac{3}{2}$. Hence, radius is $\frac{3}{4}$.

Example 2.2 **If the equation $px^2+(2-q)xy+3y^2-6qx+30y+6q=0$ represents a circle, then find the values of *p* and *q*.**

Sol. In the equation of circle, xy term does not occur and the coefficient of x^2 and y^2 are equal.

Therefore, $\quad 2-q=0 \Rightarrow q=2$ and $p=3$.

Also for this value of p and q circle is real.

Example 2.3 **A point *P* moves in such a way that the ratio of its distance from two coplanar points is always a fixed number (≠ 1). Then, identify the locus of the point.**

Sol. Let two coplanar points are (0, 0) and (*a*, 0). Under given conditions, we get

$$\frac{\sqrt{x^2+y^2}}{\sqrt{(x-a)^2+y^2}}=\lambda,\ (\lambda\neq 1)$$

(where λ is any fixed number)

$$\Rightarrow x^2+y^2+\left(\frac{\lambda^2}{\lambda^2-1}\right)(a^2-2ax)=0$$

which is the equation of a circle.

Example 2.4 **Prove that the maximum number of points with rational coordinates on a circle whose centre is $(\sqrt{3}, 0)$ is two.**

Sol. There cannot be three points on the circle with rational coordinates as for then the centre of the circle, being the circumcentre of a triangle whose vertices have rational coordinates, must have rational coordinates ($\because$ the coordinates will be obtained by solving two linear equations in x, y having rational coefficients). But the point $(\sqrt{3}, 0)$ does not have rational coordinates. Also, the equation of the circle is

$$(x-\sqrt{3})^2 + y^2 = r^2 \Rightarrow x = \sqrt{3} \pm \sqrt{r^2 - y^2}$$

For suitable r, x, where x is rational, y may have two rational values.

For example, $r = 2$, $x = 0$, $y = 1, -1$, satisfy $x = \sqrt{3} \pm \sqrt{r^2 - y^2}$.

So, we get two points (0, 1), (0, −1) which have rational coordinates.

Example 2.5 **Prove that for all values of θ, the locus of the point of intersection of the lines $x \cos\theta + y \sin\theta = a$ and $x \sin\theta - y\cos\theta = b$ is circle.**

Sol. Since point of intersection satisfies both the given lines we can find locus by eliminating θ from given equation.

Therefore, by squaring and adding, we get equation $x^2 + y^2 = a^2 + b^2$ which is equation of circle.

Example 2.6 **Find the length of the chord cut-off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$.**

Sol.

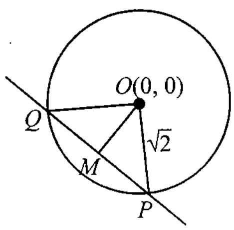


Fig. 2.2

We have, OM = length of the perpendicular from (0, 0) on $y = 2x + 1 \Rightarrow OM = \frac{1}{\sqrt{5}}$ and, OP = radius of the given circle $= \sqrt{2}$.

$$\therefore \quad PQ = 2PM = 2\sqrt{OP^2 - OM^2}$$

$$= 2\sqrt{2 - \frac{1}{5}} = \frac{6}{\sqrt{5}}$$

Example 2.7 **Let $A \equiv (-2, -2)$ and $B \equiv (2, -2)$ be two points and AB substends an angle of 45° at any point P in the plane in such a way that area of ΔAPB is 8 square unit, then find the number of possible position(s) of P.**

Sol.

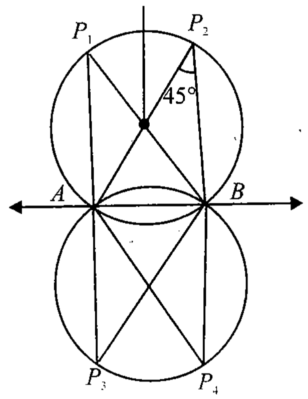


Fig. 2.3

$$\because \quad AB = 4 \text{ and area of } \Delta APB = 8$$

$$\therefore \quad \frac{1}{2} \times 4 \times h = 8 \Rightarrow h = 4;$$

h the height of ΔAPB.

From Fig. 2.3, it is clear that P lies on circle of radius $2\sqrt{2}$ unit with AB as its chord so there are four possible positions of P.

Example 2.8 **An acute triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (− 4, 3), respectively, then find $\angle QPR$.**

Sol.

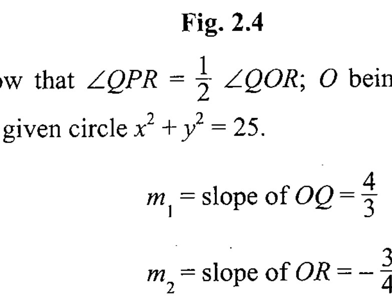


Fig. 2.4

We know that $\angle QPR = \frac{1}{2} \angle QOR$; O being the centre (0, 0) of the given circle $x^2 + y^2 = 25$.

Let $\quad m_1 = \text{slope of } OQ = \frac{4}{3}$

and $\quad m_2 = \text{slope of } OR = -\frac{3}{4}$

As $\quad m_1 m_2 = -1, \angle QOR = \frac{\pi}{2}$

$\Rightarrow \quad \angle QPR = \frac{\pi}{4}$

Example 2.9 **Two tangents to the circle $x^2 + y^2 = 4$ at the points A and B meet at $P(-4, 0)$. Then, find the area of the quadrilateral $PAOB$, where O is the origin**

Sol.

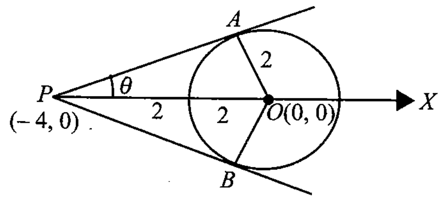


Fig. 2.5

Clearly, $\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\therefore$ $\theta = 30°$

So, $ar(\Delta POA) = \frac{1}{2} \times 2 \times 4 \times \sin 60°$

$\therefore$ $ar(\text{quad. } PAOB) = 2 \times \frac{1}{2} \times 2 \times 4 \sin 60°$

$= 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3}$.

Equation of a Circle Passing Through Three Given Points

The general equation of circle, i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three independent constants g, f and c. Hence, for determining the equation of a circle, three conditions are required.

i. The equation of the circle through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$:

Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$ (i)

If three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on the circle Eq. (i), their coordinates must satisfy its equation. Hence, solving equations

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ (ii)

$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$ (iii)

$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$ (iv)

g, f, c are obtained from (ii), (iii) and (iv).

Note:

Concyclic quadrilateral: If all the four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concyclic.

Example 2.10 **If a circle passes through the point (0, 0), (a, 0), (0, b), then find its centre.**

Sol: Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Now on passing through the points, we get three equations:

$c = 0$ (i)

$a^2 + 2ga + c = 0$ (ii)

$b^2 + 2fb + c = 0$ (iii)

On solving them, we get $g = -\frac{a}{2}, f = -\frac{b}{2}$

Hence, the centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Example 2.11 **Find the equation of circle which passes through the points (1, – 2), (4, – 3) and whose centre lies on the line $3x + 4y = 7$.**

Sol: Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

If Eq. (i) passes through the points (1, – 2) and (4, – 3), then

$5 + 2g - 4f + c = 0$ (ii)

and $25 + 8g - 6f + c = 0$ (iii)

Since the centre $(-g, -f)$ lies on the line $3x + 4y = 7$,

$\therefore$ $-3g - 4f = 7$ (iv)

Solving Eqs. (ii), (iii) and (iv), we get

$g = -\frac{47}{15}, f = \frac{3}{5}$ and $c = \frac{11}{3}$.

Substituting in Eq. (i), the equation of the circle is

$15x^2 + 15y^2 - 94x + 18y + 55 = 0.$

Example 2.12 **Show that a cyclic quadrilateral is formed by the lines $5x + 3y = 9$, $x = 3y$, $2x = y$ and $x + 4y + 2 = 0$ taken in order. Find the equation of the circumcircle.**

Sol. Solving the given equation in pairs taken in order, the coordinates of the vertices of the quadrilateral $ABCD$ are $A\left(\frac{3}{2}, \frac{1}{2}\right)$, $B(0, 0)$, $C\left(-\frac{2}{9}, -\frac{4}{9}\right)$ and $D\left(\frac{42}{17}, -\frac{19}{17}\right)$.

First, we shall find the equation of the circle passing through A, B and C. Let the equation of this circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Coordinates of the points A, B and C must satisfy Eq. (i), so substituting in Eq. (i), we have

$3g + f + c + \frac{5}{2} = 0$ (ii)

$c = 0$ (iii)

and $4g + 8f - 9c = \frac{20}{9}$ (iv)

Solving Eqs. (ii), (iii) and (iv), we get $g = -\frac{10}{9}, f = \frac{5}{6}$ and $c = 0$.

Substituting in Eq. (i), the equation of the circle through the vertices A, B, C is

$9x^2 + 9y^2 - 20x + 15y = 0$ (v)

Since the coordinates of the vertex $D\left(\frac{42}{17}, -\frac{19}{17}\right)$ also satisfy Eq. (v), hence a cyclic quadrilateral $ABCD$ is formed by the given lines, and Eq. (v) is the equation of the circumcircle of the quadrilateral.

Equation of Circle on a Given Diameter

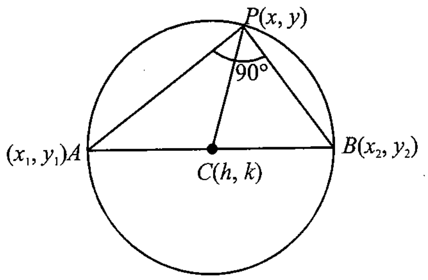


Fig. 2.6

To find the equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter. We will find the locus of point P such that $\angle APB = \pi/2$.

In the figure given above, $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter and let $P(x, y)$ be any point on the circle.

Now, slope of $AP = \frac{y - y_1}{x - x_1}$ and slope of $BP = \frac{y - y_2}{x - x_2}$

Since $\angle APB = 90°$

$\therefore$ Slope of $AP \times$ Slope of $BP = -1$.

$$\Rightarrow \frac{(y - y_1)}{(x - x_1)} \times \frac{(y - y_2)}{(x - x_2)} = -1$$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
which is the required equation of circle.

Example 2.13 **Find the equation of the circle which passes through (1, 0) and (0, 1) and has its radius as small as possible.**

Sol. The radius will be minimum, if the given points are the end points of a diameter.

Then, equation of circle is

$$(x - 1)(x - 0) + (y - 0)(y - 1) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - y = 0.$$

Example 2.14 **If the abscissa and ordinates of two points P and Q are the roots of the equations $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively, then find the equation of the circle with PQ as diameter.**

Sol. Let x_1, x_2 and y_1, y_2 be roots of $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively.

Then, $\quad x_1 + x_2 = -2a, x_1x_2 = -b^2$

and $\quad y_1 + y_2 = -2p, y_1y_2 = -q^2$

The equation of the circle with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as the end points of diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0.$$

Example 2.15 **Tangents PA and PB are drawn to $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$. Then find the equation of the circumcircle of triangle PAB.**

Sol. Clearly the points O, A, P and B are concycllic, and OP is the diameter of circle.

Thus, equation of circumcircle of triangle PAB is

$$x(x - x_1) + y(y - y_1) = 0$$

or $\quad x^2 + y^2 - xx_1 - yy_1 = 0$

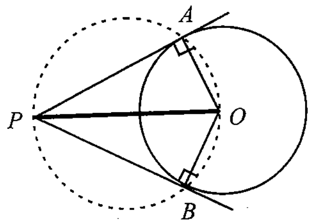


Fig. 2.7

Parametric Form of Circle

i.

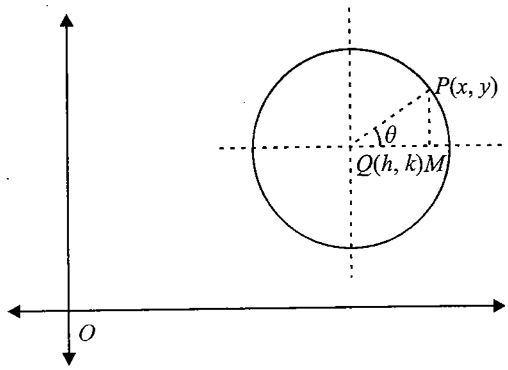


Fig. 2.8

The parametric coordinates of any point on the circle $(x - h)^2 + (y - k)^2 = r^2$ are given by $(h + r\cos\theta, k + r\sin\theta)$, where θ is parameter $(0 \le \theta \le 2\pi)$.

As from Fig. 2.8, $\cos\theta = \frac{QM}{PQ} = \frac{x - h}{r}$ and $\sin\theta = \frac{PM}{PQ} = \frac{y - k}{r}$

$\Rightarrow \quad x = h + r\cos\theta$ and $y = k + r\sin\theta$

In particular, coordinates of any point on the circle $x^2 + y^2 = r^2$ are $(r \cos\theta, r \sin\theta)$ $(0 \le \theta < 2\pi)$.

ii. The parametric coordinates of any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$ and $y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$ $(0 \le \theta < 2\pi)$.

Example 2.16 **Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$.**

Sol. Equation of the circle can be rewritten in the form

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{p}{2} + \frac{p}{\sqrt{2}}\cos\theta = \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$$

and

$$y = -\frac{p}{2} + \frac{p}{\sqrt{2}}\sin\theta = \frac{p}{2}(-1 + \sqrt{2}\sin\theta),$$

where $0 \le \theta < 2\pi$.

Example 2.17 **Find the centre of the circle $x = -1 + 2\cos\theta$, $y = 3 + 2\sin\theta$.**

Sol. Given that $\frac{x+1}{2} = \cos\theta$ and $\frac{y-3}{2} = \sin\theta$

$$\Rightarrow \quad \left(\frac{x+1}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

$$\Rightarrow \quad (x+1)^2 + (y-3)^2 = 4$$

whose centre is $(-1, 3)$.

Intercepts Made on the Axes by a Circle

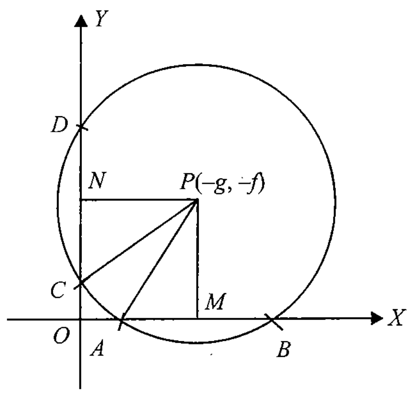


Fig. 2.9

From the diagram $PM = |f|$ and $PN = |g|$

Also $AP = CP$, radius $= \sqrt{g^2 + f^2 - c}$

$$\therefore \quad AB = 2AM = 2\sqrt{AP^2 - PM^2}$$

$$= 2\sqrt{(g^2 + f^2 - c) - f^2} = 2\sqrt{g^2 - c}$$

Similarly, $CD = 2\sqrt{f^2 - c}$

Note:

- *Intercepts are always positive.*
- *If circle touches x-axis then $|AB| = 0$ $\therefore$ $c = g^2$ and if circle touches y-axis then $|CD| = 0$ $\therefore$ $c = f^2$.*
- *If circle touches both axes, then $|AB| = 0 = |CD|$ $\therefore$ $c = g^2 = f^2$.*

Example 2.18 **Find the length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the x-axis.**

Sol. Comparing the given equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get $g = 5, f = -3$ and $c = 9$.

$\therefore$ Length of intercept on x-axis $= 2\sqrt{g^2 - c}$

$$= 2\sqrt{(5)^2 - 9} = 8$$

Example 2.19 **If the intercepts of the variable circle on the x-axis and y-axis are 2 units 4 units respectively, then find the locus of the centre of the variable circle.**

Sol.

Given that $\quad 2\sqrt{g^2 - c} = 2$ and $2\sqrt{f^2 - c} = 4$

$\Rightarrow \quad g^2 - c = 1$ and $f^2 - c = 4$

$\Rightarrow \quad f^2 - g^2 = 3$

Hence, locus is $\quad y^2 - x^2 = 3$.

DIFFERENT FORMS OF THE EQUATIONS OF A CIRCLE

When the circle passes through the origin (0, 0) and has intercepts α and β on the x-axis and y-axis, respectively:

Clearly, A and B are end points of diameter.
Hence, equation of circle is

$$(x - \alpha)(x - 0) + (y - 0)(y - \beta) = 0$$

or

$$x^2 + y^2 - \alpha x - \beta y = 0$$

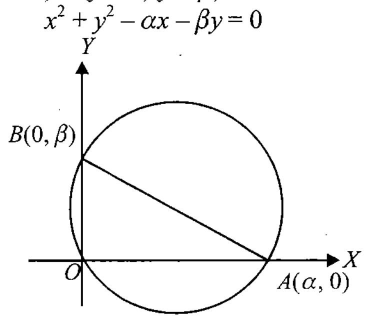


Fig. 2.10

When the circle touches x-axis:

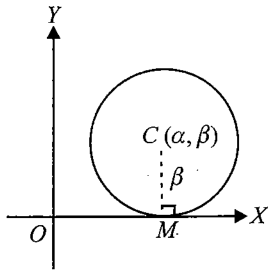


Fig. 2.11

$(x-\alpha)^2+(y-\beta)^2=\beta^2$

When the circle touches y-axis:

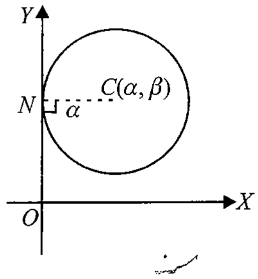


Fig. 2.12

$(x-\alpha)^2+(y-\beta)^2=\alpha^2$

When the circle touches both axes:

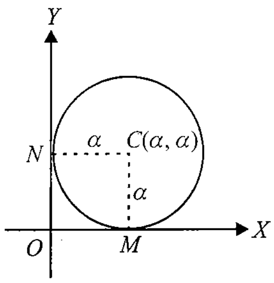


Fig. 2.13

$(x-\alpha)^2+(y-\alpha)^2=\alpha^2$

When the circle touches x-axis at $(\alpha, 0)$ and cuts off intercept on y-axis of length $2l$:

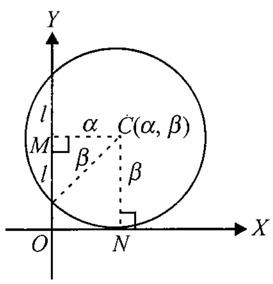


Fig. 2.14

From the figure, $\beta=\sqrt{\alpha^2+l^2}$

Hence, equation of circle is $(x-\alpha)^2+(y-\beta)^2=\beta^2$.

When the circle touches y-axis at $(0, \beta)$ and cuts off intercept on x-axis of length $2k$:

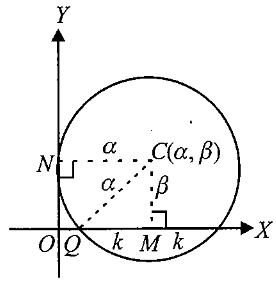


Fig. 2.15

From the figure, $\alpha=\sqrt{\beta^2+k^2}$

Hence, equation of circle is $(x-\alpha)^2+(y-\beta)^2=\alpha^2$.

Example 2.20 **Find the equation of the circle which touches both the axes and the straight line $4x+3y=6$ in the first quadrant and lies below it.**

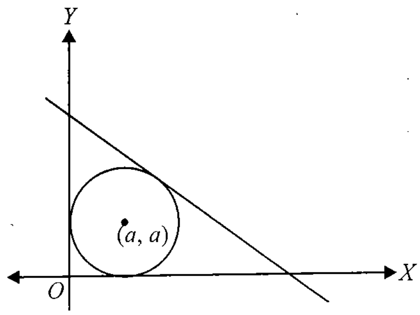


Fig. 2.16

Sol. Since the circle touches both the axes and the straight line $4x+3y=6$ in first quadrant, therefore coordinates of its centre are (a, a) and radius $=a$, where $a>0$. Since $4x+3y-6=0$ touches the circle.

$$\therefore \qquad \frac{7a-6}{\sqrt{16+9}}=\pm a$$

$$\Rightarrow \qquad 7a-6=\pm 5a \Rightarrow a=3, \frac{1}{2}$$

Since (0, 0) and (1/2, 1/2) lie on the same side of the line $4x+3y=6$, where (0, 0) and (3, 3) lie on the opposite side of the line.

Therefore, for the required circle, $a=\frac{1}{2}$. Hence, equation of the required circle is

$$(x-1/2)^2+(y-1/2)^2=(1/2)^2$$

or $\quad 4x^2+4y^2-4x-4y+1=0$

Example 2.21 **A circle touches the y-axis at the point (0, 4) and cuts the x-axis in a chord of length 6 units. Then find the radius of the circle.**

Sol. O' is centre and from the figure given below,

$$\text{Radius } (r) = \sqrt{(4)^2 + (3)^2} = 5.$$

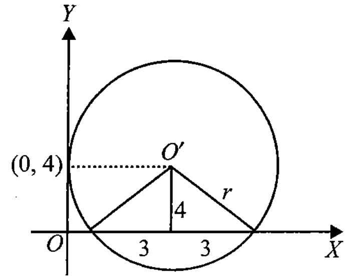


Fig. 2.17

Example 2.22 **Find the equation of the circle which is touched by $y = x$, has its centre on the positive direction of the x-axis and cuts off a chord of length 2 units along the line $\sqrt{3}y - x = 0$.**

Sol. Since the required circle has its centre on x-axis. So, let the coordinates of the centre be $(a, 0)$. The circle touches $y = x$.

Therefore, radius = length of the $\perp$ from $(a, 0)$ on $x - y = 0$ is $a/\sqrt{2}$.

The circle cuts off a chord of length 2 units along $x - \sqrt{3}y = 0$.

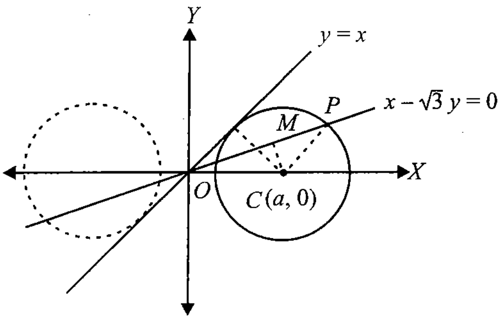


Fig. 2.18

From diagram,

$$CP^2 = CM^2 + MP^2$$

$$\left(\frac{a}{\sqrt{2}}\right)^2 = \left(\frac{a - \sqrt{3} \times 0}{\sqrt{1^2 + (\sqrt{3})^2}}\right)^2 + 1^2$$

$$\Rightarrow \quad \frac{a^2}{2} = 1 + \frac{a^2}{4} \Rightarrow a = 2$$

Thus, centre of the circle is at (2, 0) and radius $= \frac{a}{\sqrt{2}} = \sqrt{2}$

So, its equation is $x^2 + y^2 - 4x + 2 = 0$.

Example 2.23 **Find the equations of the circles passing through the point (– 4, 3) and touching the lines $x + y = 2$ and $x - y = 2$.**

Sol.

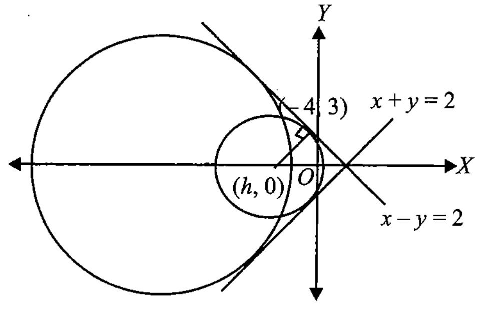


Fig. 2.19

Since the circle touches both the lines $x + y = 2$ and $x - y = 2$, its centre must lie on the x-axis. Let the centre of the circle be $(h, 0)$.

Now radius of the circle = perpendicular distance of point $(h, 0)$ from the line $x + y - 2 = 0$

$$= \frac{|h + 0 - 2|}{\sqrt{2}} = \frac{|h - 2|}{\sqrt{2}}$$

Then, equation of the circle is $(x - h)^2 + (y - 0)^2 = \frac{(h-2)^2}{2}$

Since the circle passes through the point (– 4, 3),

$$(-4 - h)^2 + (3 - 0)^2 = \frac{(h-2)^2}{2}$$

$$\Rightarrow \quad h^2 + 20h + 46 = 0$$

$$\Rightarrow \quad h = \frac{-20 \pm \sqrt{400 - 184}}{2}$$

$$= -10 \pm 3\sqrt{6}$$

POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

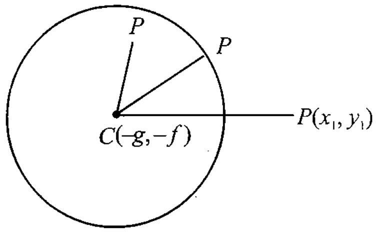


Fig. 2.20

Point P lies outside, on or inside the circle accordingly as CP >, =, < radius

or $\sqrt{(x_1 + g)^2 + (y_1 + f)^2}$ >, =, < $\sqrt{g^2 + f^2 - c}$

or $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ >, =, < 0

Maximum and Minimum Distance of a Point from the Circle

Let any point $P(x_1, y_1)$ and circle $x^2 + y^2 + 2gx + 2fy + c = 0$

The centre and radius of the circle are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

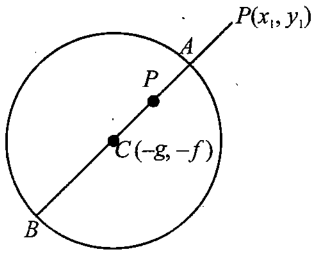


Fig. 2.21

The maximum and minimum distance from $P(x_1, y_1)$ to the circle are

$$PB = CB + PC = r + PC$$

and $$PA = |CP - CA| = |PC - r|$$ (*P* inside or outside)

where $$r = \sqrt{g^2 + f^2 - c}$$

Note: *If* $PC < r$ *then P inside,* $PC > r$ *then P outside.*

Example 2.24 **Find the greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.**

Sol. Since $S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$. So, *P* lies outside the circle. Join *P* with the centre $C(2, 1)$ of the given circle. Suppose *PC* cuts the circle at *A* and *B* (where *A* is nearer to *C*). Then, *PB* is the greatest distance of *P* from the circle.

We have: $$PC = \sqrt{(10-2)^2 + (7-1)^2} = 10$$

and $$CB = \text{radius} = \sqrt{4 + 1 + 20} = 5$$

$$\therefore \quad PB = PC + CB = 10 + 5 = 15$$

Example 2.25 **Find the values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circles $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$.**

Sol.

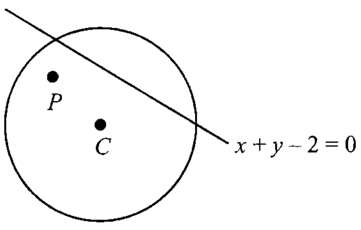


Fig. 2.22

The given circle $S(x, y) \equiv x^2 + y^2 - x - y - 6 = 0$ (i)

has centre at $$C \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$$

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle i.e.

$$S(\alpha - 1, \alpha + 1) < 0$$

$$\Rightarrow (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$$

$$\Rightarrow \quad \alpha^2 - \alpha - 2 < 0,$$

i.e. $$(\alpha - 2)(\alpha + 1) < 0$$

$$\Rightarrow \quad -1 < \alpha < 2 \quad \text{(ii)}$$

and also *P* and *C* must lie on the same side of the line (see Fig. 2.22)

$$L(x, y) \equiv x + y - 2 = 0 \quad \text{(iii)}$$

i.e. $L\left(\frac{1}{2}, \frac{1}{2}\right)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

Now, since $$L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - 2 < 0$$

Therefore, we have $L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0$,

i.e. $$\alpha < 1. \quad \text{(iv)}$$

Inequalities (ii) and (iv) together give the possible values of α as $-1 < \alpha < 1$.

Example 2.26 **Find the number of points (x, y) having integral coordinates satisfying the condition $x^2 + y^2 < 25$.**

Sol. Since $x^2 + y^2 < 25$ and *x* and *y* are integers, the possible values of *x* and *y* are $(0, \pm 1, \pm 2, \pm 3, \pm 4)$.

Thus, *x* and *y* can be chosen in nine ways each and (x, y) can be chosen in $9 \times 9 = 81$ ways.

However, we have to exclude cases $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ and $(\pm 4, \pm 4)$, i.e., $3 \times 4 = 12$ cases (as these points lies either on the circle or outside circle).

Hence, the number of points are $81 - 12 = 69$.

Example 2.27 **The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes, and the point (1, 4) is inside the circle. Find the range of the values of k.**

Sol.

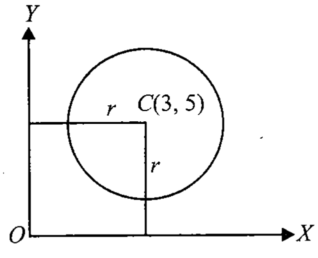


Fig. 2.23

The equation of the circle is

$$x^2 + y^2 - 6x - 10y + k = 0 \quad \text{(i)}$$

whose centre is $C(3, 5)$ and radius $r = \sqrt{34 - k}$

If the circle does not touch or intersect the x-axis, then radius $r < y$-coordinate of centre C,

or $\sqrt{34 - k} < 5$

$\Rightarrow$ $34 - k < 25$

$\Rightarrow$ $k > 9$ (ii)

Also if the circle does not touch or intersect the y-axis, then the radius $r < x$-coordinate of centre C

or $\sqrt{34 - k} < 3$

$\Rightarrow$ $34 - k < 9$

$\Rightarrow$ $k > 25$ (iii)

If the point (1, 4) is inside the circle, then its distance from centre $C < r$ (radius),

or $\sqrt{[(3-1)^2 + (5-4)^2]} < \sqrt{34 - k}$

$\Rightarrow$ $5 < 34 - k$

$\Rightarrow$ $k < 29$ (iv)

Now all the conditions (ii), (iii) and (iv) are satisfied if $25 < k < 29$ which is the required range of the values of k.

Concept Application Exercise 2.1

1. If the line $x + 2by + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then find the value of b.
2. Prove that the locus of a point that moves such that the sum of the square of its distances from the three vertices of a triangle is constant is a circle.
3. Find the number of integral values of λ for which $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius does not exceed 5.
4. If a circle whose centre is $(1, -3)$ touches the line $3x - 4y - 5 = 0$, then find its radius.
5. If one end of a diameter of the circle $2x^2 + 2y^2 - 4x - 8y + 2 = 0$ is (3, 2) the find the other end of the diameter.
6. Prove that the locus of centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and (1, 0), where 't' is a parameter is circle.
7. Find the equation of the circle which passes through the points $(3, -2)$ and $(-2, 0)$ and centre lies on the line $2x - y = 3$.
8. Find the radius of the circle $(x - 5)(x - 1) + (y - 7)(y - 4) = 0$.
9. Find the equations of the circles which pass through the origin and cut off chords of length a from each of the lines $y = x$ and $y = -x$.
10. Find the equation of the circle which touches x-axis and whose centre is (1, 2).
11. Find the equation of circle which touches both the axes and the line $x = c$.
12. Find the equation of the circle with centre at $(3, -1)$ and which cuts off an intercept of length 6 from the line $2x - 5y + 18 = 0$.
13. Find the locus of the centre of the circle which cuts off intercepts of length $2a$ and $2b$ from x-axis and y-axis, respectively.
14. Circles are drawn through the point (2, 0) to cut intercept of length 5 units on the x-axis. If their centres lie in the first quadrant, then find their equation.
15. Find the equation of the circle passing through the origin and cutting intercepts of length 3 and 4 units from the positive axes.
16. Find the point of intersection of the circle $x^2 + y^2 - 3x - 4y + 2 = 0$ with the x-axis.
17. Find the length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the x-axis.
18. Find the values of k for which points $(2k, 3k)$, (1, 0), (0, 1) and (0, 0) lie on a circle.
19. If one end of the diameter is (1, 1) and other end lies on the line $x + y = 3$, then find the locus of centre of circle.
20. Tangent drawn from the point $P(4, 0)$ to the circle $x^2 + y^2 = 8$ touches it at the point A in the first quadrant. Find the coordinates of another point B on the circle such that $AB = 4$.
21. If the join of (x_1, y_1) and (x_2, y_2) makes an obtuse angle at (x_3, y_3), then prove that $(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) < 0$.

INTERSECTION OF A LINE AND A CIRCLE

Let the equation of the circle be

$$x^2 + y^2 = a^2 \quad \text{(i)}$$

and the equation of the line be

$$y = mx + c \quad \text{(ii)}$$

Solving Eqs. (i) and (ii),

$$x^2 + (mx + c)^2 = a^2$$

or $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ (iii)

Case I: When points of intersection are real and distinct. In this case Eq. (iii) has two distinct roots.

$\therefore \qquad B^2 - 4AC > 0$

$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$

$\Rightarrow \qquad a^2 > \dfrac{c^2}{1 + m^2}$

$\Rightarrow \quad a > \dfrac{|c|}{\sqrt{(1 + m^2)}}$ = length of perpendicular from (0, 0) to $y = mx + c$

$\Rightarrow \quad a >$ length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II: When the points of intersection are coincident. In this case, Eq. (iii) has two equal roots.

$\therefore \qquad B^2 - 4AC = 0$

$\Rightarrow \qquad a = \dfrac{|c|}{\sqrt{(1 + m^2)}}$

a = length of the perpendicular from the point (0, 0) to $y = mx + c$

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III: When the points of intersection are imaginary. In this case, Eq. (iii) has imaginary roots.

$\therefore \qquad B^2 - 4AC < 0$

$\Rightarrow \qquad a^2 < \dfrac{c^2}{1 + m^2}$

$\Rightarrow \quad a < \dfrac{|c|}{\sqrt{(1 + m^2)}}$ = length of perpendicular from (0, 0) to $y = mx + c$

or $a <$ length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

Example 2.28 **Find the range of values of m for which the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ at distinct or coincident points.**

Sol. The length of the perpendicular from the centre (0, 0) to the line $= \dfrac{2}{\sqrt{1 + m^2}}$.

The radius of the circle = 1

For the line to cut the circle at distinct or coincident points, $\dfrac{2}{\sqrt{1 + m^2}} \le 1$ or $1 + m^2 \ge 4$ or $m^2 \ge 3$.

Segments of Secants, Chords and Tangent

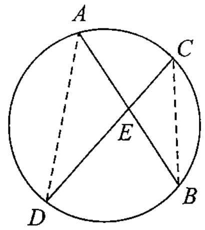


Fig. 2.24(i)

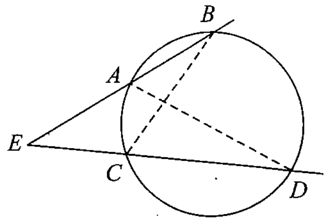


Fig. 2.24(ii)

Secants AB and CD intersect insider the circle in Fig. 2.24(i) and outside the circle in Fig. 2.24(ii).

From the figure, we have $\angle DCB = \angle DAB$ and $\angle ADC = \angle ABC$.

Hence $\Delta ADE \sim \Delta CBE$.

$\therefore \dfrac{AE}{CE} = \dfrac{DE}{BE} \quad \therefore AE \times BE = CE \times DE$

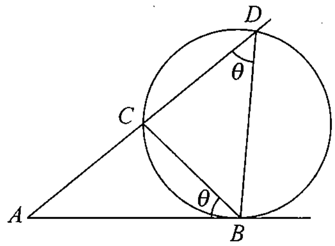


Fig. 2.24(iii)

In Fig. 2.24(iii), AD is secant and AB is tangent.

From Fig. 2.24(iii), $\Delta ABD \sim \Delta ACB$

$\therefore \dfrac{AB}{AC} = \dfrac{AD}{AB} \quad \therefore AB^2 = AC \times AD$

Example 2.29 **If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then prove that $|a_1a_2| = |b_1b_2|$.**

Sol. Let the given lines be $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$. Suppose L_1 meets the coordinate axes at P and Q and L_2 meets at R and S. Then, coordinates of P, Q, R, S are

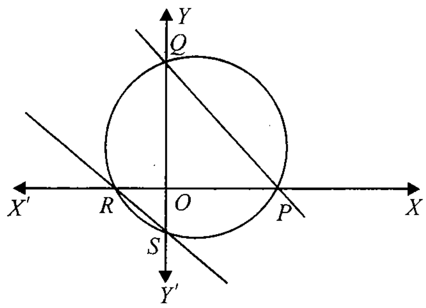


Fig. 2.25

$P\left(-\frac{c_1}{a_1}, 0\right), Q\left(0, -\frac{c_1}{b_1}\right)$

$R\left(-\frac{c_2}{a_2}, 0\right)$, and $S\left(0, -\frac{c_2}{b_2}\right)$

Since P, Q, R, S are concyclic,

$\therefore \quad OP \cdot OR = OQ \cdot OS$

$\Rightarrow \quad \left|\left(-\frac{c_1}{a_1}\right)\left(-\frac{c_2}{a_2}\right)\right| = \left|\left(-\frac{c_1}{b_1}\right)\left(-\frac{c_2}{b_2}\right)\right|$

$\Rightarrow \quad |a_1 a_2| = |b_1 b_2|$

Example 2.30 **If a line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = a^2$ at A and B, then find the value of $PA \cdot PB$.**

Sol. From the figure, $PA \cdot PB$ = constant

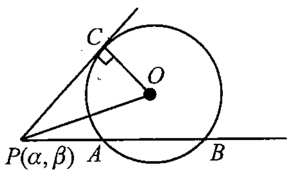


Fig. 2.26

Also $\quad PA \cdot PB = PC^2$

But $\quad PC^2 = OP^2 - OC^2$

$= \alpha^2 + \beta^2 - a^2$

$\Rightarrow \quad PA \cdot PB = \alpha^2 + \beta^2 - a^2$

TANGENT TO A CIRCLE AT A GIVEN POINT

Let PQ be *a* chord and AB be a secant passing through P.

Let P be the fixed point and Q move along the circle towards P, then the secant PQ turns about P. In the limit, when Q coincides with P, then the secant AB becomes *a* tangent to a circle at the point P.

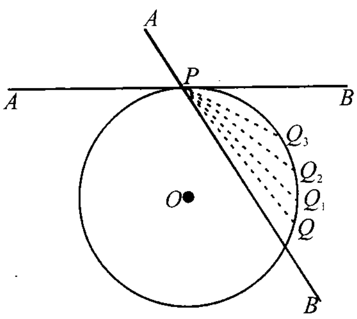


Fig. 2.27

Different Forms of the Equations of Tangents

Point Form

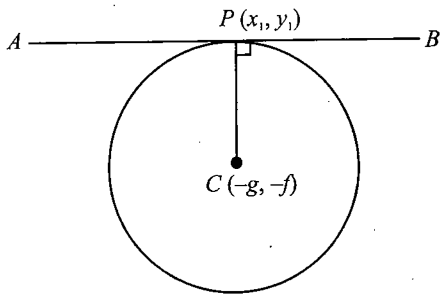


Fig. 2.38

To find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $P(x_1, y_1)$ on it.

CP is perpendicular of tangent at P

$$\text{Slope of } CP = \frac{y_1 + f}{x_1 + g}$$

$$\therefore \quad \text{Slope of tangent} = -\frac{x_1 + g}{y_1 + f}$$

Slope of tangent at point P can be obtained by differentiation also.

Differentiating equation of circle w.r.t. to x

we get $2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x_1 + g}{y_1 + f}$$

$\therefore$ equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

or $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(since point (x_1, y_1) lies on the circle $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$)

or $\quad T = 0$,

where $\quad T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Note:

1. For equation of tangent of circle at (x_1, y_1) substitute xx_1 for x^2, yy_1 for y^2, $\frac{x+x_1}{2}$ for x, $\frac{y+y_1}{2}$ for y and keep the constant as such.
2. For circle $x^2 + y^2 = a^2$ equation of tangent at point (x_1, y_1) is given by $xx_1 + yy_1 = a^2$
3. For circle $(x-h)^2 + (y-k)^2 = a^2$ equation of tangent at point (x_1, y_1) is given by $(x-h)(x-x_1) + (y-k)(y-y_1) = a^2$
4. Since parametric coordinates of circle $x^2 + y^2 = a^2$ are $(a \cos \theta, a \sin \theta)$, then equation of tangent at $(a \cos \theta, a \sin \theta)$ is $x.a \cos \theta + y.a \sin \theta = a^2$ or $x \cos \theta + y \sin \theta = a$

Slope Form

Let $y = mx + c$ is the tangent of the circle $x^2 + y^2 = a^2$

$\therefore$ Length of perpendicular from centre of circle (0, 0) on $(y = mx + c)$ = radius of circle

$$\therefore \quad \frac{|c|}{\sqrt{1+m^2}} = a \Rightarrow c = \pm a\sqrt{1+m^2}$$

substituting this value of c in $y = mx + c$, we get $y = mx \pm a\sqrt{1+m^2}$ which are the required equations of tangents.

Corollary : It also follows that $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ which is the condition of tangency.

Note:

1. If slope of tangent is given, then two parallel tangents can be drawn the circle at the ends of diameter.
2. Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is
$y + f = m(x + g) \pm \sqrt{g^2 + f^2 - c}\sqrt{1+m^2}$

Points of Contact

If circle $x^2 + y^2 = a^2$ and tangent in terms of slope $y = mx \pm a\sqrt{(1+m^2)}$

Solving $x^2 + y^2 = a^2$ and $y = mx \pm a\sqrt{1+m^2}$, simultaneously, we get

$$x = \pm \frac{am}{\sqrt{(1+m^2)}}$$

and
$$y = \mp \frac{a}{\sqrt{(1+m^2)}}$$

Thus, the coordinates of the points of contact are $\left(\pm \frac{am}{\sqrt{(1+m^2)}}, \mp \frac{a}{\sqrt{(1+m^2)}}\right)$.

Alternative Method

Let point of contact be (x_1, y_1), then tangent at (x_1, y_1) of $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Since $xx_1 + yy_1 = a^2$ and $y = mx \pm a\sqrt{1+m^2}$ are identical

$$\Rightarrow \quad \frac{x_1}{m} = \frac{y_1}{-1} = \frac{+a^2}{\pm a\sqrt{1+m^2}}$$

$$\Rightarrow \quad x_1 = \pm \frac{am}{\sqrt{1+m^2}}$$

and
$$y_1 = \mp \frac{a}{\sqrt{1+m^2}}$$

Thus, the coordinates of the points of contact are $\left(\pm \frac{am}{\sqrt{(1+m^2)}}, \mp \frac{a}{\sqrt{(1+m^2)}}\right)$.

Note:

1. *If the line $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$.*
2. *If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$.*

Tangents from a Point Outside the Circle

If circle is $x^2 + y^2 = a^2$ (i)

any tangent to the circle Eq. (i) is

$$y = mx + a\sqrt{(1+m^2)} \quad \text{(ii)}$$

If outside point is (x_1, y_1), then $y_1 = mx_1 + a\sqrt{1+m^2}$ or $(y_1 - mx_1)^2 = a^2(1 + m^2)$

or $y_1^2 + m^2 x_1^2 - 2xm_1 y_1 = a^2 + a^2m^2$

$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1 y_1 + y_1^2 - a^2 = 0$

which is quadratic in m which given two values of m.

Substituting these values of m in Eq. (ii), we get the equation of tangents.

Example 2.31 **Find the angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$.**

Sol. Any line through (0, 0) be $y - mx = 0$ and it is a tangent to circle $(x - 7)^2 + (y + 1)^2 = 25$, if

$$\frac{|-1-7m|}{\sqrt{1+m^2}} = 5$$

$\Rightarrow$ $$m = \frac{3}{4}, -\frac{4}{3}$$

Therefore, the product of both the slopes is – 1 i.e.,

$$\frac{3}{4} \times \left(-\frac{4}{3}\right) = -1$$

Hence, the angle between the two tangents is $\frac{\pi}{2}$.

Example 2.32 **Find the equation of the tangents to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the straight line $4x + 3y + 5 = 0$.**

Sol. Let equation of tangent be $4x + 3y + k = 0$, then radius = distance of centre from the line

$\Rightarrow$ $$\sqrt{9+4+12} = \left|\frac{4(3)+3(-2)+k}{\sqrt{16+9}}\right|$$

$\Rightarrow$ $$6 + k = \pm 25$$

$\Rightarrow$ $$k = 19 \text{ and } -31$$

Hence, the tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

Example 2.33 **Tangent to circle $x^2 + y^2 = 5$ at (1, – 2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Find the coordinate of the corresponding point of contact**

Sol. Equation of tangent to $x^2 + y^2 = 5$ at (1, – 2) is $x - 2y - 5 = 0$.

Putting $x = 2y + 5$ in second circle, we get $(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$

$\Rightarrow$ $$5y^2 + 10y + 5 = 0$$

$\Rightarrow$ $$y = -1$$

$\Rightarrow$ $$x = -2 + 5 = 3$$

Thus, point of contact is (3, –1).

Example 2.34 **Prove that the equation of any tangent to the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ is of the form $y = m(x-1) + 3\sqrt{1+m^2} - 2$.**

Sol. The circle is $(x-1)^2 + (y+2)^2 = 3^2$.

As any tangent to $x^2 + y^2 = 3^2$ is given by $y = mx + 3\sqrt{1+m^2}$, any tangent to the given circle will be

$$y + 2 = m(x-1) + 3\sqrt{1+m^2}.$$

Example 2.35 **If $a > 2b > 0$ then find the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$, and $(x-a)^2 + y^2 = b^2$.**

Sol. $y = mx - b\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m.

If it also touches the circle $(x-a)^2 + y^2 = b^2$, then the length of the perpendicular from its centre $(a, 0)$ on this line is equal to the radius b of the circle, which gives

$$\frac{ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}} = \pm b.$$

Taking negative value of R.H.S., we get $m = 0$, so we neglect it.

Taking the positive value of R.H.S., we get

$$ma = 2b\sqrt{1+m^2}$$

$\Rightarrow$ $$m^2(a^2 - 4b^2) = 4b^2$$

$\Rightarrow$ $$m = \frac{2b}{\sqrt{a^2-4b^2}}.$$

Length of the Tangent from a Point to a Circle

Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$, then centre and radius of circle are $(-g, -f)$ and $\sqrt{(g^2 + f^2 - c)}$, respectively, and let $P(x_1, y_1)$ be any point outside the circle.

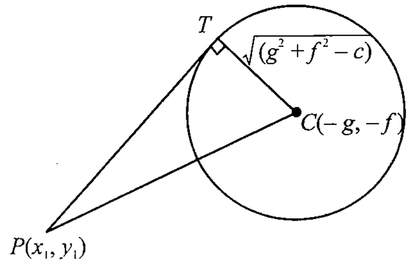


Fig. 2.29

In ΔPCT,

$$PT = \sqrt{(PC)^2 - (CT)^2}$$
$$= \sqrt{(x_1+g)^2 + (y_1+f)^2 - g^2 - f^2 + c}$$
$$= \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$$
$$= \sqrt{S_1},$$

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note:

1. *For S_1, first write the equation of circle in standard form and coefficient of x^2 = coefficient of y^2 = 1 and making R.H.S. of circle is zero, then let L.H.S. be S.*

Example 2.36 **Prove that the angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$, where $S_1 = \alpha^2 + \beta^2 - a^2$.**

Sol.

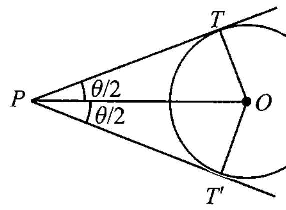


Fig. 2.30

Let PT and PT' be the tangents drawn from $P(\alpha, \beta)$ to the circle $x^2 + y^2 = a^2$ and let $\angle TPT' = \theta$. If O is the centre of the circle, then $\angle TPO = \angle T'PO = \frac{\theta}{2}$.

$$\therefore \quad \tan\frac{\theta}{2} = \frac{OT}{PT} = \frac{a}{\sqrt{S_1}}$$

$$\Rightarrow \quad \theta = 2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right).$$

Example 2.37 **Find the ratio of the length of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$, $5x^2 + 5y^2 - 48x + 64y + 300 = 0$.**

Sol. Let $P(h, k)$ be a point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$.

Then, the lengths PT_1 and PT_2 of the tangent from $P(h, k)$ to $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$, respectively, are

$$PT_1 = \sqrt{h^2 + k^2 - \frac{24}{5}h + \frac{32}{5}k + 15}$$

$$\text{and} \quad PT_2 = \sqrt{h^2 + k^2 - \frac{48}{5}h + \frac{64}{5}k + 60}$$

Since (h, k) lies on $15x^2 + 5y^2 - 48x + 64y = 0$

$$\therefore \quad h^2 + k^2 - \frac{48}{15}h + \frac{64}{15}k = 0$$

$$\therefore \quad PT_1 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{24}{5}h + \frac{32}{5}k + 15}$$

$$= \sqrt{\frac{32}{15}k - \frac{24}{15}h + 15}$$

$$\text{and,} \quad PT_2 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{48}{5}h + \frac{64}{5}k + 60}$$

$$= \sqrt{-\frac{96}{15}h + \frac{128}{15}k + 60}$$

$$= 2\sqrt{-\frac{24}{15}h + \frac{32}{15}k + 15} = 2(PT_1)$$

$\therefore PT_1 : PT_2 = 1 : 2.$

Example 2.38 **If from any point P on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$, then find the angle between the tangents.**

Sol.

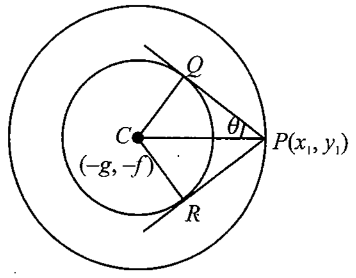


Fig. 2.31

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{(i)}$$

and the length of the tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ is $PQ = PR$

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$$

$$= \sqrt{-c + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$$

$$= (\sqrt{g^2 + f^2 - c})\cos\alpha$$

The radius of the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ is

$$CQ = CR$$

$$= \sqrt{g^2 + f^2 - c\sin^2\alpha - (g^2 + f^2)\cos^2\alpha}$$

$$= (\sqrt{g^2 + f^2 - c})\sin\alpha$$

In ΔCPQ,

$$\tan\theta = \frac{CQ}{PQ}$$

$$= \frac{\sqrt{g^2 + f^2 - c}\,\sin\alpha}{\sqrt{g^2 + f^2 - c}\,\cos\alpha} = \tan\alpha \Rightarrow \theta = \alpha$$

Example 2.39 **If the distance from the origin of the centres of three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$ $(i = 1, 2, 3)$ are in G.P., then prove that the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in G.P.**

Sol. The centres of the given circles are $(-\lambda_i, 0)$ $(i = 1, 2, 3)$. The distances from the origin of the centres are λ_i, $(i = 1, 2, 3)$. It is given that $\lambda_2^2 = \lambda_1\lambda_3$. Let $P(h, k)$ be any point on the circle $x^2 + y^2 = c^2$. Then $h^2 + k^2 = c^2$.

Now, L_i = length of the tangent from (h, k) to $x^2 + y^2 + 2\lambda_i x - c^2 = 0$

$$= \sqrt{h^2 + k^2 + 2\lambda_i h - c^2}$$

$$= \sqrt{c^2 + 2\lambda_i h - c^2} \qquad [\because h^2 + k^2 = c^2]$$

$$= \sqrt{2\lambda_i h}, i = 1, 2, 3$$

Therefore,

$$L_2^2 = 2\lambda_2 h = 2h(\sqrt{\lambda_1 \lambda_3}) \; [\because \lambda_2^2 = \lambda_1 \lambda_3]$$

$$= \sqrt{2\lambda_1 h}\,\sqrt{2\lambda_3 h} = L_1 L_3$$

Hence, L_1, L_2, L_3 are in G.P.

Pair of Tangents

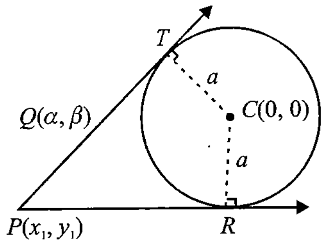


Fig. 2.32

Let the circle be $x^2 + y^2 = a^2$

Its centre and radius are $C(0, 0)$ and a, respectively. Let the given external point be $P(x_1, y_1)$.

From point $P(x_1, y_1)$, two tangents PT and PR can be drawn to the circle, touching circle at T and R, respectively.

Let $Q(\alpha, \beta)$ on PT, then equation of PQ is

$$y - y_1 = \frac{\beta - y_1}{\alpha - x_1}(x - x_1)$$

or $\quad y(\alpha - x_1) - (\beta - y_1)x - \alpha y_1 + \beta x_1 = 0$

Length of perpendicular from $C(0, 0)$ on $PT = a$(radius)

$$\Rightarrow \frac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a$$

or $\quad (\beta x_1 - \alpha y_1)^2 = a^2\{(\alpha - x_1)^2 + (\beta - y_1)^2\}$

$\therefore$ Locus of $Q(\alpha, \beta)$ is $(yx_1 - xy_1)^2 = a^2\{(x - x_1)^2 + (y - y_1)^2\}$

$$\Rightarrow y^2x_1^2 + x^2y_1^2 - 2xy\,x_1y_1 = a^2\{x^2 + x_1^2 - 2xx_1 + y^2 + y_1^2 - 2yy_1\}$$

$$\Rightarrow y^2(x_1^2 + y_1^2 - a^2) + x^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\Rightarrow (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\Rightarrow SS_1 = T^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2 + y^2 = a^2$.

where $S = x^2 + y^2 - a^2$, $S_1 = x_1^2 + y_1^2 - a^2$,
and $T = xx_1 + yy_1 - a^2$.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point of contact. Normal always passes through the centre of the circle.

Point Form

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point of contact.

The normal of the circle always passes through the centre of the circle.

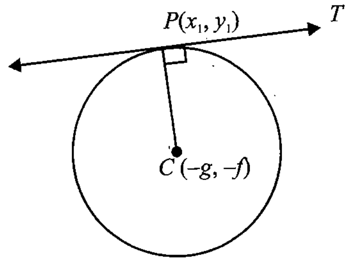


Fig. 2.33

To find the equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on it.

Since normal passes through the centre, we have slope of normal $CP = \dfrac{y_1 + f}{x_1 + g}$

Hence, equation of normal is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$

or $\qquad \dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

Example 2.40 **Find the equations of the normals to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinate is -1.**

Sol. The abscissa of point is found by substituting the ordinates and solving for abscissa

$\Rightarrow \qquad x^2 - 8x + 15 = 0$

$\Rightarrow \qquad x = 5$ or 3

$\Rightarrow$ Points are $(5, -1)$ and $(3, -1)$.

Normal is given by $\dfrac{x - 5}{5 - 4} = \dfrac{y + 1}{-1 - 1}$

$\Rightarrow \quad 2x + y - 9 = 0$

and $\quad \dfrac{x-3}{3-4} = \dfrac{y+1}{-1-1}$

$\Rightarrow \quad 2x - y - 7 = 0$

Example 2.41 **Find the equation of the normal to the circle $x^2 + y^2 - 2x = 0$ parallel to the line $x + 2y = 3$.**

Sol. Any line parallel to $x + 2y = 3$ is $x + 2y + \lambda = 0$ and for this to be a normal to the given circle, must pass through its centre $(1, 0)$, i.e., $\lambda = -1$.

So, normal is $x + 2y - 1 = 0$.

Concept Application Exercise 2.2

1. Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which makes equal intercepts on the positive co-ordinates axes.
2. If the length of tangent drawn from the point (5, 3) to the circle $x^2 + y^2 + 2x + ky + 17 = 0$ be 7, then find the value of k.
3. If the line $lx + my + n = 0$ is tangent to the circle $x^2 + y^2 = a^2$, then find the condition.
4. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. Then find its equations.
5. Find the equation of the normal to the circle $x^2 + y^2 = 9$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
6. Find the equations of tangents to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ which are perpendicular to the line $5x + 12y + 8 = 0$.
7. An infinite number of tangents can be drawn from (1, 2) to the circle $x^2 + y^2 - 2x - 4y + \lambda = 0$, then find the value of λ.
8. If circle $x^2 + y^2 - 4x - 8y - 5 = 0$ intersect the line $3x - 4y = m$ in two distinct points, then find the values of m.
9. If a line passing through origin touches the circle $(x - 4)^2 + (y + 5)^2 = 25$, then find its slope.
10. The tangent at any point P on the circle $x^2 + y^2 = 4$, meets the coordinate axes in A and B, then find the locus of the midpoint of AB.
11. Find the locus of a point which moves so that the ratio of the of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 2:3.
12. Find the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$.

CHORD OF CONTACT

From any external point $A(x_1, y_1)$ draw pair of tangents AP and AQ touching the circle at $P(x', y')$ and $Q(x'', y'')$, respectively. Then PQ is the chord of contact with P, Q as its points of contact.

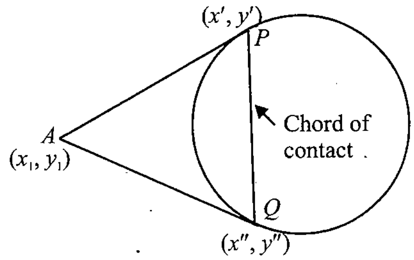


Fig. 2.34

Let the circle be $x^2 + y^2 = a^2$.

Then equations of tangents AP and AQ are $xx' + yy' = a^2$ and $xx'' + yy'' = a^2$, respectively.

Since both tangents AP and AQ pass through $A(x_1, y_1)$, then $x_1x' + y_1y' = a^2$ and $x_1x'' + y_1y'' = a^2$.

Points $P(x', y')$ and $Q(x'', y'')$ lie on $xx_1 + yy_1 = a^2$.

$\therefore$ Equation of chord of contact PQ is $xx_1 + yy_1 = a^2$.

or $T = 0$, where $T = xx_1 + yy_1 - a^2$

- *Equation of chord of contact at (x_1, y_1) with circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ or $T = 0$.*

Example 2.42 **If the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then prove that a, b, c are in G.P.**

Sol. Let (h, k) be a point on $x^2 + y^2 = a^2$. Then

$$h^2 + k^2 = a^2 \qquad \text{(i)}$$

The equation of the chord of contact of tangents drawn from (h, k) to $x^2 + y^2 = b^2$ is

$$hx + ky = b^2 \qquad \text{(ii)}$$

This touches the circle $x^2 + y^2 = c^2$. Therefore $\left|\dfrac{-b^2}{\sqrt{h^2 + k^2}}\right| = c$

$\Rightarrow \quad \left|\dfrac{-b^2}{\sqrt{a^2}}\right| = c \qquad$ [Using Eq. (i)]

$\Rightarrow \quad b^2 = ac \Rightarrow a, b, c$ are in G.P.

Example 2.43 **If the straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ in points P and Q, then find the coordinates of the point of intersection of tangents drawn at P and Q to the circle $x^2 + y^2 = 25$.**

Sol.

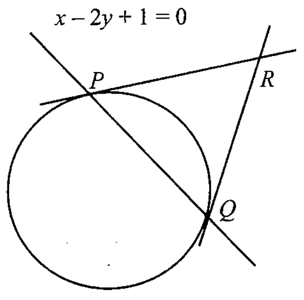


Fig. 2.35

Let $R(h, k)$ be the point of intersection of tangents drawn at P and Q to the given circle. Then PQ is the chord of contact of tangents drawn from R to $x^2 + y^2 = 25$.

So, its equation is

$$hx + ky - 25 = 0 \qquad \text{(i)}$$

It is given that the equation of PQ is $x - 2y + 1 = 0$ (ii)

Since Eq. (i) and (ii) represent the same line, therefore

$$\frac{h}{1} = \frac{k}{-2} = \frac{-25}{1}$$

$\Rightarrow$ $h = -25, k = 50$

Hence, the required point is (–25, 50).

Example 2.44 **If the chord of contact of tangents drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre, then prove that $h^2 + k^2 = 2a^2$.**

Sol.

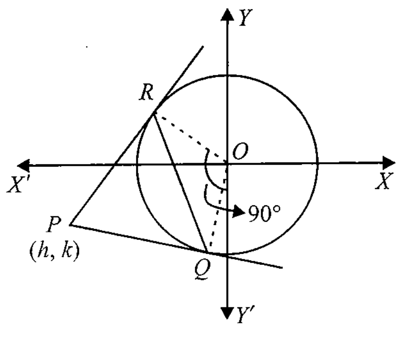


Fig. 2.36

As shown in diagram $\angle ROQ = \pi/2$ also $\angle PRO = \angle PQO = \pi/2$.

Then quadrilateral $PROQ$ is square and hence $PR = RO$

$$\Rightarrow \quad \sqrt{h^2 + k^2 - a^2} = a \Rightarrow h^2 + k^2 = 2a^2$$

Example 2.45 **Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point P on the line $2x + y - 4 = 0$. The corresponding chord of contact passes through a fixed point whose coordinates are**

a. $\left(\frac{1}{4}, \frac{1}{2}\right)$ **b.** $\left(\frac{1}{2}, 1\right)$

c. $\left(\frac{1}{2}, \frac{1}{4}\right)$ **d.** $\left(1, \frac{1}{2}\right)$

Sol.

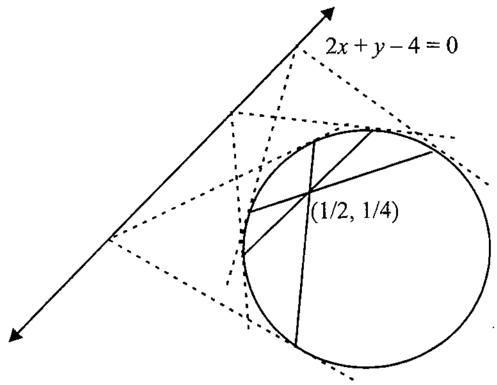


Fig. 2.37

Let any point on the line $2x + y - 4 = 0$ be $P \equiv (a, 4 - 2a)$.

Equation of chord of contact of the circle $x^2 + y^2 = 1$ with respect to point P is

$$x . a + y . (4 - 2a) = 1$$

$$\Rightarrow (4y - 1) + a(x - 2y) = 0.$$

This line always passes through a point of intersection of the lines $4y - 1 = 0$ and $x - 2y = 0$ which is fixed point whose coordinate are $y = \frac{1}{4}$ and $x = 2y = \frac{1}{2}$.

Example 2.46 **Find the area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact.**

Sol.

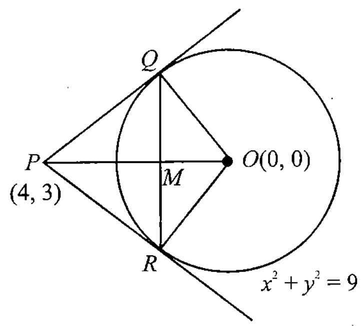


Fig. 2.38

The equation of the chord of contact of tangents drawn from $P(4, 3)$ to $x^2 + y^2 = 9$ is $4x + 3y = 9$.

The equation of PO is $y = \frac{3}{4}x$.

Now, OM = (length of the $\perp$ from (0, 0) on $4x + 3y - 9 = 0$ is $\frac{9}{5}$

$$QR = 2.\ QM = 2\sqrt{OQ^2 - OM^2}$$

$$= 2\sqrt{9 - \frac{81}{25}} = \frac{24}{5}$$

Now, $PM = OP - OM = 5 - \frac{9}{5} = \frac{16}{5}$

So, area of $\Delta PQR = \frac{1}{2}\left(\frac{24}{5}\right)\left(\frac{16}{5}\right)$.

$= \left(\frac{192}{25}\right)$ sq. units.

EQUATION OF THE CHORD BISECTED AT A GIVEN POINT

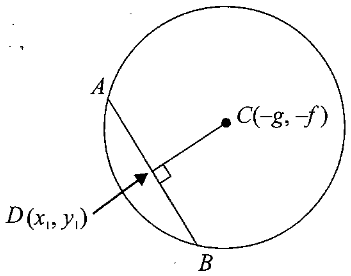


Fig. 2.39

Let any chord AB of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ be bisected at $D(x_1, y_1)$.

Then slope of $DC = \frac{y_1 + f}{x_1 + g}$

Therefore, slope of the chord AB is $-\frac{x_1 + g}{y_1 + f}$

then equation of AB is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$

$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow T = S_1$.

where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Note:

Chord bisected at point (x_1, y_1) is the farthest from centre among all the chords passing through the point (x_1, y_1). Also for such chord, the length of the chord is minimum.

Example 2.47 **Find the equation to the chord of the circle $x^2 + y^2 = 9$ whose middle point is (1, –2).**

Sol. The required equation is

$$x - 2y - 9 = 1 + 4 - 9$$

[Using $S' = T$]

or $x - 2y - 5 = 0$

Example 2.48 **A variable chord is drawn through the origin to the circle $x^2 + y^2 - 2ax = 0$. Find the locus of the centre of the circle drawn on this chord as diameter.**

Sol. Let (h, k) be the coordinates of the centre of the circle of which the given chord is the diameter. Then (h, k) is the mid-point of the chord and so its equation is

$$T = S_1$$

i.e. $h^2 + k^2 - 2ah = hx + ky - a(x + h)$

$\Rightarrow x(h - a) + ky = h^2 + k^2 - ah$

It passes through (0, 0), therefore $h^2 + k^2 - ah = 0$

So, the locus of (h, k) is $x^2 + y^2 - ax = 0$.

Alternative Method:

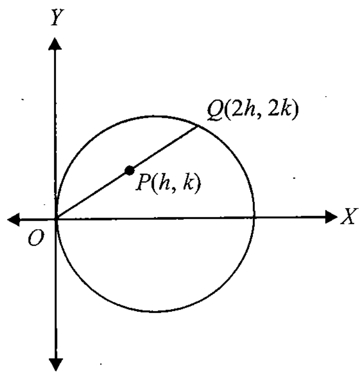


Fig. 2.40

Since point P is midpoint of chord OQ, Q has coordinates $(2h, 2k)$, which lies on the given circle.

$\therefore (2h)^2 + (2k)^2 - 2a(2h) = 0$ or $x^2 + y^2 - ax = 0$.

Example 2.49 **Find the equation of chord of the circle $x^2 + y^2 = a^2$ passing through the point (2, 3) farthest from the centre.**

Sol.

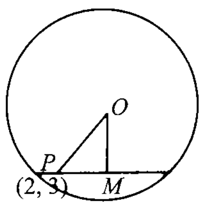


Fig. 2.41

Let $P(2, 3)$ be given point, M be the middle point of a chord of the circle $x^2 + y^2 = a^2$ through P.

Then the distance of the centre O of the circle from the chord is OM.

and $(OM)^2 = (OP)^2 - (PM)^2$ which is maximum when PM is minimum,

i.e. P coincides with M, which is the middle point of the chord.

Hence, the equation of the chord is $T = S_1$, i.e. $2x + 3y - a^2 = (2)^2 + (3)^2 - a^2 \Rightarrow 2x + 3y = 13$

Concept Application Exercise 2.3

1. Find the middle point of the chord of the circle $x^2 + y^2 = 25$ intercepted on the line $x - 2y = 2$.
2. The line $9x + y - 18 = 0$ is the chord of contact of the point $P(h, k)$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$, for

   a. $\left(\frac{24}{5}, \frac{-4}{5}\right)$ b. $P(3, 1)$

   c. $P(-3, 1)$ d. $\left(-\frac{2}{5}, \frac{12}{5}\right)$
3. Tangents are drawn to the circle $x^2 + y^2 = 9$ at the points where it is met by the circle $x^2 + y^2 + 3x + 4y + 2 = 0$. Find the point of intersection of these tangents.
4. Find the distance between the chords of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f).
5. A line $lx + my + n = 0$ meets the circle $x^2 + y^2 = a^2$ at the points P and Q. If the tangents drawn at the points P and Q meet at R, then find the coordinates of R.

DIRECTOR CIRCLE AND ITS EQUATION

Director Circle: The locus of the point of intersection of two perpendicular tangents to a given circle is known as its director circle.

Equation of Director Circle: The equation of any tangent to the circle $x^2 + y^2 = a^2$ is

$$y = mx + a\sqrt{(1 + m^2)} \quad \text{(i)}$$

Let $P(h, k)$ be the point of intersection of tangents, then $P(h, k)$ lies on Eq. (i)

$$\therefore \quad k = mh + a(\sqrt{1 + m^2})$$

or $\quad (k - mh)^2 = a^2(1 + m^2)$

or $m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$

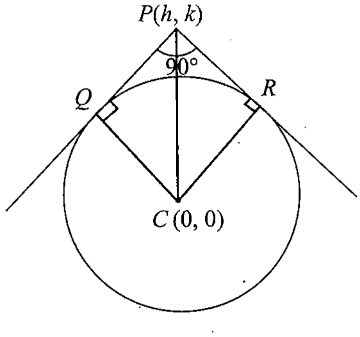


Fig. 2.42

This is quadratic equation in m, let two roots are m_1 and m_2.

But tangents are perpendiculars, therefore $m_1 m_2 = -1$

$$\Rightarrow \frac{k^2 - a^2}{h^2 - a^2} = -1 \Rightarrow k^2 - a^2 = -h^2 + a^2 \Rightarrow h^2 + k^2 = 2a^2$$

Hence, locus of $P(h, k)$ is $x^2 + y^2 = 2a^2$.

Equation of director circle for circle $(x - p)^2 + (y - q)^2 = a^2$ is given by $(x - p)^2 + (y - q)^2 = 2a^2$.

Alternative Method:

From the figure,

$CRPQ$ is a square

$\therefore \quad CQ = CP \cos 45°$

or $\quad 2a^2 = h^2 + k^2$

or $\quad x^2 + y^2 = 2a^2$ which is the required locus.

INTERSECTION OF TWO CIRCLES

Different cases of intersection of two circles:

Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ (i)

and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ (ii)

with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2, respectively. Then following cases may arise:

Case I:

When $|C_1C_2| > r_1 + r_2$, i.e., the distance between the centres is greater than the sum of radii, then two circles neither intersect not touch each other .

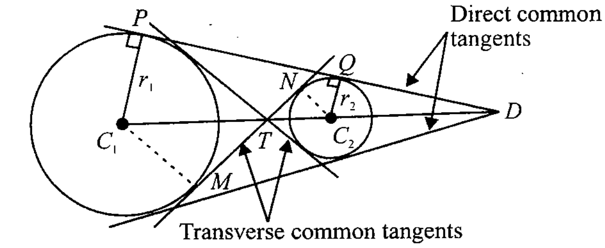


Fig. 2.43

In this case, four common tangents can be drawn to two circles, in which two are direct common tangents and the other two are transverse common tangents.

From Fig. 2.43, ΔC_1MT and ΔC_2NT are similar. Hence, $\frac{C_1T}{C_2T} = \frac{C_1M}{C_2N} = \frac{r_1}{r_2}$. Using this, we can find point T.

Similarly, ΔC_1PD and C_2QD are similar. Hence, $\frac{C_1D}{C_2D} = \frac{C_1P}{C_2Q} = \frac{r_1}{r_2}$

To find equations of common tangents:

Now assume the equation of tangent of any circle in the form of the slope $(y + f) = m(x + g) + a\sqrt{(1 + m^2)}$ (where a is the radius of the circle).

T and D will satisfy the assumed equation. Thus 'm' is obtained. We can find the equation of common tangent if we substitute the value of m in the assumed equation.

Case II:

When $|C_1C_2| = r_1 + r_2$, i.e, the distance between the centres is equal to the sum of radii, then two circles touch externally.

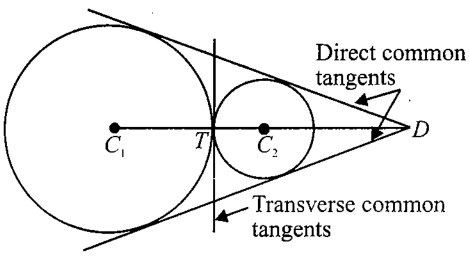


Fig. 2.44

In this case, two direct common tangents are real and distinct while the transverse tangents are coincident.

In such cases, the point of contact T divides the line joining C_1 and C_2 internally in the ratio $r_1 : r_2 \Rightarrow \frac{C_1T}{C_2T} = \frac{r_1}{r_2}$.

Then coordinates of T are $\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$

The equation of tangent at point T is $S_1 - S_2 = 0$, where $S_1 = 0$ and $S_2 = 0$ are equations of circles.

Case III:

When $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$, i.e., the distance between the centres is less than sum of radii then two circles intersect at two distinct points.

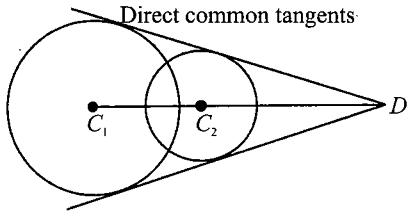


Fig. 2.45

In this case, two direct common tangents are real and distinct while the transverse tangents are imaginary.

Here, also point D divides C_1C_2 externally, $\frac{C_1D}{C_2D} = \frac{r_1}{r_2}$

Case IV:

When $|C_1C_2| = |r_1 - r_2|$, i.e., the distance between the centres is equal to the difference of the radii, then two circles touch internally.

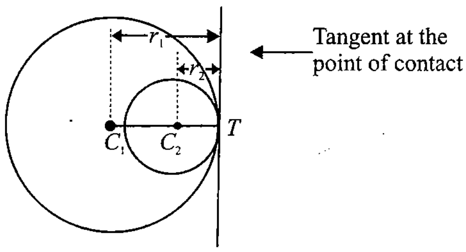


Fig. 2.46

In this case, there is only one common tangent.

If circles are represented by $S_1 = 0$ and $S_2 = 0$, then equation of common tangent is $S_1 - S_2 = 0$.

In such cases, the point of contact T divides the line joining C_1 and C_2 externally in the ratio $r_1 : r_2 \Rightarrow \frac{C_1T}{C_2T} = \frac{r_1}{r_2}$.

Then coordinates of T are $\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$.

Case V:

When $|C_1C_2| < |r_1 - r_2|$, i.e., the distance between the centres is less than the difference of the radii.

In this case, all the four common tangents are imaginary.

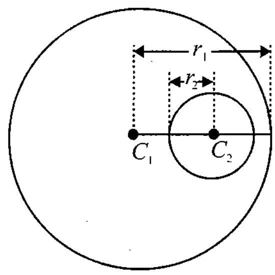


Fig. 2.47

Length of an External Common Tangent and Internal Common Tangent to Two Circles

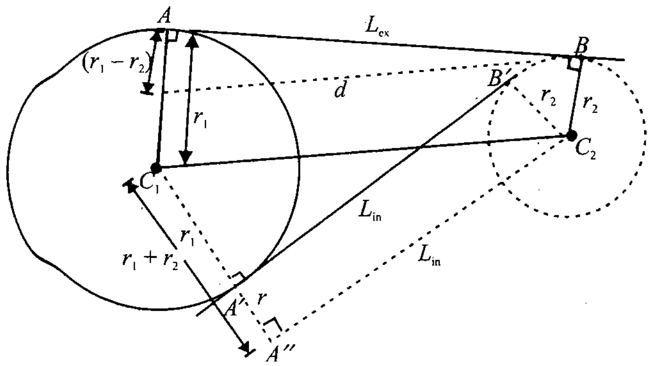


Fig. 2.48

Length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$

and length of internal common tangent

$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$

[Applicable only when $d > (r_1 + r_2)$]

where d is the distance between the centres of two circles

and r_1 and r_2 are the radii of two circles where $|C_1C_2| = d$.

Example 2.50 **Find the number of common tangents to circles $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 9 = 0$.**

Sol. For $x^2 + y^2 + 2x + 8y - 23 = 0$

$\therefore \quad C_1(-1, -4), r_1 = 2\sqrt{10}$

For $x^2 + y^2 - 4x - 10y + 9 = 0 \therefore C_2(2, 5), r_2 = 2\sqrt{5}$

Now $\quad C_1C_2$ = distance between centres

$\therefore \quad C_1C_2 = \sqrt{9 + 81} = 3\sqrt{10} = 9.486$

and $\quad r_1 + r_2 = 2(\sqrt{10} + \sqrt{5}) = 10.6$

$r_1 - r_2 = 2\sqrt{5}(\sqrt{2} - 1)$

$= 2 \times 2.2 \times 0.4$

$= 4.4 \times 0.4 = 1.76$

$\Rightarrow \quad r_1 - r_2 < C_1C_2 < r_1 + r_2$

$\Rightarrow$ Two circles intersect at two distinct points

$\Rightarrow$ Two tangents can be drawn.

Example 2.51 **Find the equation of a circle with centre (4, 3) touching the circle $x^2 + y^2 = 1$.**

Sol. Let the circle be $x^2 + y^2 - 8x - 6y + k = 0$ touching the circle $x^2 + y^2 = 1$. Then the equation of the common tangent is $S_1 - S_2 = 0 \Rightarrow 8x + 6y - 1 - k = 0$

This is a tangent to the circle $x^2 + y^2 = 1$. Therefore,

$\pm 1 = \dfrac{k+1}{\sqrt{8^2 + 6^2}} \Rightarrow k + 1 = \pm 10 \Rightarrow k = -11$ or 9

Hence, the circles are $x^2 + y^2 - 8x - 6y + 9 = 0$ and, $x^2 + y^2 - 8x - 6y - 11 = 0$.

Alternative Solution:

The given circle is $x^2 + y^2 = 1$, which has centre $C_1(0, 0)$ and radius $r_1 = 1$. The required circle has centre $C_2(4, 3)$ and radius r_2

If the circle are touching externally then, $r_1 + r_2 = C_1C_2$

$\Rightarrow r_2 = 5 - 1 = 4$ If circles are touching internally then

$r_2 - r_1 = C_1C_2 \Rightarrow r_2 = 6$

Thus, required circles are $(x - 4)^2 + (y - 3)^2 = 16$ or $(x - 4)^2 + (y - 3)^2 = 36$.

Example 2.52 **Find the condition if circles whose equations are $x^2 + y^2 + c^2 = 2ax$ and $x^2 + y^2 + c^2 - 2by = 0$ will touch one another externally.**

Sol. The two circles are $x^2 + y^2 - 2ax + c^2 = 0$ and $x^2 + y^2 - 2by + c^2 = 0$

Centres: $\quad C_1(a, 0) \quad C_2(0, b)$

Radii: $\quad r_1 = \sqrt{a^2 - c^2}, r_2 = \sqrt{b^2 - c^2}$

Since the two circles touch each other externally, therefore $C_1C_2 = r_1 + r_2$

$\Rightarrow \quad \sqrt{a^2 + b^2} = \sqrt{a^2 - c^2} + \sqrt{b^2 - c^2}$

$\Rightarrow a^2 + b^2 = a^2 - c^2 + b^2 - c^2 + 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

$\Rightarrow \quad c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

$\Rightarrow \quad a^2b^2 = c^2(a^2 + b^2) \Rightarrow \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$.

Example 2.53 **Find the equation of the smaller circle that touches the circle $x^2 + y^2 = 1$ and passes through the point (4, 3).**

Sol.

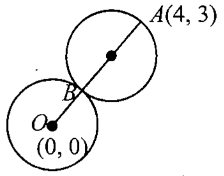


Fig. 2.49

For smallest circle OA will become common normal $OA = 5$
$\Rightarrow AB = 4$.

Equation of line OA is $y = \frac{3}{4}x$. Putting this value of 'y' in $x^2 + y^2 = 1$, we get

$$x^2 + \frac{9x^2}{16} = 1 \Rightarrow x = \pm\frac{4}{5}$$

$\Rightarrow B \equiv \left(\frac{4}{5}, \frac{3}{5}\right)$. Thus, required circle is $\left(x - \frac{4}{5}\right)(x - 4) + (y - 3)\left(y - \frac{3}{5}\right) = 0$

or $x^2 + y^2 - \frac{24}{5}x - \frac{18}{5}y + 5 = 0$.

Example 2.54 **Show that the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ touches each other. Find the coordinates of the point of contact and the equation of the common tangent at the point of contact.**

Sol.

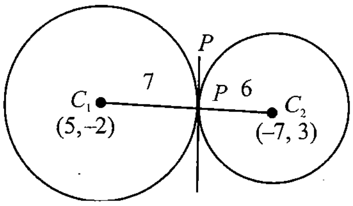


Fig. 2.50

$$C_1C_2 = \sqrt{[(5+7)^2 + (-2-3)^2]}$$
$$= 13 = r_1 + r_2$$

Hence, the two circles touch externally.

Coordinates of the point of contact:

If P is the point of contact of the two circle, then P will divide C_1C_2 internally in the ratio $r_1 : r_2$, i.e. 7 : 6.

$\therefore$ Coordinates of P are $\left(\frac{7.(-7) + 6.5}{7+6}, \frac{7.3 + 6.(-2)}{7+6}\right)$ or $(-19/13, 9/13)$

Equation of the common tangent:

Since the two circles touch each other, $S_1 - S_2 = 0$ is the common tangent at the point of contact which is $-24x + 10y - 42 = 0$ or $12x - 5y + 21 = 0$.

Example 2.55 **Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.**

Sol. Here for the first circle, centre is $C_1(1, 3)$ and radius $r_1 = 1$;

and for second circles, centre is $C_2(-3, 1)$ and radius $r_2 = 3$.

Thus, $C_1C_2 = 2\sqrt{5}$ and $r_1 + r_2 = 4 \Rightarrow C_1C_2 > r_1 + r_2, r_1 \neq r_2$; thus we have 4 common tangents.

To find direct common tangents:

The coordinates of the point P dividing line C_1C_2 in the ratio $r_1 : r_2$, i.e. 1 : 3 externally, are

$$\left(\frac{1.(-3) - 3.1}{1-3}, \frac{1.1 - 3.3}{1-3}\right) \text{ or } (3, 4)$$

Therefore, equation of any line through point $P(3, 4)$ is

$y - 4 = m_1(x - 3)$

or $m_1x - y + 4 - 3m_1 = 0$. (i)

If Eq. (i) is tangent to first circle, then length of $\perp$ from centre $C_1(1, 3)$ of (i) $= r_1$ (radius)

$$\Rightarrow \frac{|m_1 - 3 + 4 - 3m_1|}{\sqrt{(m_1^2 + 1)}} = 1 \Rightarrow (1 - 2m_1)^2 = m_1^2 + 1$$

$$\Rightarrow 3m_1^2 - 4m_1 = 0 \Rightarrow m_1 = 0, 4/3$$

Substituting $m_1 = 0$ and 4/3 in Eq. (i), the equation of direct common tangents are $y = 4$ and $4x - 3y = 0$.

To find transverse common tangents:

The coordinates of the point Q dividing the line C_1C_2 in the ratio $r_1 : r_2$, i.e. 1 : 3 internally, are (0, 5/2).

$\therefore$ Equation of any line through $Q(0, 5/2)$ is $y - 5/2 = m_2(x - 0)$ or $m_2x - y + 5/2 = 0$. (ii)

If Eq. (ii) is tangent to first circles, then length of $\perp$ from centre $C_1(1, 3)$ on Eq. (ii) $= r_1$ (radius)

$$\Rightarrow \frac{|m_2 - 3 + 5/2|}{\sqrt{(m_1^2 + 1)}} = 1 \Rightarrow (2m_2 - 1)^2 = 4(m_2^2 + 1)$$

$$\Rightarrow -4m_2 - 3 = 0$$

$\therefore$ $m_2 = -3/4$ and ∞, as coeff. of $m_2^2 = 0$.

Substituting $m_2 = -3/4$ and ∞ in Eq. (ii), the equations of transverse common tangents are

$$3x + 4y - 10 = 0 \text{ and } x = 0.$$

ANGLE OF INTERSECTION OF TWO CIRCLES

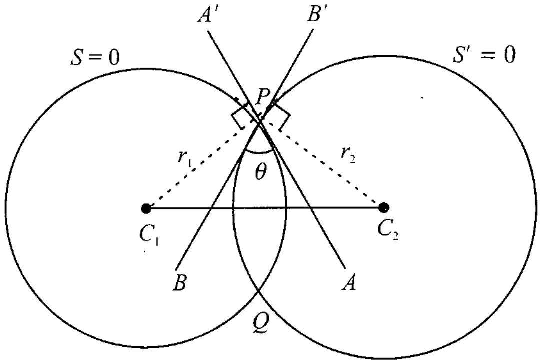


Fig. 2.51

Let the two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ intersect each other at the point P and Q. The angle θ between two circles $S = 0$ and $S' = 0$ is defined as the angle between the tangents to the two circles at the point of intersection. θ must be taken acute angle.

C_1 and C_2 are the centres of circles $S = 0$ and $S' = 0$, then $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ and radii of circles $S = 0$ and $S' = 0$ are $r_1 = \sqrt{g^2 + f^2 - c}$ and $r_2 = \sqrt{g_1^2 + f_1^2 - c_1}$

Let $d = |C_1C_2|$ = Distance between their centres

$$= \sqrt{(-g + g_1)^2 + (-f + f_1)^2}$$

$$= \sqrt{(g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1)}$$

$$C_1P \perp AA' \Rightarrow \angle C_1PB = \frac{\pi}{2} - \theta$$

$$C_2P \perp BB' \Rightarrow \angle C_2PA = \frac{\pi}{2} - \theta$$

$$\Rightarrow \quad \angle C_1PC_2 = \frac{\pi}{2} - \theta + \frac{\pi}{2} - \theta + \theta = \pi - \theta$$

Now, in ΔPC_1C_2, $\cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$ (i)

If the angle between the circles is 90°, i.e., $\theta = 90°$, then the circles are said to be **orthogonal circles** or we say that the cirlces cut each other **orthogonally.**

Then, from Eq. (i),

$$0 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\Rightarrow \quad r_1^2 + r_2^2 - d^2 = 0$$

$$\Rightarrow \quad r_1^2 + r_2^2 = d^2$$

$$\Rightarrow \quad g^2 + f^2 - c + g_1^2 + f_1^2 - c_1$$

$$= g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1$$

$$\Rightarrow \quad 2gg_1 + 2ff_1 = c + c_1$$

Example 2.56 **Find the angle at which the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect.**

Sol. The angle of intersection of two circles is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - C_1C_2}{2r_1r_2},$$

where r_1, r_2 are radii of two circles and C_1C_2 is the distance between their centres.

Here, $r_1 = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = r_2$ and $C_1C_2 = 1$

$$\Rightarrow \quad \cos\theta = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

Example 2.57 **If the circles $x^2 + y^2 + 2a'x + 2b'y + c' = 0$ and $2x^2 + 2y^2 + 2ax + 2by + c = 0$ intersect orthogonally, then prove that $aa' + bb' = c + \frac{c'}{2}$**

Sol. The given circles are $x^2 + y^2 + 2a'x + 2b'y + c' = 0$ and $x^2 + y^2 + ax + by + \frac{c}{2} = 0$

These two intersect orthogonally,

$$\therefore \quad 2\left(a'.\frac{a}{2} + b'.\frac{b}{2}\right) = c' + \frac{c}{2} \Rightarrow aa' + bb' = c' + \frac{c}{2}$$

Example 2.58 **A circle passes through the origin and has its centre on $y = x$. If it cuts $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then find the equation of the circle.**

Sol. Let the required circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

This passes through (0, 0), therefore $c = 0$

The centre $(-g, -f)$ of Eq. (i) lies on $y = x$, therefore $g = f$.

Since Eq. (i) cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, therefore

$$2(-2g - 3f) = c + 10 \Rightarrow -10g = 10$$

$[\because g = f \text{ and } c = 0]$

$$\Rightarrow \quad g = f = -1.$$

Hence, the required circle is $x^2 + y^2 - x - y = 0$

RADICAL AXIS

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

and $S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ (ii)

Let $P(x_1, y_1)$ be a point such that $|PA| = |PB|$

$$\Rightarrow \sqrt{x_1^2+y_1^2+2gx_1+2gy_1+c}$$
$$=\sqrt{x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1}$$

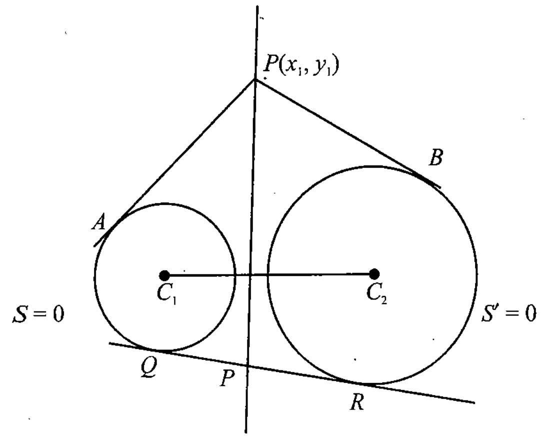


Fig. 2.52

On squaring, we get

$$x_1^2+y_1^2+2gx_1+2fy_1+c=x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1$$

$$\Rightarrow\ 2(g-g_1)x_1+2(f-f_1)y_1+c-c_1=0$$

Therefore, locus of $P(x_1, y_1)$ is

$$2(g-g_1)x+2(f-f_1)y+c-c_1=0$$

which is the required equation of radical axis of the given circles. Clearly, this is a straight line.

Properties of the Radical Axis

1. Radical axis is perpendicular to the line joining the centres of the given circles.

   Slope of $C_1C_2=\dfrac{-f_1+f}{-g_1+g}=\dfrac{f-f_1}{g-g_1}=m_1$ (say)

   Slope of radical axis is $-\dfrac{(g-g_1)}{(f-f_1)}=m_2$ (say)

   $\therefore\ m_1m_2=-1$

   Hence, C_1C_2 and radical axis are perpendicular to each other.

2. The radical axis bisects common tangents of two circles:

   Let QR be the common tangent. If it meets the radical axis P, then PQ and PR are two tangents to the circles. Hence, $PQ = PR$ since length of tangents are equal from any point on radical axis. Hence, radical axis bisects the common tangent QR.

Note:

Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

3. If two circles cut a third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle, or the locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the given two circles.

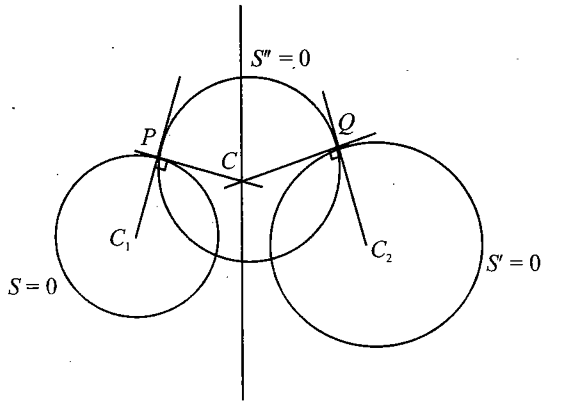


Fig. 2.53

Since circle $S'' = 0$ intersects the circle $S = 0$ and $S' = 0$ orthogonally

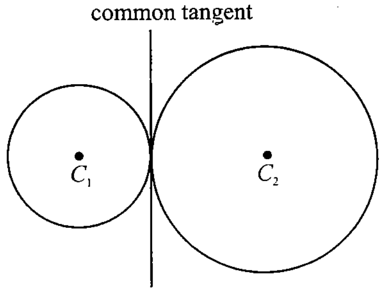
$C_1P \perp CP$ and $C_2Q \perp CQ$. (where C is centre of the circle $S'' = 0$)

But $CP = CQ$ = radius of the circle $S'' = 0$

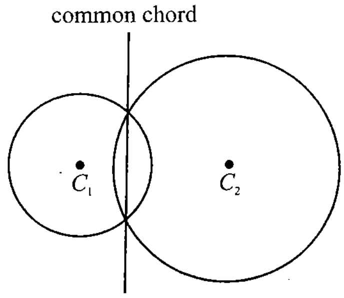
Hence, C lies on the radical axis of the circles $S = 0$ and $S' = 0$, as CP and CQ are also the length of tangents from C to the given circles.

4. The position of the radical axis of the two circles geometrically is shown below:

Radical axis is common tangent



Radical axis is common chord



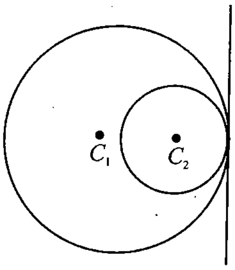


Fig. 2.54

Radical Centre

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be

$S_1 = 0$ (i)
$S_2 = 0$ (ii)
$S_3 = 0$ (iii)

Let OL, OM and ON be radical axes of the pair sets of circles $\{S_1 = 0, S_2 = 0\}$, $\{S_3 = 0, S_1 = 0\}$ and $\{S_2 = 0, S_3 = 0\}$ respectively.

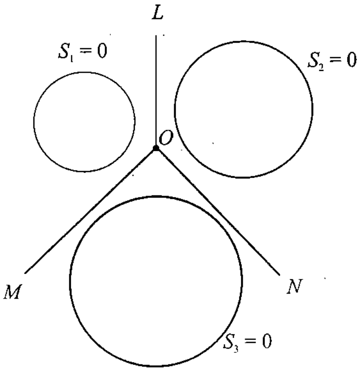


Fig. 2.55

Equations of OL, OM and ON are respectively

$S_1 - S_2 = 0$ (iv)
$S_3 - S_1 = 0$ (v)
$S_2 - S_3 = 0$ (vi)

Let the straight lines (iv) and (v) i.e., OL and OM meet in O. The equation of any straight line passing through O is

$$(S_1 - S_2) + \lambda (S_3 - S_1) = 0$$

where λ is any constant.

For $\lambda = 1$, this equation becomes $S_2 - S_3 = 0$ which is, by (vi), equation of ON.

Thus the third radical axis also passes through the point where (iv) and (v) meet. In the above figure, O is the **radical centre**.

Properties of Radical Centre

1. Coordinates of radical centre can be found by solving the equations $S_1 = S_2 = S_3 = 0$.
2. The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle. Draw perpendicular from A on BC.

   $\therefore \quad \angle ADB = \angle ADC = \pi/2$

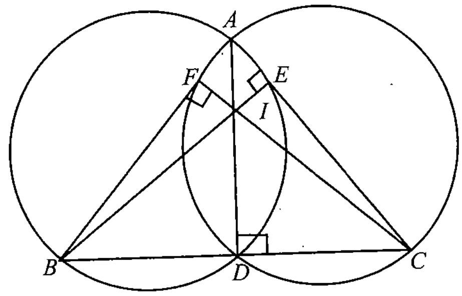


Fig. 2.56

Therefore, the circles whose diameters are AB and AC passes through D and A. Hence, AD is their radical axis. Similarly, the radical axis of the circles on AB and BC as diameter is the perpendicular line from B on CA and radical axis of the circles on BC and CA as diameter is the perpendicular line from C on AB. Hence, the radical axis of three circles meet in a point. This point I is radical centre but here radical centre is the point of intersection of altitudes, i.e., AD, BE and CF. Hence, radical centre = orthocentre.

3. The radical centre of three given circles will be the centre of a fourth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

   Let the fourth circle be $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is centre of this circle and r be the radius. The centre of circle is the radical centre of the given circles and r is the length of tangent from (h, k) to any of the given three circles.

Example 2.59 **If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, show that either $g = 3/4$ or $f = 2$.**

Sol. The radical axis of the given circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + (3/2)x + 4y + c = 0$,

is $(2g - 3/2)x + (2f - 4)y = 0$ or $(4g - 3)x + 4(f - 2)y = 0$ (i)

This radical axis (i) touches the circles $x^2 + y^2 + 2x + 2y + 1 = 0$, (ii)

if the length of $\perp$ from centre $(-1, -1)$ on the line (i) = radius of circle (ii),

i.e. $$\frac{(4g-3)(-1) + 4(f-2)(-1)}{\sqrt{[(4g-3)^2 + 16(f-2)^2]}} = \pm \sqrt{(1+1-1)}$$

$\Rightarrow \quad [(4g-3)+4(f-2)]^2 = (4g-3)^2 + 16(f-2)^2$

$\Rightarrow \quad 8(4g-3)(f-2) = 0$

$\Rightarrow \quad g = 3/4$ or $f = 2$

Example 2.60 **The equation of the three circles are given $x^2+y^2=1$, $x^2+y^2-8x+15=0$, $x^2+y^2+10y+24=0$.**

Determine the coordinates of the point P such that the tangents drawn from it to the circles are equal in length.

Sol. We know that the point from which lengths of tangents are equal in length is radical centre of the given three circles. Now radical axis of the first two circles is

$(x^2+y^2-1)-(x^2+y^2-8x+15)=0,$

i.e., $x-2=0,$ (i)

and radical axis of the second and third circles is

$(x^2+y^2-8x+15)-(x^2+y^2+10y+24)=0,$

i.e., $8x+10y+9=0$ (ii)

Solving Eqs. (i) and (ii), the coordinates of the radical centre, i.e. of point P are $P(2, -5/2)$.

Example 2.61 **The line $Ax+By+C=0$ cuts the circle $x^2+y^2+ax+by+c=0$ in P and Q. The line $A'x+B'x+C'=0$ cuts the circle $x^2+y^2+a'x+b'y+c'=0$ in R and S. If P, Q, R, S are concyclic, then show that**

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Sol. P and Q are the points of intersection of the line $L_1 \equiv Ax + By + C = 0$ (i)

and the circles $S_1 \equiv x^2+y^2+ax+by+c=0,$ (ii)

R and S are the points of intersection of the line $L_2 \equiv A'x+B'y+C'=0$ (iii)

and the circle $S_2 \equiv x^2+y^2+a'x+b'y+c'=0$ (iv)

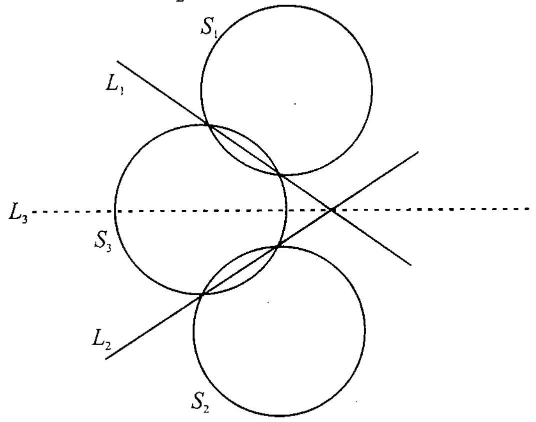


Fig. 2.57

Radical axis of circle $S_1=0$ and $S_2=0$ is $S_1-S_2=0,$

i.e. $L_3 \equiv (a-a')x+(b-b')y+(c-c')=0$ (v)

If P, Q, R, S are concyclic and $S_3=0$ is the equation of this circle through P, Q, R, S, line (i) is the radical axis of circles $S_1=0$ and $S_3=0$ and line (ii) is the radical axis of the circles $S_2=0$ and $S_3=0$.

Thus, the straight lines given by Eqs. (v), (i) and (iii) are the radical axes of circles $S_1=0$, $S_2=0$ and $S_3=0$ taken in pairs.

Since the radical axes of three circles taken in pairs are concurrent or parallel, $\therefore$ we have

$$\therefore \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Example 2.62 **Find the equation of a circle which cuts the three circles $x^2+y^2-3x-6y+14=0$, $x^2+y^2-x-4y+8=0$, $x^2+y^2+2x-6y+9=0$ orthogonally.**

Sol. The circle having centre at the radical centre of three given circles and radius equal to the length of the tangent from it to any one of three circles cuts all the three circles orthogonally. The given circles are

$x^2+y^2-3x-6y+14=0$ (i)

$x^2+y^2-x-4y+8=0$ (ii)

$x^2+y^2+2x-6y+9=0$ (iii)

The radical axes (i), (ii) and (iii) are, respectively

$x+y-3=0$ (iv)

and $3x-2y+1=0$ (v)

Solving Eqs. (iv) and (v), we get $x=1, y=2$

Thus, the coordinates of the radical centre are (1, 2).

The length of the tangent from (1, 2) to Eq. (i) is $r = \sqrt{1+4-3-12+14} = 2$

Hence, the required circle is $(x-1)^2+(y-2)^2=2^2$ or $x^2+y^2-2x-4y+1=0$.

COMMON CHORD OF TWO CIRCLES

The common chord joining the point of intersection of two given circles is called their common chord.

If $S=0$ and $S'=0$ be two intersecting circles, the equation of their common chord is

$$S - S' = 0$$

Let $S \equiv x^2+y^2+2gx+2fy+c=0$

and $S' \equiv x^2+y^2+2g'x+2f'y+c'=0$

be two circles intersecting at P and Q.

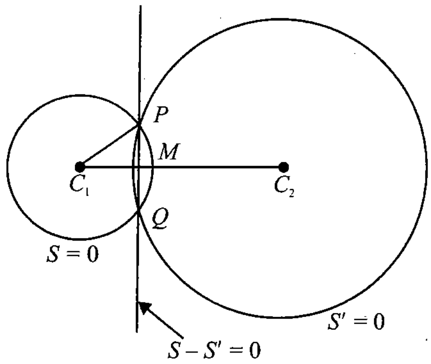


Fig. 2.58

Then PQ is their common chord.

$\therefore \quad S - S' = 0$

$\Rightarrow 2(g - g')\,x + 2(f - f')\,y + c - c' = 0$

is the common chord of two circles $S = 0$ and $S' = 0$.

Length of the Common Chord

$$PQ = 2(PM) = 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where C_1P = radius of the circle $S = 0$ and C_1M is the length of perpendicular form C_1 on common chord PQ.

Note:

1. *The length of common chord PQ of two circles is maximum when it is a diameter of the smaller circle.*
2. *If circle is described on the common chord as a diameter then centre of the circle passing through P and Q lie on the common chord of two circles i.e., $S - S' = 0$.*
3. *If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.*

Example 2.63 **If the tangents are drawn to the circle $x^2 + y^2 = 12$ at the point where it meets the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, then find the point of intersection of these tangents.**

Sol.

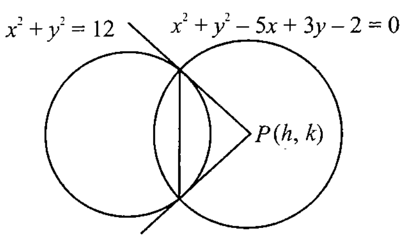


Fig. 2.59

Let (h, k) be the point of intersection of the tangents.

Then the chord of contact of tangents is the common chord of the circle $x^2 + y^2 = 12$

and $x^2 + y^2 - 5x + 3y - 2 = 0$

i.e. $\quad 5x - 3y - 10 = 0$

Also, the equation of the chord of contact w.r.t. P is is $hx + ky - 12 = 0$

Equations $hx + ky - 12 = 0$ and $5x - 3y - 10 = 0$ represent the same line, $\therefore \frac{h}{5} = \frac{k}{-3} = \frac{-12}{-10} \Rightarrow h = 6, k = \frac{-18}{5}$

Hence, the required point is $(6, -18/5)$.

Example 2.64 **Find the angle which the common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin.**

Sol.

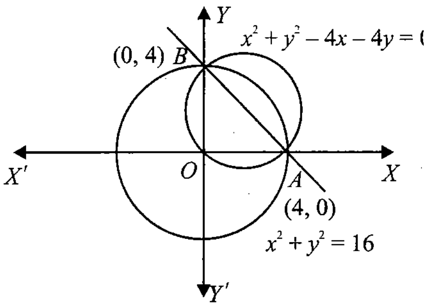


Fig. 2.60

The equation of the common chord of the circles $x^2 + y^2 - 4x - 4y = 0$

and $x^2 + y^2 = 16$ is $x + y = 4$ which meets $x^2 + y^2 = 16$ at $A(4, 0)$ and $B(-4, 0)$. Obviously $OA \perp OB$.

Hence, the common chord AB makes a right angle at the centre of the circle $x^2 + y^2 = 16$.

Example 2.65 **Find the length of the common chord of the circles $x^2 + y^2 + 2x + 6y = 0$ and $x^2 + y^2 - 4x - 2y - 6 = 0$**

Sol.

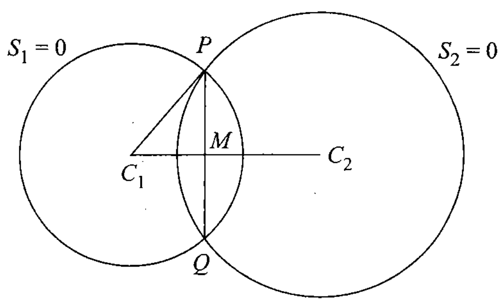


Fig. 2.61

The equation of common chord PQ of the circles

$$S_1 : x^2 + y^2 + 2x + 6y = 0$$

and $$S_2 : x^2 + y^2 - 4x - 2y - 6 = 0$$

is $$S_1 - S_2 = 0 \text{ or } 6x + 8y + 6 = 0 \text{ or } 3x + 4y + 3 = 0$$

centre of S_1 is $(-1, -3)$, radius $= \sqrt{1+9} = \sqrt{10}$

C_1M = length of the $\perp$ from $(-1, -3)$ to $3x + 4y + 3 = 0$

$$= \frac{|-3-12+3|}{\sqrt{9+16}} = \frac{12}{5}$$

Now $$PQ = 2PM = 2\sqrt{C_1P^2 - C_1M^2}$$

$$= 2\sqrt{10 - \frac{144}{25}}$$

$$= 2\sqrt{\left(\frac{106}{25}\right)}$$

$$= \frac{2\sqrt{106}}{5}$$

Example 2.66 **If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisect the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then prove that**

$$2g'(g - g') + 2f'(f - f') = c - c'$$

Sol. It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle

$x^2 + y^2 + 2g'x + 2f'y + c' = 0$, therefore the common chord of these two circles passes through the centre

$(-g', -f')$ of $x^2 + y^2 + 2g'x + 2f'y + c' = 0$

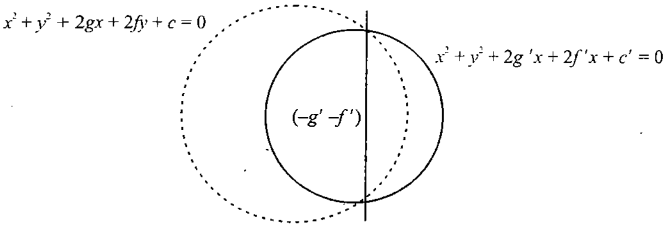


Fig. 2.62

The equation of the common chord of the two given circles is $2x(g - g') + 2y(f - f') + c - c' = 0$.

This passes through $(-g', -f')$

$\therefore -2g'(g - g') - 2f'(f - f') + c - c' = 0$

$\Rightarrow 2g'(g - g') + 2f'(f - f') = c - c'$

Concept Application Exercise 2.4

1. The circles $x^2 + y^2 - 12x - 12y = 0$ and $x^2 + y^2 + 6x + 6y = 0$

   a. touch each other externally
   b. touch each other internally
   c. intersect in two points
   d. none of these

2. If the circle $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct point P and Q, then find the values of a for which the line $5x + by - a = 0$ passes through P and Q.

3. Which of the following is a point on the common chord of the circle $x^2 + y^2 + 2x - 3y + 6 = 0$ and $x^2 + y^2 + x - 8y - 31 = 0$

   a. $(1, -2)$ **b.** $(1, 4)$
   c. $(1, 2)$ **d.** $(1, -4)$

4. Consider the circles $x^2 + (y - 1)^2 = 9$, $(x - 1)^2 + y^2 = 25$. They are such that

   a. These circles touch each other
   b. One of these circles lies entirely inside the other
   c. Each of these circles lies outside the other
   d. They intersect in two points

5. If the circles of same radius a and centres at $(2, 3)$ and $(5, 6)$ cut orthogonally, then find a.

6. If the two circles $2x^2 + 2y^2 - 3x + 6y + k = 0$ and $x^2 + y^2 - 4x + 10y + 16 = 0$ cut orthogonally, then find the value of k.

7. Find the condition that the circle $(x - 3)^2 + (y - 4)^2 = r^2$ lies entirely within the circle $x^2 + y^2 = R^2$.

8. Find the radical centre of the circles $x^2 + y^2 + 4x + 6y = 19$, $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x - 2y = 5$.

9. Find the equation of the circle which intersects circles $x^2 + y^2 + x + 2y + 3 = 0$, $x^2 + y^2 + 2x + 4y + 5 = 0$ and $x^2 + y^2 - 7y - 8y - 9 = 0$ at right angle.

10. Two circles 'C_2' and 'C_1' intersect in such a way that their common chord is of maximum length. Centre of C_1 is $(1, 2)$ and it's radius is 3 units. Radius of C_2 is 5 units. If slope of common chord is $\frac{3}{4}$, then find the centre of C_2.

11. The equation of a circle is $x^2 + y^2 = 4$. Find the centre of the smallest circle touching the circle and the line $x + y = 5\sqrt{2}$.

12. Consider four circles $(x \pm 1)^2 + (y \pm 1)^2 = 1$. Find the equation of smaller circle touching these four circles.

13. Find the equation of the circle whose radius is 3 and which touches internally the circle

    $x^2 + y^2 - 4x - 6y - 12 = 0$ at the point $(-1, -1)$.

14. Find the number of common tangents that can be drawn to the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$.

15. Two fixed circles with radii r_1 and $r_2 (r_1 > r_2)$, respectively, touch each other externally. Then

identify the locus of the point of intersection of their direct common tangents.

16. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents ($a > b \geq 2$), then prove that $\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$.

17. If the radius of the circle $(x-1)^2 + (y-2)^2 = 1$ and $(x-7)^2 + (y-10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 and 0.4 unit/sec, then at what value of t will they touch each other?

FAMILY OF CIRCLES

1. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as
$S + \lambda S' = 0$ (where λ is a parameter, $\lambda \neq -1$)

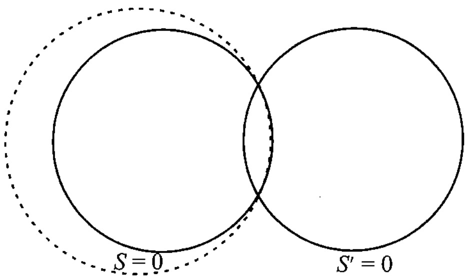


Fig. 2.63

2. The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$
(where λ is a parameter)

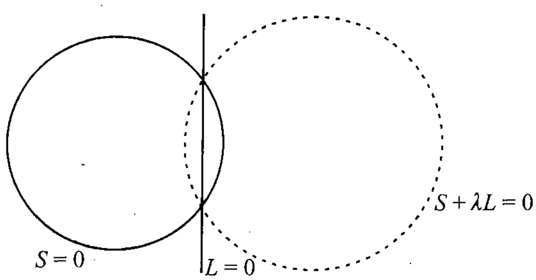


Fig. 2.64

3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$
(where λ is a parameter)

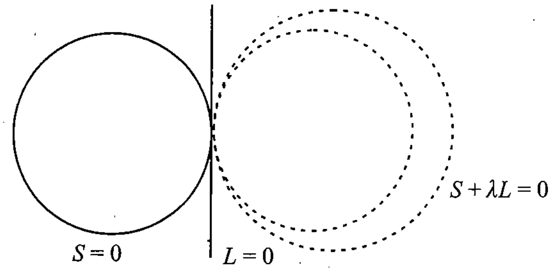


Fig. 2.65

4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$S + \lambda L = 0$$

(where λ is a parameter)

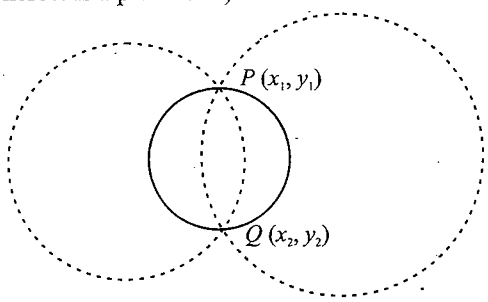
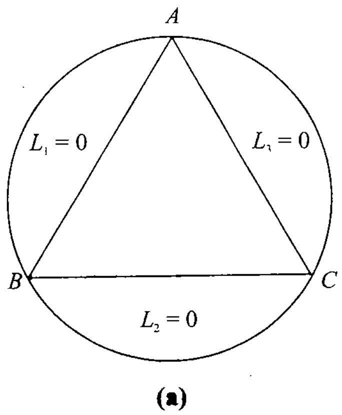


Fig. 2.66

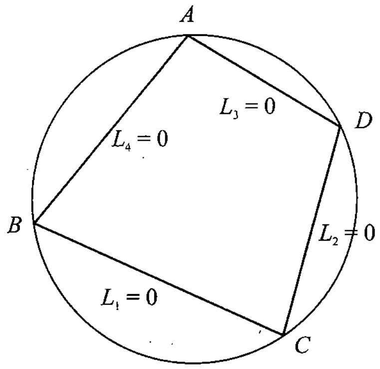
Here $S = 0$ is equation of circle with P and Q as end point of diameter and $L = 0$ is line through points P and Q.

5. The equation of family of circles which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is
$(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$ and if m is infinite, the family of cirlces is
$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$
(where λ is a parameter).
Here $(x - x_1)^2 + (y - y_1)^2 = 0$ is point circle at point (x_1, y_1)

6.



(a)



(b)

Fig. 2.67

Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.

Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Example 2.67 **If the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ cuts $x^2 + y^2 + 4x + 3y + 2 = 0$ in A and B, then find the equation of the circle on AB as diameter.**

Sol. The equation of the common chord AB of the two circles is $2x + 1 = 0$. [Using $S_1 - S_2 = 0$]

The equation of the required circle is $(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0$. [Using $S_1 + \lambda(S_2 - S_1) = 0$]

$\Rightarrow x^2 + y^2 + 2x(\lambda + 1) + 3y + \lambda + 1 = 0$

Since AB is a diameter of this circle, therefore centre lies on it.

So, $-2\lambda - 2 + 1 = 0 \Rightarrow \lambda = -1/2$

Thus, the required circle is $x^2 + y^2 + x + 3y + 1/2 = 0$

or $2x^2 + 2y^2 + 2x + 6y + 1 = 0$

Example 2.68 **Show that the equation of the circle passing through (1, 1) and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is $4x^2 + 4y^2 + 30x - 13y - 25 = 0$.**

Sol. Equation of the circle passing through the points of intersection of the given circle is

$$(x^2 + y^2 + 13x - 3y) + \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0 \quad \text{(i)}$$

If this circle passes through the point (1, 1), then

$(1 + 1 + 13 - 3) + \lambda(2 + 2 + 4 - 7 - 25) = 0$

$\Rightarrow \quad \lambda = 1/2$

Substituting $\lambda = 1/2$ in Eq. (i), the equation of the required circle is $4x^2 + 4y^2 + 30x - 13y - 25 = 0$.

Example 2.69 **Find the equation of the smallest circle passing through the intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$.**

Sol. Any circle passing through the points of intersection of the given line and circle has the equation

$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0$. Its centre $= \left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

The circle is the smallest if $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ is on the chord $x + y = 1$.

$\Rightarrow \quad -\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \Rightarrow \lambda = -1$

Putting this value for λ, the equation of the smallest circle is $x^2 + y^2 - 9 - (x + y - 1) = 0$.

Example 2.70 **C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0), respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Find the equation of the common tangent to C_1 and C_3 which does not pass through C_2.**

Sol.

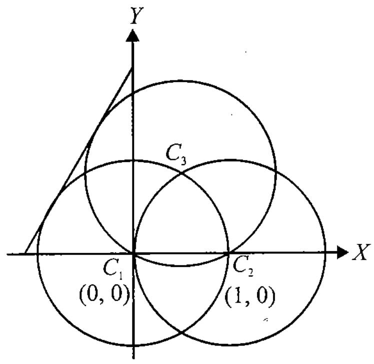


Fig. 2.68

Equation of any circle through (0, 0) and (1, 0) is

$$(x-1)(x-0) + (y-0)(y-0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$

If it represents C_3, its radius = 1

$\Rightarrow \quad 1 = (1/4) + (\lambda^2/4)$

$\Rightarrow \quad \lambda = \pm\sqrt{3}$

As the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3

is $\quad x^2 + y^2 - x - \sqrt{3}y = 0$.

Since C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and $(1/2, \sqrt{3}/2)$.

So, let the equation of a common tangent be $\sqrt{3}x - y + k = 0$.

It will touch C_1, if $\frac{|k|}{\sqrt{3+1}} = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}x - y + 2 = 0$.

Example 2.71 **Find the radius of the smallest circle which touches the straight-line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$. Compute upto one place of decimal only.**

Sol. Equation of the circle touching the line

$3x - y - 6 = 0$ at $(1, -3)$ is given by $(x-1)^2 + (y+3)^2 + \lambda(3x - y - 6) = 0$

i.e. $x^2 + y^2 + (3\lambda - 2)x + (6 - \lambda)y + 10 - 6\lambda = 0$ (i)

Equation (i) touches $y = x$,

$\therefore\ 2x^2 + (2\lambda + 4)x + 10 - 6\lambda = 0$

i.e. $x^2 + (\lambda + 2)x + 5 - 3\lambda = 0$ (ii)

has equal roots

$\Rightarrow (\lambda+2)^2 - 4(5 - 3\lambda) = \lambda^2 + 16\lambda - 16 = 0$

$\Rightarrow \lambda = \frac{1}{2}[-16 \pm \sqrt{(256+64)}] = -8 \pm \sqrt{(80)}$

Now $(\text{radius})^2 = R^2 = \frac{1}{4}[(3\lambda-2)^2 + (6-\lambda)^2 - (10 - 6\lambda)4]$
$= \frac{1}{4}[10\lambda^2]$

$\Rightarrow R = \frac{\sqrt{10}\lambda}{2} = \frac{\sqrt{10}}{2}(-8 \pm 4\sqrt{5}) = |-4\sqrt{10} \pm 10\sqrt{2}|$

$\Rightarrow$ radius of smaller circle $= |-4\sqrt{10} + 10\sqrt{2}| = 1.5$ approx.

Example 2.72 **A variable circle which always touches the line $x + y - 2 = 0$ at $(1, 1)$ cuts the circle $x^2 + y^2 + 4x + 5y - 6 = 0$. Prove that all the common chords of intersection pass through a fixed point. Find that points.**

Sol. Any circle which touches the line $x + y - 2 = 0$ at $(1, 1)$ will be of the form

$$(x-1)^2 + (y-1)^2 + \lambda(x + y - 2) = 0$$

or $x^2 + y^2 + (\lambda - 2)x + (\lambda - 2)y + 2 - 2\lambda = 0$

The common chord of this circle and $x^2 + y^2 + 4x + 5y - 6 = 0$ will be

$(\lambda - 6)x + (\lambda - 7)y + 8 - 2\lambda = 0$ or $(-6x - 7y + 8) + \lambda(x + y - 2) = 0$

which is a family of lines, each member of which will be passing through a fixed point, which is the point of intersection of the lines $-6x - 7y + 8 = 0$ and $x + y - 2 = 0$ which is $(6, -4)$.

Example 2.73 **Let S_1 be a circle passing through $A(0, 1)$, $B(-2, 2)$ and S_2 is a circle of radius $\sqrt{10}$ units such that AB is common chord of S_1 and S_2. Find the equation of S_2.**

Sol. Equation of line AB is

$$y - 2 = \frac{2-1}{-2-0}(x+2) = -\frac{1}{2}(x+2)$$

$\Rightarrow x + 2y - 2 = 0$ (i)

Equation of circle whose diagonally opposite points are A and B:

$$(x-0)(x+2) + (y-1)(y-2) = 0$$

$\Rightarrow\ x^2 + y^2 + 2x - 3y + 2 = 0$ (ii)

Family of circles passing through the points of intersection of Eqs. (i) and (ii)

$$x^2 + y^2 + 2x - 3y + 2 + \lambda(x + 2y - 2) = 0$$

$\Rightarrow\ x^2 + y^2 + (2+\lambda)x + (2\lambda - 3)y + 2 - 2\lambda = 0$ (iii)

Equation (iii), represents a circle of radius $\sqrt{10}$ units

$$\Rightarrow \sqrt{\left(-\frac{2+\lambda}{2}\right)^2 + \left(-\frac{2\lambda-3}{2}\right)^2 - 2 + 2\lambda} = \sqrt{10}$$

$\Rightarrow (4 + 4\lambda + \lambda^2) + (4\lambda^2 + 9 - 12\lambda) + 8\lambda - 8 = 40$
$\Rightarrow \lambda = \pm\sqrt{7}$

Hence, required circles are

$$x^2 + y^2 + 2x - 3y + 2 \pm \sqrt{7}(x + 2y - 2) = 0$$

There are two such circles possible.

Example 2.74 **If C_1, C_2 and C_3 belong to a family of circles through the points (x_1, y_1) and (x_2, y_2), prove that the ratio of the lengths of the tangent from any point on C_1 to the circles C_2 and C_3 is constant.**

Sol. Equations of the circles through (x_1, y_1) and (x_2, y_2) are

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda_r \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$
$(r = 1, 2, 3)$

Let (h, k) be a point on C_1.

$\Rightarrow\ \phi(h, k) + \lambda_1 \psi(h, k) = 0$

where $\phi(h, k) = (h - x_1)(h - x_2) + (k - y_1)(k - y_2)$

and $\psi(h, k) = \begin{vmatrix} h & k & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$

Let T_2 be the length of the tangent from (h, k) to C_2 and T_3 be the length of the tangent from (h, k) to C_3.

$$\Rightarrow T_2 = \sqrt{\phi(h,k) + \lambda_2 \psi(h,k)},\ T_3 = \sqrt{\phi(h,k) + \lambda_3 \psi(h,k)}$$

$$\Rightarrow \frac{T_2}{T_3} = \frac{\sqrt{\phi(h,k) + \lambda_2 \psi(h,k)}}{\sqrt{\phi(h,k) + \lambda_3 \psi(h,k)}} = \frac{\sqrt{(\lambda_2 - \lambda_1)\psi(h,k)}}{\sqrt{(\lambda_3 - \lambda_1)\psi(h,k)}}$$

$= \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}}$ which is independent of the choice of (h, k) and hence a constant.

PROBLEMS BASED ON LOCUS

Example 2.75 **If a line segment $AM = a$ moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, then find the locus of M.**

Sol.

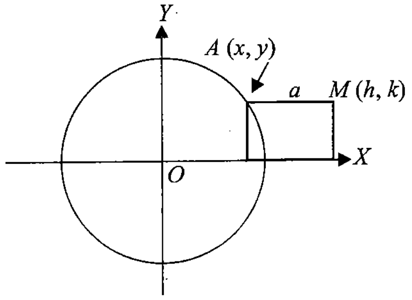


Fig. 2.69

Let the coordinates of A be (x, y) and M be (h, k)

Since AM is parallel to OX,

$$h = x + a \text{ and } k = y$$

$$\Rightarrow \quad x = h - a \text{ and } y = k$$

As $A(x, y)$ lies on the circle

$$x^2 + y^2 = a^2, \text{ we have}$$

$$(h - a)^2 + k^2 = a^2 \Rightarrow h^2 - 2ah + k^2 = 0$$

$\Rightarrow$ Locus of $M(h, k)$ is

$$x^2 + y^2 = 2ax.$$

Example 2.76 **The tangents to $x^2 + y^2 = a^2$ having inclinations α and β intersect at P. If $\cot\alpha + \cot\beta = 0$, then find the locus of P.**

Sol. Let the coordinates of P be (h, k). Let the equation of a tangent from $P(h, k)$ to the circle $x^2 + y^2 = a^2$ be

$$y = mx + a\sqrt{1 + m^2}..$$

Since $P(h, k)$ lies on $\quad y = mx + a\sqrt{1 + m^2}.$

$$\therefore \quad k = mh + a\sqrt{1 + m^2}$$

$$\Rightarrow \quad (k - mh)^2 = a\sqrt{(1 + m^2)}$$

$$\Rightarrow m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$$

This is a quadratic in m. Let the two roots be m_1 and m_2. Then, $m_1 + m_2 = \dfrac{2hk}{h^2 - a^2}$

But $\tan\alpha = m_1$, $\tan\beta = m_2$ and it is given that $\cot\alpha + \cot\beta = 0$

$$\therefore \quad \frac{1}{m_1} + \frac{1}{m_2} = 0 \Rightarrow m_1 + m_2 = 0$$

$$\Rightarrow \quad \frac{2hk}{h^2 - a^2} = 0 \Rightarrow hk = 0$$

Hence, the locus of (h, k) is

$$xy = 0.$$

Example 2.77 **Find the locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by $\pi/3$.**

Sol.

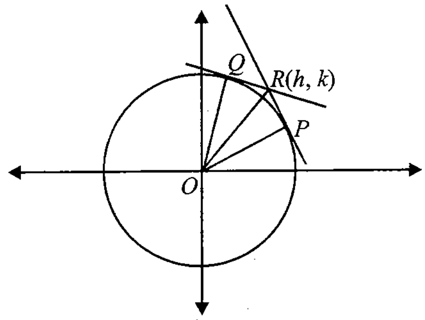


Fig. 2.70

Let the parametric angles of two points on the circle

$$x^2 + y^2 = a^2 \text{ be } \theta \text{ and } \pi/3 + \theta.$$

Then, the two points are $P(a\cos\theta, a\sin\theta)$ and $Q(a\cos(\pi/3 + \theta), a\sin(\pi/3 + \theta))$

In the figure $\quad \angle POQ = \pi/3$ and $\angle POR = \pi/6$.

In ΔOPR, $\quad OP = OR\cos 30°$

$$\Rightarrow \quad a = \sqrt{h^2 + k^2}\,\frac{\sqrt{3}}{2}$$

$\Rightarrow$ Locus of $R(h, k)$ is

$$3(x^2 + y^2) = 4a^2.$$

Example 2.78 **Find the locus of the centre of a circle touching the circle $x^2 + y^2 - 4y - 2x = 4$ internally and tangents on which from (1, 2) is making a 60° angle with each other.**

Sol.

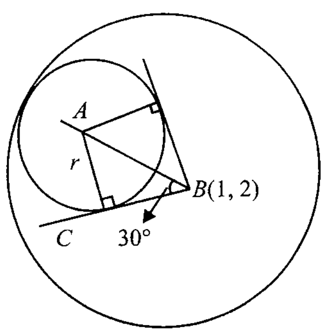


Fig. 2.71

Let r and R be radius of required and given circle respectively and let centre is (h, k).

By given condition $\sqrt{(h-1)^2+(k-2)^2} = R - r$

Now, $\frac{r}{AB} = \sin 30°$

$\Rightarrow \quad r = AB \sin 30° = (R-r)\frac{1}{2}$

$(AB = R - r)$

$\Rightarrow \quad \sqrt{(h-1)^2+(k-2)^2} = R - \frac{R}{3} = \frac{2R}{3}$

Now, $R = 3$

$\Rightarrow \quad \sqrt{(h-1)^2+(k-2)^2} = 2$

$\Rightarrow$ Locus is

$(x-1)^2 + (y-2)^2 = 4$

Example 2.79 **Two rods of lengths a and b slide along the x-axis and y-axis, respectively, in such a manner that their ends are concyclic. Find the locus of the centre of the circle passing through the end points.**

Sol.

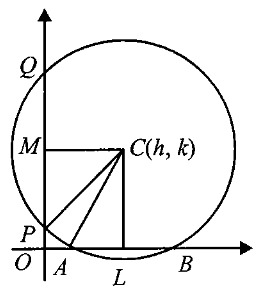


Fig. 2.72

Let $C(h, k)$ be the centre of the circle passing through the end points of the rod AB and PQ of lengths a and b, respectively,

CL and CM be perpendiculars from C on AB and PQ, respectively.

Then $AL = (1/2)\, AB = a/2$,

$PM = (1/2)\, PQ = b/2$

and $CA = CP$ (radii of the same circle)

$\Rightarrow \quad k^2 + \frac{a^2}{4} = h^2 + \frac{b^2}{4}$

$\Rightarrow \quad 4(h^2 - k^2) = a^2 - b^2$

So that locus of (h, k) is $4(x^2 - y^2) = a^2 - b^2$.

Example 2.80 **A circle with centre at the origin and radius equal to a meets the axis of x at A and B.$P(\alpha)$ and $Q(\beta)$ are two points on the circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. Find the locus of the point of intersection of AP and BQ.**

Sol.

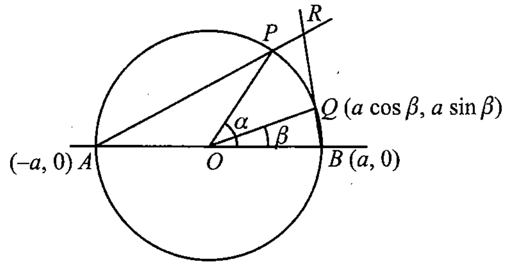


Fig. 2.73

Coordinates of A are $(-a, 0)$ and of P are $(a \cos \alpha, a \sin \alpha)$

$\therefore$ Equation of AP is $\quad y = \frac{a \sin \alpha}{a(\cos \alpha + 1)}(x + a)$

or $\quad y = \tan(\alpha/2)(x + a) \quad$ (i)

Similarly equation of BQ is

$y = \frac{a \sin \beta}{a(\cos \beta - 1)}(x - a)$

or $\quad y = -\cot(\beta/2)(x - a) \quad$ (ii)

We now eliminate α, β from Eqs. (i) and (ii)

$\therefore \quad \tan(\alpha/2) = \frac{y}{a+x}, \tan(\beta/2) = \frac{a-x}{y}$

Now $\quad \alpha - \beta = 2\gamma$

$\Rightarrow \quad \tan \gamma = \frac{\tan(\alpha/2) - \tan(\beta/2)}{1 + \tan(\alpha/2)\tan(\beta/2)}$

$= \frac{\frac{y}{a+x} - \frac{a-x}{y}}{1 + \frac{y}{a+x}\cdot\frac{a-x}{y}}$

$\Rightarrow \quad \tan \gamma = \frac{y^2 - (a^2 - x^2)}{(a+x)y + (a-x)y}$

$= \frac{x^2 + y^2 - a^2}{2ay}$

$\Rightarrow \quad x^2 + y^2 - 2ay \tan \gamma = a^2.$

Example 2.81 **Find the locus of the midpoint of the chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at the point $(c, 0)$.**

Sol.

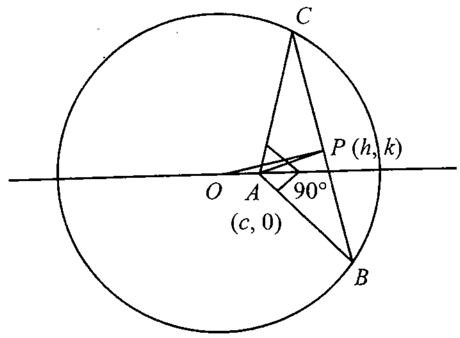


Fig. 2.74

Let $P(h, k)$ be the midpoint of a chord BC which subtends a right angle at $A(c, 0)$.

Then, clearly

$$AP = PC = PB = \sqrt{[(h-c)^2 + k^2]} \quad \text{(i)}$$

Also $$PC = \sqrt{(a^2 - OP^2)}$$

$$= \sqrt{[a^2 - (h^2 + k^2)]} \quad \text{(ii)}$$

From Eq. (i) and (ii), generalizing (h, k), we get the locus of P as

$$(x-c)^2 + y^2 = a^2 - (x^2 + y^2)$$

i.e. $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$.

Example 2.82 **A variable circle passes through the point $A(a, b)$ and touches the x-axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.**

Sol.

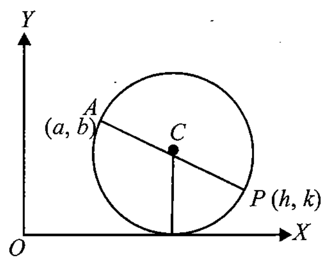


Fig. 2.75

$|y\text{-coord. of centre}| = \text{radius}$

$$\Rightarrow \quad \left(\frac{k+b}{2}\right)^2 = \frac{(h-a)^2 + (k-b)^2}{4}$$

$\Rightarrow$ Locus of $P(h, k)$ is

$$(x-a)^2 = 4by.$$

Concept Application Exercise 2.5

1. Find the locus of the midpoint of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$, which makes an angle of 120° at the centre
2. A tangent is drawn to each of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Show that if the two tangents are mutually perpendicular, the locus of their point of intersection is a circle concentric with the given circles.
3. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a variable triangle OAB. Sides OA and OB lie along the x-and y-axis, respectively, where 'O' is the origin. Find the locus of the midpoint of side AB.
4. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant, prove that its locus is a circle.
5. 'P' is the variable point on the circle with centre at C. CA and CB are perpendicular from C on x-axis and y-axis respectively. Show that the locus of the centroid of triangle PAB is a circle with centre at the centroid of triangle CAB and radius equal to the one third of the radius of the given circle.
6. Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of x, equidistant from the point $(k, 0)$. Show that the locus of their intersection is $ky^2 = a^2(k - x)$.
7. A straight line moves so that the product of length of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of the perpendiculars from each of these points upon the straight-line is a unique circle.

EXERCISES

Subjective Type

Solutions on page 2.56

1. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N, respectively. If the distance from C to the line-segment MN is equal to 5 units, find the area of the rectangle $ABCD$.
2. Let ABC be a triangle right angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB, AC and the circle S internally. Further, let S_2 be the circle touching the lines AB and AC produce and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2, respectively, show that $r_1r_2 = 4$ area(ΔABC).
3. Find the range of parameter 'a' for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.
4. Find the locus of the centres of the circles $x^2 + y^2 - 2ax - 2by + 2 = 0$, where '$a$' and '$b$' are parameters, if the tangents from the origin to each of the circles are orthogonal.
5. Three concentric circles, of which biggest is $x^2 + y^2 = 1$, have their radii in A.P. If the line $y = x + 1$ cuts all the circles in real and distinct points, then find the interval in which the common difference of A.P. will lie.

6. Let $A = (-1, 0)$, $B = (3, 0)$ and PQ be any line passing through $(4, 1)$ having slope m. Find the range of 'm' for which there exist two points on PQ at which AB subtends a right angle.

7. The equation of radical axis of two circles is $x + y = 1$. One of the circles has the ends of a diameter at the points $(1, -3)$ and $(4, 1)$ and the other passes through the point $(1, 2)$. Find the equations of these circles.

8. S is a circle having centre at $(0, a)$ and radius $b(b < a)$. A variable circle centred at $(\alpha, 0)$ and touching circle S, meets the X-axis at M and N. Find a point P on the Y-axis, such that $\angle MPN$ is a constant for any choice of α.

9. $S(x, y) = 0$ represents a circle. The equation $S(x, 2) = 0$ gives two identical solutions $x = 1$ and the equation $S(1, y) = 0$ gives two solutions $y = 0, 2$. Find the equation of the circle.

10. Find the equation of a family of circles touching the lines $x^2 - y^2 + 2y - 1 = 0$.

11. A and B are two points in xy-plane, which are $2\sqrt{2}$ unit distance apart and subtend an angle of 90° at the point $C(1, 2)$ on the line $x - y + 1 = 0$, which is larger than any angle subtended by the line segment AB at any other point on the line. Find the equation(s) of the circle through the points A, B and C.

12. From the variable point A on a circle $x^2 + y^2 = 2a^2$, two tangents are drawn to the circle $x^2 + y^2 = a^2$ which meet the curve at B and C. Find the locus of the circumcentre of ΔABC.

13. Find the circle of minimum radius which passes through the point $(4, 3)$ and touches the circle $x^2 + y^2 = 4$ externally.

14. Two variable chords AB and BC of a circle $x^2 + y^2 = r^2$ are such that $AB = BC = r$. Find the locus of point of intersection of tangents at 'A' and 'C'.

15. If $3x + y = 0$ is a tangent to a circle whose centre is $(2, -1)$, then find the equation of the other tangent to the circle from the origin.

16. Find the length of the chord of contact with respect to the point on the director circle of circle $x^2 + y^2 + 2ax - 2by + a^2 - b^2 = 0$.

17. A circle $x^2 + y^2 + 4x - 2\sqrt{2}\,y + c = 0$ is the director circle of circle S_1, and S_2, is the director circle of circle S_1 and so on. If the sum of radii of all these circles is 2, then find the value of c.

18. Consider three circles C_1, C_2 and C_3 such that C_2 is the director circle of C_1 and C_3 is the director circle of C_2. Tangents to C_1 from any point on C_3 intersect C_2 at P and Q. Find the angle between the tangents to C_2 at P and Q. Also identify the locus of the point of intersection of tangents at P and Q.

19. From a point P on the normal $y = x + c$ of the circle $x^2 + y^2 - 2x - 4y + 5 - \lambda^2 = 0$, two tangents are drawn to the same circle touching it at point B and C. If area of quadrilateral $OBPC$ (where O is the centre of the circle) is 36 sq. units. Find the possible values of λ, it is given that point P is at a distance $|\lambda|(\sqrt{2} - 1)$ from the circle.

20. Find the centre of the smallest circle which cut circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 8x + 8y - 33 = 0$ orthogonally.

21. Perpendiculars are drawn, respectively, from the points P and Q to the chords of contact of the points Q and P with respect to a circle. Prove that the ratio of the lengths of perpendiculars is equal to the ratio of the distances of the points P and Q from the centre of the circles.

22. Find the number of such points $(a + 1, \sqrt{3}\,a)$, where $a \in Z$, lying inside the region bounded by the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 2x - 15 = 0$.

23. If eight distinct points can be found on the curve $|x| + |y| = 1$ such that from each point two mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, then find the range of a.

24. A circle of radius 5 units has diameter along the angle bisector of the lines $x + y = 2$ and $x - y = 2$. If chord of contact from origin makes an angle of 45° with the positive direction of x-axis, find the equation of the circle.

25. Let AB be the chord of contact of the point $(5, -5)$ w.r.t. the circle $x^2 + y^2 = 5$, then find the locus of the orthocentre of the triangle PAB, where P be any point moving on the circle.

26. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$. AB be the chord of contact of this point w.r.t. the circle $x^2 + y^2 - 2x = 0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle.

27. AB is a diameter of a circle, CD is a chord parallel to AB and $2CD = AB$. The tangent at B meets the line AC produced at E. Prove that $AE = 2AB$.

28. Two parallel tangents to a given circle are cut by a third tangent in the points R and Q. Show that the lines from R and Q to the centre of the circle are mutually perpendicular.

29. A circle of radius 1 unit touches positive x-axis and positive y-axis at A and B, respectively. A variable line passing through origin intersects the circle in two points D and E. If the area of the triangle DEB is maximum when the slope of the line is m, then find the value of m^{-2}.

Objective Type

Solutions on page 2.62

Each question has four choices a, b, c and d, out of which *only one* answer is correct.

1. The number of rational point(s) (a point (a, b) is called rational, if a and b both are rational numbers) on the circumference of a circle having centre (π, e) is

   a. at most one **b.** at least two
   c. exactly two **d.** infinite

2. If the equation of any two diagonals of a regular pentagon belongs to family of lines $(1 + 2\lambda)y - (2 + \lambda)x + 1 - \lambda = 0$ and their lengths are sin 36°, then locus of centre of circle circumscribing the given pentagon (the triangles formed by these diagonals with sides of pentagon have no side common) is

   a. $x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$

   b. $x^2 + y^2 - 2x - 2y + \cos^2 72° = 0$

   c. $x^2 + y^2 - 2x - 2y + 1 + \cos^2 72° = 0$

   d. $x^2 + y^2 - 2x - 2y + \sin^2 72° = 0$

3. If OA and OB are equal perpendicular chord of the circles $x^2 + y^2 - 2x + 4y = 0$, then equations of OA and OB are where O is origin.

   a. $3x + y = 0$ and $3x - y = 0$

   b. $3x + y = 0$ or $3y - x = 0$

   c. $x + 3y = 0$ and $y - 3x = 0$

   d. $x + y = 0$ or $x - y = 0$

4. Equation of chord of the circle $x^2 + y^2 - 3x - 4y - 4 = 0$, which passes through the origin such that the origin divides it in the ratio 4 : 1, is

   a. $x = 0$ b. $24x + 7y = 0$

   c. $7x + 24y = 0$ d. $7x - 24y = 0$

5. The line $2x - y + 1 = 0$ is tangent to the circle at the point (2, 5) and the centre of the circles lies on $x - 2y = 4$. The radius of the circle is

   a. $3\sqrt{5}$ b. $5\sqrt{3}$ c. $2\sqrt{5}$ d. $5\sqrt{2}$

6. In a triangle ABC, right angled at A, on the leg AC as diameter, a semicircle is described. If a chord joins A with the point of intersection D of the hypotenuse and the semicircle, then the length of AC equals to

   a. $\dfrac{AB.AD}{\sqrt{AB^2 + AD^2}}$ b. $\dfrac{AB.AD}{AB + AD}$

   c. $\sqrt{AB.AD}$ d. $\dfrac{AB \cdot AD}{\sqrt{AB^2 - AD^2}}$

7. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

   a. $8\sqrt{3}$ sq. units b. $4\sqrt{3}$ sq. units

   c. $6\sqrt{3}$ sq. units d. None

8. The locus of the centre of the circles such that the point (2, 3) is the midpoint of the chord $5x + 2y = 16$ is

   a. $2x - 5y + 11 = 0$ b. $2x + 5y - 11 = 0$

   c. $2x + 5y + 11 = 0$ d. None

9. Two congruent circles with centres at (2, 3) and (5, 6), which intersect at right angles, have radius equal to

   a. $2\sqrt{2}$ b. 3 c. 4 d. None

10. A circle of radius unity is centred at origin. Two particles start moving at the same time from the point (1, 0) and move around the circle in opposite direction. One of the particle moves counterclockwise with constant speed v and the other moves clockwise with constant speed $3v$. After leaving (1, 0), the two particles meet first at a point P, and continue until they meet next at point Q. The coordinates of the point Q are

    a. (1, 0) b. (0, 1) c. (0, –1) d. (–1, 0)

11. The value of 'c' for which the set $\{(x, y) \mid x^2 + y^2 + 2x \le 1\} \cap \{(x, y) \mid x - y + c \ge 0\}$ contains only one point in common is

    a. $(-\infty, -1] \cup [3, \infty)$ b. $\{-1, 3\}$

    c. $\{-3\}$ d. $\{-1\}$

12. A circle is inscribed into a rhombous $ABCD$ with one angle 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P is any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to

    a. 12 b. 11

    c. 9 d. None of these

13. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents $(a > b \ge 2)$, then

    a. $\theta = 2\cos^{-1}\left(\dfrac{a-b}{a+b}\right)$

    b. $\theta = 2\tan^{-1}\left(\dfrac{a+b}{a-b}\right)$

    c. $\theta = 2\sin^{-1}\left(\dfrac{a+b}{a-b}\right)$

    d. $\theta = 2\sin^{-1}\left(\dfrac{a-b}{a+b}\right)$

14. B and C are fixed points having co-ordinates (3, 0) and (–3, 0), respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation

    a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 2$

    c. $9(x^2 + y^2) = 1$ d. $9(x^2 + y^2) = 4$

15. $ABCD$ is a square of unit area. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. The radius of the circle is

    a. $2 - \sqrt{2}$ b. $\sqrt{2} - 1$

    c. $\dfrac{1}{2}$ d. $\dfrac{1}{\sqrt{2}}$

16. A pair of tangents are drawn to a unit circle with centre at the origin and these tangents intersect at A enclosing an angle of 60°. The area enclosed by these tangents and the arc of the circle is

    a. $\dfrac{2}{\sqrt{3}} - \dfrac{\pi}{6}$ b. $\sqrt{3} - \dfrac{\pi}{3}$

    c. $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{6}$ d. $\sqrt{3}\left(1 - \dfrac{\pi}{6}\right)$

17. A straight line with slope 2 and y-intercept 5 touches the circle, $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are

a. $(-6, 11)$ b. $(-9, -13)$
c. $(-10, -15)$ d. $(-6, -7)$

18. A circle of constant radius 'a' passes through origin 'O' and cuts the axes of co-ordinates in points P and Q, then the equation of the locus of the foot of perpendicular from O to PQ is

a. $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$

b. $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$

c. $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$

d. $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$

19. A line meets the co-ordinate axes in A and B. A circle is circumscribed about the triangle OAB. If d_1 and d_2 are the distances of the tangent to the circle at the origin O from the points A and B, respectively, then the diameter of the circle is

a. $\frac{2d_1 + d_2}{2}$ b. $\frac{d_1 + 2d_2}{2}$
c. $d_1 + d_2$ d. $\frac{d_1 d_2}{d_1 + d_2}$

20. If a circle of constant radius $3k$ passes through the origin 'O' and meets co-ordinate axes at A and B, then the locus of the centroid of the triangle OAB is

a. $x^2 + y^2 = (2k)^2$ b. $x^2 + y^2 = (3k)^2$
c. $x^2 + y^2 = (4k)^2$ d. $x^2 + y^2 = (6k)^2$

21. The equation of a line inclined at an angle $\pi/4$ to the X-axis, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is

a. $2x - 2y - 3 = 0$ b. $2x - 2y + 3 = 0$
c. $x - y + 6 = 0$ d. $x - y - 6 = 0$

22. Let C be a circle with two diameters intersecting at an angle of 30°. A circle S is tangent to both the diameters and to C, and has radius unity. The largest radius of C is

a. $1 + \sqrt{6} + \sqrt{2}$ b. $1 + \sqrt{6} - \sqrt{2}$
c. $\sqrt{6} + \sqrt{2} - 11$ d. None of these

23. A straight line l_1 with equation $x - 2y + 10 = 0$ meets the circle with equation $x^2 + y^2 = 100$ at B in the first quadrant. A line through B, perpendicular to l_1 cuts y-axis at $P(0, t)$. The value of 't' is

a. 12 b. 15 c. 20 d. 25

24. Let a and b represent the length of a right triangle's legs. If d is the diameter of a circle inscribed into the triangle, and D is the diameter of a circle circumscribed on the triangle, then $d + D$ equals

a. $a + b$ b. $2(a + b)$
c. $\frac{1}{2}(a + b)$ d. $\sqrt{a^2 + b^2}$

25. If the chord $y = mx + 1$ of the circles $x^2 + y^2 = 1$ subtends an angle 45° at the major segment of the circle, then value of m is

a. 2 b. -2
c. -1 d. None of these

26. A variable chord of circle $x^2 + y^2 = 4$ is drawn from the point $P(3, 5)$ meeting the circle at the points A and B. A point Q is taken on this chord such that $2PQ = PA + PB$. Locus of 'Q' is

a. $x^2 + y^2 + 3x + 4y = 0$
b. $x^2 + y^2 = 36$
c. $x^2 + y^2 = 16$
d. $x^2 + y^2 - 3x - 5y = 0$

27. In triangle ABC, equation of side BC is $x - y = 0$. Circumcentre and orthocentre of the triangle are (2, 3) and (5, 8), respectively. Equation of circumcircle of the triangle is

a. $x^2 + y^2 - 4x + 6y - 27 = 0$
b. $x^2 + y^2 - 4x - 6y - 27 = 0$
c. $x^2 + y^2 + 4x + 6y - 27 = 0$
d. $x^2 + y^2 + 4x + 6y - 27 = 0$

28. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut-off by the line $x + y = 2$, is

a. $(-\infty, 5\sqrt{2})$
b. $(4\sqrt{2} - \sqrt{14}, 5\sqrt{2})$
c. $(4\sqrt{2} - \sqrt{14}, 4\sqrt{2} + \sqrt{14})$
d. None of these

29. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the coordinate axis. The co-ordinates of its vertices are

a. $(-6, -9), (-6, 5), (8, -9), (8, 5)$
b. $(-6, 9), (-6, -5), (8, -9), (8, 5)$
c. $(-6, -9), (-6, 5), (8, 9), (8, 5)$
d. $(-6, -9), (-6, 5), (8, -9), (8, -5)$

30. $(-6, 0)$, $(0, 6)$ and $(-7, 7)$ are the vertices of a ΔABC. The incircle of the triangle has the equation

a. $x^2 + y^2 - 9x - 9y + 36 = 0$
b. $x^2 + y^2 + 9x - 9y + 36 = 0$
c. $x^2 + y^2 + 9x + 9y - 36 = 0$
d. $x^2 + y^2 + 18x - 18y + 36 = 0$

31. If O is the origin and OP, OQ are the tangents from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$, then circumcenter of the triangle OPQ is

a. $(3, -2)$ b. $\left(\frac{3}{2}, -1\right)$

c. $\left(\frac{3}{4}, -\frac{1}{2}\right)$ d. $\left(-\frac{3}{2}, 1\right)$

32. The locus of the midpoint of a line segment that is drawn from a given external point P to a given circle with centre O (where O is origin) and radius r, is

a. a straight line perpendicular to PO

b. a circle with centre P and radius r

c. a circle with centre P and radius $2r$

d. a circle with centre at the midpoint PO and radius $r/2$

33. The difference between the radii of the largest and the smallest circles which have their centre on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$ and pass through the point (a, b) lying outside the given circle, is

a. 6 b. $\sqrt{(a+1)^2 + (b+2)^2}$

c. 3 d. $\sqrt{(a+1)^2 + (b+2)^2} - 3$

34. An isosceles triangles ABC is inscribed in a circle $x^2 + y^2 = a^2$ with the vertex A at $(a, 0)$ and the base angle B and C each equal to 75°, then coordinates of an end point of the base are

a. $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$ b. $\left(-\frac{\sqrt{3}a}{2}, a\right)$

c. $\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ d. $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

35. The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is

a. $(\sqrt{2} + 2)a$ b. $2\sqrt{2}a$

c. $(\sqrt{2} + 1)a$ d. $(2 + \sqrt{2})a$

36. The locus of a point which moves such that the sum of the squares of its distance from three vertices of a triangle is constant is a/an

a. circle b. straight line

c. ellipse d. None of these

37. A circle passes through the points $A(1, 0)$, $B(5, 0)$ and touches the y-axis at $C(0, h)$. If $\angle ACB$ is maximum then

a. $h = 3\sqrt{5}$ b. $h = 2\sqrt{5}$ c. $h = \sqrt{5}$ d. $h = 2\sqrt{10}$

38. A circle with centre (a, b) passes through the origin. The equation of the tangent to the circle at the origin is

a $ax - by = 0$ b. $ax + by = 0$

c. $bx - ay = 0$ d. $bx + ay = 0$

39. The area of the triangle formed by joining the origin to the points of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

a. 3 b. 4 c. 5 d. 6

40. If (α, β) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at $(2, -2)$, then the greatest values of α is

a. $4 - \sqrt{2}$ b. 6 c. $4 + 2\sqrt{2}$ d. $4 + \sqrt{2}$

41. A region in the x–y plane is bounded by the curve $y = \sqrt{25 - x^2}$ and the line $y = 0$. If the point $(a, a + 1)$ lies in the interior of the region, then

a. $a \in (-4, 3)$ b. $a \in (-\infty, -1) \in (3, \infty)$

c. $a \in (-1, 3)$ d. None of these

42. There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in Z$. If the two circles have exactly two common tangents then the number of possible values of n is

a. 2 b. 8

c. 9 d. None of these

43. C_1 is a circle of radius 1 touching the x-axis and the y-axis. C_2 is another circle of radius > 1 and touching the axes as well as the circle C_1. Then, the radius of C_2 is

a. $3 - 2\sqrt{2}$ b. $3 + 2\sqrt{2}$

c. $3 + 2\sqrt{3}$ d. None of these

44. Equation of incircle of equilateral triangle ABC where $B \equiv (2, 0)$ $C \equiv (4, 0)$ and A lies in fourth quadrant is

a. $x^2 + y^2 - 6x + \frac{2y}{\sqrt{3}} + 9 = 0$

b. $x^2 + y^2 - 6x - \frac{2y}{\sqrt{3}} + 9 = 0$

c. $x^2 + y^2 + 6x + \frac{2y}{\sqrt{3}} + 9 = 0$

d. None of these

45. $f(x, y) = x^2 + y^2 + 2ax + 2Ly + c = 0$ represents a circle. If $f(x, 0) = 0$ has equal roots, each being 2 and $f(0, y) = 0$ has 2 and 3 as it's roots, then centre of circle is

a. $(2, 5/2)$ b. Data are not sufficient

c. $(-2, -5/2)$ d. Data are inconsistent

46. The area bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ and the pair of lines $\sqrt{3}(x^2 + y^2) = 4xy$, is equal to

a. $\frac{\pi}{2}$ b. $\frac{5\pi}{2}$ c. 3π d. $\frac{\pi}{4}$

47. The straight line $x \cos\theta + y \sin\theta = 2$ will touch the circle $x^2 + y^2 - 2x = 0$, if

a. $\theta = n\pi, n \in I$ b. $A = (2n + 1)\pi, n \in I$

c. $\theta = 2n\pi, n \in I$ d. None of these

48. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the side, then its radius is

a. 3 b. 2 c. $\frac{3}{2}$ d. 1

49. Consider a family of circle which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval

a. $k \ge \frac{1}{2}$ **b.** $-\frac{1}{2} \le k \le \frac{1}{2}$

c. $k \le \frac{1}{2}$ **d.** $0 < k < \frac{1}{2}$

50. If the conics whose equations are $S \equiv \sin^2\theta\, x^2 + 2hxy + \cos^2\theta\, y^2 + 32x + 16y + 19 = 0$, $S' \equiv \cos^2\theta\, x^2 + 2h'xy + \sin^2\theta\, y^2 + 16x + 32y + 19 = 0$ intersects in four concyclic points then, (where $\theta \in R$)

a. $h + h' = 0$ **b.** $h = h'$

c. $h + h' = 1$ **d.** None of these

51. The range of values of $\lambda(\lambda > 0)$ such that the angle θ between the pair of tangents drawn from $(\lambda, 0)$ to the circle $x^2 + y^2 = 4$ lies in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ is

a. $\left(\frac{4}{\sqrt{3}}, 2\sqrt{2}\right)$ **b.** $(0, \sqrt{2})$

c. $(1, 2)$ **d.** None of these

52. The circles which can be drawn, to pass through $(1, 0)$ and $(3, 0)$ and to touch the y-axis, intersect at an angle θ, then $\cos\theta$ is equal to

a. $\frac{1}{2}$ **b.** $-\frac{1}{2}$ **c.** $\frac{1}{4}$ **d.** $-\frac{1}{4}$

53. Locus of midpoints of the chords of contact of $x^2 + y^2 = 2$ from the points on the line $3x + 4y = 10$ is a circle with centre P. If O be the origin then OP is equal to

a. 2 **b.** 3 **c.** $\frac{1}{2}$ **d.** $\frac{1}{3}$

54. Suppose $ax + by + c = 0$, where a, b, c are in A.P. be normal to a family of circles. The equation of the circle of the family intersects the circle $x^2 + y^2 - 4x - 4y - 1 = 0$ orthogonally is

a. $x^2 + y^2 - 2x + 4y - 3 = 0$

b. $x^2 + y^2 + 2x - 4y - 3 = 0$

c. $x^2 + y^2 - 2x + 4y - 5 = 0$

d. $x^2 + y^2 - 2x - 4y + 3 = 0$

55. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B, then area of triangle OAB (O being the origin) is

a. $\frac{3\sqrt{3}\, r^2}{4}$ b. $\frac{\sqrt{3}\, r^2}{4}$

c. $\frac{3r^2}{4}$ d. r^2

56. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$. AB be the chord of contact of this point w.r.t. the circle $x^2 + y^2 - 2x = 0$. The locus of the circumcnetre of the triangle CAB (C being centre of the circles) is

a. $2x^2 + 2y^2 - 4x + 1 = 0$

b. $x^2 + y^2 - 4x + 2 = 0$

c. $x^2 + y^2 - 4x + 1 = 0$

d. $2x^2 + 2y^2 - 4x + 3 = 0$

57. Two circles of radii 'a' and 'b' touching each other externally, are inscribed in the area bounded by $y = \sqrt{1 - x^2}$ and the x-axis. if $b = \frac{1}{2}$, then a is equal to

a. $\frac{1}{4}$ **b.** $\frac{1}{8}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{\sqrt{2}}$

58. From an arbitrary point 'P' on the circle $x^2 + y^2 = 9$, tangents are drawn to the circle $x^2 + y^2 = 1$, which meet $x^2 + y^2 = 9$ at A and B. Locus of the point of intersection of tangents at A and B to the circle $x^2 + y^2 = 9$ is

a. $x^2 + y^2 = \left(\frac{27}{7}\right)^2$ **b.** $x^2 - y^2 = \left(\frac{27}{7}\right)^2$

c. $y^2 - x^2 = \left(\frac{27}{7}\right)^2$ **d.** None of these

59. If r_1 and r_2 are the radii of smallest and largest circles which passes through $(5, 6)$ and touches the circle $(x - 2)^2 + y^2 = 4$, then $r_1 r_2$ is

a. $\frac{4}{41}$ **b.** $\frac{41}{4}$ **c.** $\frac{5}{41}$ **d.** $\frac{41}{6}$

60. Minimum radius of circle which is orthogonal with both the circles $x^2 + y^2 - 12x + 35 = 0$ and $x^2 + y^2 + 4x + 3 = 0$ is

a. 4 **b.** 3 **c.** $\sqrt{15}$ **d.** 1

61. If $C_1 : x^2 + y^2 = (3 + 2\sqrt{2})^2$ be a circle and PA and PB are pair of tangents on C_1, where P is any point on the director circle of C_1, then the radius of smallest circle which touch C_1 externally and also the two tangents PA and PB is

a. $2\sqrt{3} - 3$ **b.** $2\sqrt{2} - 1$ **c.** $2\sqrt{2} - 1$ **d.** 1

62. From a point $R(5, 8)$ two tangents RP and RQ are drawn to a given circle $S = 0$ whose radius is 5. If circumcentre of the triangle PQR is $(2, 3)$, then the equation of circle $S = 0$ is

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 + x + 2y - 10 = 0$

c. $x^2 + y^2 - x - 2y - 20 = 0$

d. $x^2 + y^2 - 4x - 6y - 12 = 0$

63. On the line segment joining $(1, 0)$ and $(3, 0)$, an equilateral triangle is drawn having its vertex in the fourth quadrant, then radical centre of the circles described on its sides as diameter is

a. $\left(3, -\frac{1}{\sqrt{3}}\right)$ **b.** $(3, -\sqrt{3})$

c. $\left(2, -\frac{1}{\sqrt{3}}\right)$ **d.** $(2, -\sqrt{3})$

64. If the tangents are drawn from any point on the line $x+y=3$ to the circle $x^2+y^2=9$, then the chord of contact passes through the point

a. (3, 5) b. (3, 3)
c. (5, 3) d. None of these

65. If the radius of the circumcircle of the triangle TPQ, where PQ is chord of contact corresponding to point T with respect to circle $x^2+y^2-2x+4y-11=0$, is 12 units, then minimum distance of T from the director circle of the given circle is

a. 6 b. 12
c. $6\sqrt{2}$ d. $12-4\sqrt{2}$

66. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the co-ordinates axes cut at right angles, then

a. $a^2-6ab+b^2=0$ b. $a^2+2ab-b^2=0$
c. $a^2-4ab+b^2=0$ d. $a^2-8ab+b^2=0$

67. The number of common tangent(s) to the circles $x^2+y^2+2x+8y-23=0$ and $x^2+y^2-4x-10y-19=0$ is

a. 1 b. 2 c. 3 d. 4

68. Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Then $\sin\theta$ is equal to

a. $\frac{24}{25}$ b. $\frac{12}{25}$
c. $\frac{3}{4}$ d. None of these

69. The locus of the midpoints of the chords of the circle $x^2+y^2-ax-by=0$ which subtend a right angle at $\left(\frac{a}{2}, \frac{b}{2}\right)$ is

a. $ax+by=0$
b. $ax+by=a^2+b^2$
c. $x^2+y^2-ax-by+\frac{a^2+b^2}{8}=0$
d. $x^2+y^2-ax-by-\frac{a^2+b^2}{8}=0$

70. From points (3 , 4), chords are drawn to the circle $x^2+y^2-4x=0$. The locus of the midpoints of the chords is

a. $x^2+y^2-5x-4y+6=0$
b. $x^2+y^2+5x-4y+6=0$
c. $x^2+y^2-5x+4y+6=0$
d. $x^2+y^2-5x-4y-6=0$

71. The locus of the centers of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-5x+4y-2=0$ orthogonally is

a. $9x+10y-7=0$ b. $x-y+2=0$
c. $9x-10y+11=0$ d. $9x+10y+7=0$

72. The angle at which the circles $(x-1)^2+y^2=10$ and $x^2+(y-2)^2=5$ intersect is

a. $\frac{\pi}{6}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{2}$

73. A tangent at a point on the circle $x^2+y^2=a^2$ intersects a concentric circle C at two points P and Q. The tangents to the circle X at P and Q meet at a point on the circle $x^2+y^2=b^2$, then the equation of circle is

a. $x^2+y^2=ab$ b. $x^2+y^2=(a-b)^2$
c. $x^2+y^2=(a+b)^2$ d. $x^2+y^2=a^2+b^2$

74. Tangents are drawn to the circle $x^2+y^2=1$ at the points where it is met by the circles, $x^2+y^2-(\lambda+6)x+(8-2\lambda)y-3=0$, λ being the variable. The locus of the point of intersection of these tangents is

a. $2x-y+10=0$ b. $x+2y-10=0$
c. $x-2y+10=0$ d. $2x+y-10=0$

75. If point A and B are (1, 0) and B (0, 1).If point C is on the circle $x^2+y^2=1$, then locus of the orthocentre of the triangle ABC is

a. $x^2+y^2=4$
b. $x^2+y^2-x-y=0$
c. $x^2+y^2-2x-2y+1=0$
d. $x^2+y^2+2x-2y+1=0$

76. If the line $x\cos\theta+y\sin\theta=2$ is the equation of a transverse common tangent to the circles $x^2+y^2=4$ and $x^2+y^2-6\sqrt{3}x-6y+20=0$, then the value of θ is

a. $5\pi/6$ b. $2\pi/3$ c. $\pi/3$ d. $\pi/6$

77. Any circle through the point of intersection of the lines $x+\sqrt{3}y=1$ and $\sqrt{3}x-y=2$ if intersects these lines at points P and Q, then the angle subtended by the arc PQ at its centre is

a. 180°
b. 90°
c. 120°
d. Depends on centre and radius

78. If the angle between tangents drawn to $x^2+y^2+2gx+2fy+c=0$ form (0, 0) is $\pi/2$, then

a. $g^2+f^2=3c$ b. $g^2+f^2=2c$
c. $g^2+f^2=5c$ d. $g^2+f^2=4c$

79. The common chord of the circle $x^2+y^2+6x+8y-7=0$ and a circle passing through the origin, and touching the line $y=x$, always passes through the point

a. (−1/2, 1/2) b. (1, 1)
c. (1/2, 1/2) d. None of these

80. The chord of contact of tangents from three points A, B, C to the circle $x^2+y^2=a^2$ are concurrent, then A, B, C will

a. be concyclic
b. be collinear
c. form the vertices of a triangle
d. None of these

81. The chord of contact of tangents from a point P to a circle passes through Q. If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to

a. $\frac{l_1 + l_2}{2}$ **b.** $\frac{l_1 - l_2}{2}$

c. $\sqrt{l_1^2 + l_2^2}$ **d.** $2\sqrt{l_1^2 + l_2^2}$

82. If $C_1: x^2 + y^2 - 20x + 64 = 0$ and $C_2: x^2 + y^2 + 30x + 144 = 0$. Then the length of the shortest line segment PQ which touches C_1 at P and to C_2 at Q is

a. 20 **b.** 15 **c.** 22 **d.** 27

83. The ends of a quadrant of a circle have the coordinates (1, 3) and (3, 1). Then the centre of such a circle is

a. (2, 2) **b.** (1,1) **c.** (4, 4) **d.** (2, 6)

84. If the line $ax + by = 2$ is a normal to the circle $x^2 + y^2 - 4x - 4y = 0$ and a tangent to the circle $x^2 + y^2 = 1$, then

a. $a = \frac{1}{2}, b = \frac{1}{2}$

b. $a = \frac{1+\sqrt{7}}{2}, b = \frac{1-\sqrt{7}}{2}$

c. $a = \frac{1}{4}, b = \frac{3}{4}$

d. $a = 1, b = \sqrt{3}$

85. Radius of the tangent circle that can be dawn to pass through the point (0, 7), (0, 6) and touching the x-axis is

a. $\frac{5}{2}$ **b.** $\frac{3}{2}$ **c.** $\frac{7}{2}$ **d.** $\frac{9}{2}$

86. The equation of the tangent to the circle $x^2 + y^2 = a^2$, which makes a triangle of area a^2 with the coordinate axes, is

a. $x \pm y = a\sqrt{2}$ **b.** $x \pm y = \pm a\sqrt{2}$

c. $x \pm y = 2a$ **d.** $x + y = \pm 2a$

87. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by $y = x$ at P such that $OP = 6\sqrt{2}$, then the value of c is

a. 36 **b.** 144

c. 72 **d.** None of these

88. The circle $x^2 + y^2 = 4$ cuts the line joining the points $A(1, 0)$ and $B(3, 4)$ in two points P and Q. Let $\frac{BP}{PA} = \alpha$ and $\frac{BQ}{QA} = \beta$. Then α and β are roots of the quadratic equation

a. $3x^2 + 2x - 21 = 0$ **b.** $3x^2 + 2x + 21 = 0$

c. $2x^2 + 3x - 21 = 0$ **d.** None of these

89. The area of the triangle formed by the positive x-axis and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

a. $2\sqrt{3}$ sq. units **b.** $3\sqrt{2}$ sq. units

c. $\sqrt{6}$ sq. units **d.** None of these

90. Let P be point on the circle $x^2 + y^2 = 9$, Q a point on the line $7x + y + 3 = 0$, and the perpendicular bisector of PQ be the line $x - y + 1 = 0$. Then the coordinate of P are

a. (0, −3) **b.** (0, 3)

c. $\left(\frac{72}{25}, -\frac{21}{25}\right)$ **d.** $\left(-\frac{72}{25}, \frac{21}{25}\right)$

91. A straight line moves such that the algebraic sum of the perpendicular drawn to it from two fixed points is equal to $2k$. Then, the straight line always touches a fixed circle of radius

a. $2k$ **b.** $k/2$

c. k **d.** None of these

92. The coordinates of the middle point of the chord cut-off by $2x - 5y + 18 = 0$ by the circle

$x^2 + y^2 - 6x + 2y - 54 = 0$ are

a. (1, 4) **b.** (2, 4) **c.** (4, 1) **d.** (1, 1)

93. The locus of a point from which the lengths of the tangents to the circles $x^2 + y^2 = 4$ and $2(x^2 + y^2) - 10x + 3y - 2 = 0$ are equal to

a. a straight line inclined at $\pi/4$ with the line joining the centres of the circles

b. a circle

c. an ellipse

d. a straight line perpendicular to the line joining the centres of the circles

94. The equation of circumcircle of an equilateral triangle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and one vertex of the triangle is (1, 1). The equation of incircle of the triangle is

a. $4(x^2 + y^2) = g^2 + f^2$

b. $4(x^2 + y^2) + 8gx + 8fy = (1 - g)(1 + 3g) + (1 - f)(1 + 3f)$

c. $4(x^2 + y^2) + 8gx + 8fy = g^2 + f^2$

d. None of these

95. A light ray gets reflected from the $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and point of incident is (−2, −4), then equation of incident ray is

a. $4y + 3x + 22 = 0$

b. $3y + 4x + 20 = 0$

c. $4y + 2x + 20 = 0$

d. $y + x + 6 = 0$

96. Tangents PA and PB drawn to $x^2 + y^2 = 9$ from any arbitrary point 'P' on the line $x + y = 25$. Locus of midpoint of chord AB is

a. $25(x^2 + y^2) = 9(x + y)$

b. $25(x^2 + y^2) = 3(x + y)$

c. $5(x^2 + y^2) = 3(x + y)$

d. None of these

97. The circles having radii r_1 and r_2 intersect orthogonally. Length of their common chord is

a. $\dfrac{2r_1r_2}{\sqrt{r_1^2+r_2^2}}$ **b.** $\dfrac{\sqrt{r_1^2+r_2^2}}{2r_1r_2}$

c. $\dfrac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$ **d.** $\dfrac{\sqrt{r_1^2+r_2^2}}{r_1r_2}$

98. If the pair of straight line $xy\sqrt{3}-x^2=0$ is tangent to the circle at P and Q from origin O such that area of smaller sector formed by CP and CQ is 3π sq. unit, where C is the centre of circle, then OP equals to

a. $(3\sqrt{3})/2$ **b.** $3\sqrt{3}$ **c.** 3 **d.** $\sqrt{3}$

99. The two circles which passes through $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ will intersect each other at right angle, if

a. $a^2 = c^2(2m + 1)$ **b.** $a^2 = c^2(2 + m^2)$

c. $c^2 = a^2(2 + m^2)$ **d.** $c^2 = a^2(2m + 1)$

100. The condition that the chord $x\cos\alpha + y\sin\alpha - p = 0$ of $x^2+y^2-a^2=0$ may subtend a right angle at the centre of the circle is

a. $a^2 = 2p^2$ **b.** $p^2 = 2a^2$ **c.** $a = 2p$ **d.** $p = 2a$

101. The locus of the midpoints of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin, is

a. $x + y = 2$ **b.** $x^2 + y^2 = 1$

c. $x^2 + y^2 = 2$ **d.** $x + y = 1$

102. If the chord of contact of tangents from a point P to a given circle passes through Q, then the circle on PQ as diameter

a. cuts the given circle orthogonally
b. touches the given circle externally
c. touches the given circle internally
d. None of these

103. Let these base AB of a triangle ABC be fixed and the vertex C lie on a fixed circle of radius r. Lines through A and B are drawn to intersect CB and CA, respectively, at E and F such that $CE : EB = 1 : 2$ and $CF : FA = 1 : 2$. If the point of intersection P of these lines lies on the median through AB for all positions of AB then the locus of P is

a. a circle of radius $\frac{r}{2}$
b. a circle of radius $2r$
c. a parabola of latus rectum $4r$
d. a rectangular hyperbola

104. The number of integral values of y for which the chord of the circle $x^2 + y^2 = 125$ passing through the point $P(8, y)$ gets bisected at the point $P(8, y)$ and has integral slope is

a. 8 **b.** 6 **c.** 4 **d.** 2

105. Consider a square $ABCD$ of side length 1. Let P be the set of all segments of length 1 with end points on adjacent sides of square $ABCD$. The midpoints of segments in P enclose a region with area A, the value of A is

a. $\frac{\pi}{4}$ **b.** $1-\frac{\pi}{4}$

c. $4-\frac{\pi}{4}$ **d.** None of these

106. The range of values of α for which the line $2y = gx + \alpha$ is a normal to the circle $x^2 + y^2 + 2gx + 2gy - 2 = 0$ for all values of g is

a. $[1, \infty)$ **b.** $[-1, \infty)$ **c.** $(0, 1)$ **d.** $(-\infty, 1]$

107. Six points (x_i, y_i), $i = 1, 2, \ldots, 6$ are taken on the circle $x^2 + y^2 = 4$ such that $\sum_{i=1}^{6} x_i = 8$ and $\sum_{i=1}^{6} y_i = 4$. The line segment joining orthocentre of a triangle made by any three points and the centroid of the triangle made by other three points passes through a fixed points (h, k), then $h + k$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

108. A circle with radius $|a|$ and centre on y-axis slides along it and a variable lines through $(a, 0)$ cuts the circle at points P and Q. The region in which the point of intersection of tangents to the circle at points P and Q lies is represented by

a. $y^2 \geq 4(ax - a^2)$ **b.** $y^2 \leq 4(ax - a^2)$

c. $y \geq 4(ax - a^2)$ **d.** $y \leq 4(ax - a^2)$

109. If the angle of intersection of the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ is θ, then equation of the line passing through $(1, 2)$ and making an angle θ with the y-axis is

a. $x = 1$ **b.** $y = 2$

c. $x + y = 3$ **d.** $x - y = 3$

110. The centres of a set of circle, each of radius 3, lies on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

a. $4 \leq x^2 + y^2 \leq 64$ **b.** $x^2 + y^2 \leq 25$

c. $x^2 + y^2 \geq 25$ **d.** $3 \leq x^2 + y^2 \leq 9$

111. The co-ordinates of two points P and Q are (x_1, y_1) and (x_2, y_2) and O is the origin. If circles be described on OP and OQ as diameters then length of their common chord is

a. $\dfrac{|x_1y_2 + x_2y_1|}{PQ}$ **b.** $\dfrac{|x_1y_2 - x_2y_1|}{PQ}$

c. $\dfrac{|x_1x_2 + y_1y_2|}{PQ}$ **d.** $\dfrac{|x_1x_2 + y_1y_2|}{PQ}$

112. Consider a circle $x^2 + y^2 + ax + by + c = 0$ lying completely in first quadrant. If m_1 and m_2 are the maximum and minimum values of y/x for all ordered pairs (x, y) on the circumference of the circle, then the value of $(m_1 + m_2)$ is

a. $\frac{a^2-4c}{b^2-4c}$　　b. $\frac{2ab}{b^2-4c}$

c. $\frac{2ab}{4c-b^2}$　　d. $\frac{2ab}{b^2-4ac}$

113. If the circumference of the circle $x^2+y^2+8x+8y-b=0$ is bisected by the circle $x^2+y^2-2x+4y+a=0$, then $a+b$ equals to

a. 50　　b. 56　　c. −56　　d. −34

114. The equation of the circle passing through the point of intersection of the circle $x^2+y^2=4$ and the line $2x+y=1$ and having minimum possible radius is

a. $5x^2+5y^2+18x+6y-5=0$
b. $5x^2+5y^2+9x+8y-15=0$
c. $5x^2+5y^2+4x+9y-5=0$
d. $5x^2+5y^2-4x-2y-18=0$

115. The locus of the centre of the circle touching the line $2x-y=1$ at (1, 1) is

a. $x+3y=2$　　b. $x+2y=0$
c. $x+y=2$　　d. None of these

116. The distance from the centre of the circle $x^2+y^2=2x$ to the common chord of the circles $x^2+y^2+5x-8y+1=0$ and $x^2+y^2-3x+7y-25=0$ is

a. 2　　b. 4　　c. $\frac{34}{13}$　　d. $\frac{26}{17}$

117. The equation of the circle passing through the point of intersection of the circles $x^2+y^2-4x-2y=8$ and $x^2+y^2-2x-4y=8$ and the point (−1, 4) is

a. $x^2+y^2+4x+4y-8=0$
b. $x^2+y^2-3x+4y+8=0$
c. $x^2+y^2+x+y-8=0$
d. $x^2+y^2-3x-3y-8=0$

118. If the radius of the circle $(x-1)^2+(y-2)^2=1$ and $(x-7)^2+(y-10)^2=4$ are increasing uniformly w.r.t. time as 0.3 and 0.4 unit/sec, then they will touch each other at t equal to

a. 45 sec　　b. 90 sec　　c. 11 sec　　d. 135 sec

119. The equation of a circle which has normals $(x-1)\times(y-2)=0$ and a tangent $3x+4y=6$ is

a. $x^2+y^2-2x-4y+4=0$
b. $x^2+y^2-2x-4y+5=0$
c. $x^2+y^2=5$
d. $(x-3)^2+(y-4)^2=5$

Multiple Correct Answers Type

Solutions on page 2.82

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. The circle $x^2+y^2+2a_1x+c=0$ lies completely inside the circle $x^2+y^2+2a_2x+c=0$, then

a. $a_1a_2>0$　　b. $a_1a_2<0$　　c. $c>0$　　d. $c<0$

2. Consider the circle $x^2+y^2-10x-6y+30=0$. Let O be the centre of the circle and tangent at $A(7, 3)$ and $B(5, 1)$ meet at C. Let $S=0$ represents family of circles passing through A and B, then

a. area of quadrilateral $OACB=4$
b. the radical axis for the family of circles $S=0$ is $x+y=10$
c. the smallest possible circle of the family $S=0$ is $x^2+y^2-12x-4y+38=0$
d. the coordinates of point C are (7, 1)

3. If the circle $x^2+y^2+2x+2ky+6=0$ and $x^2+y^2+2ky+k=0$ intersect orthogonally, then k is

a. 2　　b. −2　　c. $-\frac{3}{2}$　　d. $\frac{3}{2}$

4. A$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a point on the circle $x^2+y^2=1$ and B is another point on the circle such that arc length $AB=\frac{\pi}{2}$ units. Then, co-ordinates of B can be

a. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$　　b. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

c. $\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$　　d. None of these

5. Let x, y be real variable satisfying the $x^2+y^2+8x-10y-40=0$. Let $a=\max\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$ and $b=\min\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$, then

a. $a+b=18$　　b. $a+b=\sqrt{2}$
c. $a-b=4\sqrt{2}$　　d. $a\cdot b=73$

6. Three sides of a triangle have the equations $L_i\equiv y-m_ix=0$; $i=1, 2, 3$. Then $L_1L_2+\lambda L_2L_3+\mu L_3L_1=0$, where $\lambda\neq 0$, $\mu\neq 0$, is the equation of the circumcircle of the triangle if

a. $1+\lambda+\mu=m_1m_2+\lambda m_2m_3+\lambda m_3m_1$
b. $m_1(1+\mu)+m_2(1+\lambda)+m_3(\mu+\lambda)=0$
c. $\frac{1}{m_3}+\frac{1}{m_1}+\frac{1}{m_2}=1+\lambda+\mu$
d. None of these

7. If equation $x^2+y^2+2hxy+2gx+2fy+c=0$ represents a circle, then the condition for that circle to pass through three quadrants only but not passing through the origin is

a. $f^2>c$　　b. $g^2>c$
c. $c>0$　　d. $h=0$

8. The points on the line $x=2$ from which the tangents drawn to the circle $x^2+y^2=16$ are at right angles is (are)

a. $(2, 2\sqrt{7})$　　b. $(2, 2\sqrt{5})$

c. $(2, -2\sqrt{7})$ **d.** $(2, -2\sqrt{5})$

9. Co-ordinates of the centre of a circle, whose radius is 2 unit and which touches the line pair $x^2 - y^2 - 2x + 1 = 0$, are

a. $(4, 0)$ **b.** $(1 + 2\sqrt{2}, 0)$

c. $(4, 1)$ **d.** $(1, 2\sqrt{2})$

10. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then a is

a. $-\frac{4}{3}$ **b.** 0 **c.** 1 **d.** $\frac{4}{3}$

11. Point M moved on the circle $(x - 4)^2 + (y - 8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle, cuts the x-axis at the point $(-2, 0)$. The co-ordinates of a point on the circle at which the moving point broke away is

a. $\left(\frac{42}{5}, \frac{36}{5}\right)$ **b.** $\left(-\frac{2}{5}, \frac{44}{5}\right)$

c. $(6, 4)$ **d.** $(2, 4)$

12. The equation of a tangent to the circle $x^2 + y^2 = 25$ passing through $(-2, 11)$ is

a. $4x + 3y = 25$ **b.** $3x + 4y = 38$

c. $24x - 7y + 125 = 0$ **d.** $7x + 24y = 250$

13. If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the radii corresponding to the points of contact is 15, then a value of c is

a. 9 **b.** 4 **c.** 5 **d.** 25

14. The equation of the circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is

a. 1 **b.** 2 **c.** 3 **d.** 6

15. The equations of tangents to the circle $x^2 + y^2 - 6x - 6y + 9 = 0$ drawn from the origin are

a. $x = 0$ **b.** $x = y$ **c.** $y = 0$ **d.** $x + y = 0$

16. If a circle passes through the point of intersection of the lines $x + y + 1 = 0$ and $x + \lambda y - 3 = 0$ with the co-ordinates axes, then

a. $\lambda = -1$

b. $\lambda = 1$

c. $\lambda = 2$

d. λ can have any real value

17. Which of the following lines have the intercepts of equal lengths on the circle, $x^2 + y^2 - 2x + 4y = 0$?

a. $3x - y = 0$

b. $x + 3y = 0$

c. $x + 3y + 10 = 0$

d. $3x - y - 10 = 0$

18. The equation of the lines parallel to $x - 2y = 1$ which touches (touch) the circle $x^2 + y^2 - 4x - 2y - 15 = 0$ is (are)

a. $x - 2y + 2 = 0$

b. $x - 2y - 10 = 0$

c. $x - 2y - 5 = 0$

d. $x - 2y + 10 = 0$

19. The circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 4x + 4y - 1 = 0$

a. touch internally

b. touch externally

c. have $3x + 4y - 1 = 0$ as the common tangent at the point of contact

d. have $3x + 4y + 1 = 0$ as the common tangent at the point of contact

20. The circles $x^2 + y^2 + 2x + 4y - 20 = 0$ and $x^2 + y^2 + 6x - 8y + 10 = 0$

a. are such that the number of common tangents on them is 2

b. are orthogonal

c. are such that the length of their common tangent is $5(12/5)^{1/4}$

d. are such that the length of their common chord is $5\sqrt{\frac{3}{2}}$

21. A point $P(\sqrt{3}, 1)$ moves on the circle $x^2 + y^2 = 4$ and after covering a quarter of the circle leaves it tangentially. The equation of a line along which the point moves after leaving the circle is

a. $y = \sqrt{3}\,x + 4$ **b.** $\sqrt{3}\,y = x + 4$

c. $y = \sqrt{3}x - 4$ **d.** $\sqrt{3}y = x - 4$

22. The equation of a circle of radius 1 touching the circles $x^2 + y^2 - 2|x| = 0$ is

a. $x^2 + y^2 + 2\sqrt{2}\,x + 1 = 0$

b. $x^2 + y^2 - 2\sqrt{3}\,y + 2 = 0$

c. $x^2 + y^2 + 2\sqrt{3}\,y + 2 = 0$

d. $x^2 + y^2 - 2\sqrt{2} + 1 = 0$

23. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then $a =$

a. $-4/3$ **b.** 0 **c.** 1 **d.** 4/3

24. The range of values of 'a' such that angle θ between the pair of tangent drawn from $(a, 0)$ to the circle $x^2 + y^2 = 1$ satisfies $\frac{\pi}{2} < \theta < \pi$, lies in

a. $(1, 2)$ **b.** $(1, \sqrt{2})$

c. $(-\sqrt{2}, -1)$ **d.** $(-2, -1)$

25. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1

a. $x + y = 0$ **b.** $x - y = 0$

c. $x + 7y = 0$ **d.** $x - 7y = 0$

26. The centre(s) of the circle(s) passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is/are

a. $\left(\frac{3}{2}, \frac{1}{2}\right)$ **b.** $\left(\frac{1}{2}, \frac{3}{2}\right)$

c. $\left(\frac{1}{2}, 2^{1/2}\right)$ **d.** $\left(\frac{1}{2}, -2^{1/2}\right)$

Reasoning Type

Solutions on page 2.86

Each question has four choices a, b, c and d, out of which only one is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.

b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.

c. Statement 1 is True and Statement 2 is False.

d. Statement 1 is False and Statement 2 is True.

1. Statement 1 : The number of circles that pass through the points (1, – 7) and (– 5, 1) and of radius 4, is two.

Statement 2: The centre of any circle that pass through the points A and B lies on the perpendicular bisector of AB.

2. Statement 1: The chord of contact of tangent from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will be collinear.

Statement 2: Lines $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$ always pass through a fixed point for $k \in R$.

3. Statement 1: Circles $x^2 + y^2 = 144$ and $x^2 + y^2 - 6x - 8y = 0$ do not have any common tangent.

Statement 2: If two circles are concentric, then they do not have common tangents.

4. Statement 1: The least and greatest distances of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are 5 and 15 units, respectively.

Statement 2: A point (x_1, y_1) lies outside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, if $S_1 > 0$, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

5. Statement 1: Number of circles passing through (1, 2), (4, 8) and (0, 0) is one.

Statement 2: Every triangle has one circumcircle.

6. Let C_1 be the circle with centre O_1 (0, 0) and radius 1 and C_2 be the circle with centre O_2 $(t, t^2 + 1)$ $(t \in R)$ and radius 2.

Statement 1: Circles C_1 and C_2 always have at least one common tangent for any value of t.

Statement 2: For the two circles, $O_1O_2 \geq |r_1 - r_2|$, where r_1 and r_2 are their radii for any value of t.

7. Tangents are drawn from the origin to the circle $x^2 + y^2 - 2hx - 2hy + h^2 = 0$ $(h \geq 0)$.

Statement 1: Angle between the tangents is $\pi/2$.

Statement 2: The given circle is touching the co-ordinate axes.

8. Consider two circles $x^2 + y^2 - 4x - 6y - 8 = 0$ and $x^2 + y^2 - 2x - 3 = 0$.

Statement 1: Both circles intersect each other at two distinct points.

Statement 2: Sum of radii of two circles in greater than distance between the centres of two circles.

9. From the point $P(\sqrt{2}, \sqrt{6})$, tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$.

Statement 1: Area of the quadrilateral $OAPB$ (being origin) is 4.

Statement 2: Area of square is a^2 where a is length of side.

10. Statement 1: Centre of the circle having $x + y = 3$ and $x - y = 1$ as its normal is (1, 2).

Statement 2: Normals to the circle always passes through its centre.

11. Statement 1: The circle having equation $x^2 + y^2 - 2x + 6y + 5 = 0$ intersects both the coordinate axes.

Statement 2: The lengths of x and y intercepts made by the circle having equation $x^2 + y^2 + 2gx + 2fy + c = 0$ are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$, respectively.

12. Statement 1: Number of circles touching lines $x + y = 1$, $2x - y = 5$ and $3x + 5y - 1 = 0$ is four.

Statement 2: In any triangle, four circles can be drawn touching all the three sides of triangle.

13. Statement 1: Chord of contact of the circle $x^2 + y^2 = 1$ w.r.t. points (2, 3). (3, 5) and (1, 1) are concurrent.

Statement 2: Points (1, 1), (2, 3) and (3, 5) are collinear.

14. Statement 1: The equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents, for different values of 'a', a system of circles passing through two fixed points lying on the x-axis.

Statement 2: $S = 0$ is a circle and $L = 0$ is a straight line, then $S + \lambda L = 0$ represents the family of circles passing through the points of intersection of circle and straight line (where λ is arbitrary parameter).

15. **Statement 1:** The chord of contact of tangent from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will be collinear.

    Statement 2: A, B, C always lies on the normal to the circle $x^2 + y^2 = a^2$

16. **Statement 1:** Points $A(1, 0)$, $B(2, 3)$, $C(5, 3)$and $D(6, 0)$ are concyclic.

    Statement 2: Points A, B, C, D forms isosceles trapezium or AB and CD meet in E, then $EA \cdot EB = EC \cdot ED$.

17. **Statement 1:** The equations of the straight lines joining origin to the points of intersection of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$ is $x - y = 0$.

    Statement 2: $y + x = 0$ is common chord of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$.

18. **Statement 1:** Two orthogonal circles intersect to generate a common chord which subtends complimentary angles at their circumferences.

    Statement 2: Two orthogonal circles intersect to generate a common chord which subtends supplementary angles at their centres.

19. **Statement 1:** The point $(a, -a)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$ whenever $a \in (-1, 4)$

    Statement 2: Point (x_1, y_1) lies inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$.

20. **Statement 1:** If circle with centre $P(t, 4 - 2t)$, $t \in R$ cuts the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2x - y - 12 = 0$; then both the intersections are orthogonal.

    Statement 2: Length of tangent from P for $t \in R$ is same for both the given circles.

21. **Statement 1:** Let $S_1 : x^2 + y^2 - 10x - 12y - 39 = 0$

    $S_2 : x^2 + y^2 - 2x - 4y + 1 = 0$

    and $S_3 : 2x^2 + 2y^2 - 20x - 24y + 78 = 0$

    The radical centre of these circles taken pairwise as $(-2, -3)$.

    Statement 2: Point of intersection of three radical axis of three circles taken in pairs is known as radical centre.

22. **Statement 1:** The equation of chord through the point $(-2, 4)$ which is farthest from the centre of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$ is $x + y - 2 = 0$.

    Statement 2: In notations, the equation of such chord of the circle $S = 0$ bisected at (x_1, y_1) must be $T = S_1$.

23. **Statement 1:** If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g = fg'$.

    Statement 2: Two circles touch other, if line joining their centres is perpendicular to all possible common tangents.

24. **Statement 1:** The circles $x^2 + y^2 + 2px + r = 0$, $x^2 + y^2 + 2qy + r = 0$ touch, if $\frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{r}$.

    Statement 2: Two circles with centre C_1, C_2 and radii r_1, r_2 touch each other if $|r_1 \pm r_2| = c_1c_2$.

Linked Comprehension Type

Solutions on page 2.88

Based upon each paragraph, there multiple choice questions have to be answered. Each face equation has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1–3

Each side of a square has length 4 units and its centre is at (3, 4). If one of the diagonals is parallel to the line $y = x$, then answer the following questions.

1. Which of the following is not the vertex of the square?

   a. (1, 6) **b.** (5, 2) **c.** (1, 2) **d.** (4, 6)

2. The radius of the circle inscribed in the triangle formed by any three vertices is

   a. $2\sqrt{2}(\sqrt{2} + 1)$ **b.** $2\sqrt{2}(\sqrt{2} - 1)$

   c. $2(\sqrt{2} + 1)$ **d.** None of these

3. The radius of the circle inscribed in the triangle formed by any two vertices of square and the centre is

   a. $2(\sqrt{2} - 1)$ **b.** $2(\sqrt{2} + 1)$

   c. $\sqrt{2}(\sqrt{2} - 1)$ **d.** None of these

For Problems 4–6

Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the center of the circumcircle of triangle PAB, then answer the following questions.

4. Range of $y = f(x)$ is

   a. $[-2, 1]$ **b.** $[-1, 4]$ **c.** $[0, 2]$ **d.** $[2, 3]$

5. Period of $y = f(x)$ is

   a. 2π **b.** 3π

   c. π **d.** Not defined

6. Which of the following is true?

   a. $f(x) = 4$ has real roots

   b. $f(x) = 1$ has real roots

   c. Range of $y = f^{-1}(x)$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

   d. None of these

For Problems 7–9

Consider a family of circles passing through the points (3, 7) and (6, 5). Answer the following questions.

7. Number of circles which belong to the family and also touching x-axis are

a. 0 **b.** 1 **c.** 2 **d.** Infinite

8. If each circle in the family cuts the circle $x^2+y^2-4x-6y-3=0$, then all the common chords pass through the fixed point which is

a. (1, 23) **b.** (2, 23/2)
c. (– 3, 3/2) **d.** None of these.

9. If circle which belong the given family cuts the circle $x^2+y^2=29$ orthogonally then centre of that circle is

a. (1/2, 3/2) **b.** (9/2, 7/2)
c. (7/2, 9/2) **d.** (3, –7/9)

For Problems 10–12

Consider the relation $4l^2-5m^2+6l+1=0$, where $l, m \in R$, then the line $lx+my+1=0$ touches a fixed circle whose

10. Centre and radius of circle one

a. (2, 0), 3 **b.** (–3, 0), $\sqrt{3}$
c. (3, 0), $\sqrt{5}$ **d.** None of these

11. Tangents PA and PB are drawn to the above fixed circle from the points P on the line $x+y-1=0$. Then chord of contact AB passes though the fixed point

a. $\left(\frac{1}{2}, -\frac{5}{2}\right)$ **b.** $\left(\frac{1}{3}, \frac{4}{3}\right)$
c. $\left(-\frac{1}{2}, \frac{3}{2}\right)$ **d.** None of these

12. Number of tangents which can be drawn from the point (2, – 3) are

a. 0 **b.** 1 **c.** 2 **d.** 1 or 2

For Problems 13–15

A circle C whose radius is 1 unit, touches x-axis at point A. The centre Q of C lies in first quadrant. The tangent from origin O to the circle touches it at T and a point P lies on it such that ΔOAP is a right-angled triangle at A and its perimeter is 8 units.

13. The length of PQ is

a. $\frac{1}{2}$ **b.** $\frac{4}{3}$
c. $\frac{5}{3}$ **d.** None of these

14. Equation of circle C is

a. $\{x-(2+\sqrt{3})\}^2+(y-1)^2=1$
b. $\{x-(\sqrt{3}-\sqrt{2})\}^2+(y-1)^2=1$
c. $(x-\sqrt{3})^2+(y-1)^2=1$
d. None of these

15. Equation of tangent OT is

a. $x-\sqrt{3}y=0$ **b.** $x-\sqrt{2}y=0$
c. $y-\sqrt{3}x=0$ **d.** None of these

For Problems 16–18

P is a variable point on the line $L=0$. Tangents are drawn to the circles $x^2+y^2=4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.

16. If $L \equiv 2x+y-6=0$, then the locus of circumcentre of ΔPQR is

a. $2x-y=4$ **b.** $2x+y=3$
c. $x-2y=4$ **d.** $x+2y=3$

17. If $P \equiv (6, 8)$, then the area of ΔQRS is

a. $\frac{(6)^{3/2}}{25}$ sq. units **b.** $\frac{(24)^{3/2}}{25}$ sq. units
c. $\frac{48\sqrt{6}}{25}$ sq. units **d.** $\frac{192\sqrt{6}}{25}$ sq. units

18. If $P \equiv (3, 4)$, then coordinates of S are

a. $\left(-\frac{46}{25}, \frac{63}{25}\right)$ **b.** $\left(\frac{-51}{25}, \frac{-68}{25}\right)$
c. $\left(-\frac{46}{25}, \frac{68}{25}\right)$ **d.** $\left(-\frac{68}{25}, \frac{51}{25}\right)$

For Problems 19–21

To the circle $x^2+y^2=4$, two tangents are drawn from $P(-4, 0)$, which touches the circle at T_1 and T_2, a rhombus $PT_1P'T_2$ is completed.

19. Circumcentre of the triangle PT_1T_2 is at

a. (– 2, 0) **b.** (2, 0)
c. $\left(\frac{\sqrt{3}}{2}, 0\right)$ **d.** None of these

20. Ratio of the area of triangle PT_1P' to that the $P'T_1T_2$ is

a. 2 : 1 **b.** 1 : 2
c. $\sqrt{3}$: 2 **d.** None of these

21. If P is taken to be at $(h, 0)$ such that P' lies on the circle, the area of the rhombus is

a. $6\sqrt{3}$ **b.** $2\sqrt{3}$
c. $3\sqrt{3}$ **d.** None of these

For Problems 22–24

Let α chord of a circle be that chord of the circle which subtends an angle α at the centre.

22. If $x+y=1$ is a chord of $x^2+y^2=1$, then α is equal to

a. $\frac{\pi}{4}$ **b.** $\frac{\pi}{2}$
c. $\frac{\pi}{6}$ **d.** $x+y=1$ is not a chord

23. If slope of a $\frac{\pi}{3}$ chord of $x^2+y^2=4$ is 1, then its equation is

a. $x-y+\sqrt{6}=0$ **b.** $x-y=2\sqrt{3}$
c. $x-y=\sqrt{3}$ **d.** $x-y+\sqrt{3}=0$

24. Distance of $\frac{2\pi}{3}$ chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

a. 1 **b.** 2 **c.** $\sqrt{2}$ **d.** $\frac{1}{\sqrt{2}}$

For Problems 25–27

Two variable chords AB and BC of a circles $x^2 + y^2 = a^2$ are such that $AB = BC = a$, and M and N are the midpoints of AB and BC, respectively, such that line joining MN intersect the circles at P and Q, where P is closer to AB and O is the centre of the circle.

25. $\angle OAB$ is

a. 30° **b.** 60° **c.** 45° **d.** 15°

26. Angles between tangents at A and C is

a. 90° **b.** 120° **c.** 60° **d.** 150°

27. Locus of point of intersection of tangents at A and C is

a. $x^2 + y^2 = a^2$ **b.** $x^2 + y^2 = 2a^2$

c. $x^2 + y^2 = 4a^2$ **d.** $x^2 + y^2 = 8a^2$

For Problems 28–30

Given two circles intersecting orthogonally having length of common chord 24/5 units. Radius of one of the circles is 3 units.

28. Radius of other circle is

a. 6 units **b.** 5 units **c.** 2 units **d.** 4 units

29. Angle between direct common tangent is

a. $\sin^{-1}\frac{24}{25}$ **b.** $\sin^{-1}\frac{4\sqrt{6}}{25}$

c. $\sin^{-1}\frac{4}{5}$ **d.** $\sin^{-1}\frac{12}{25}$

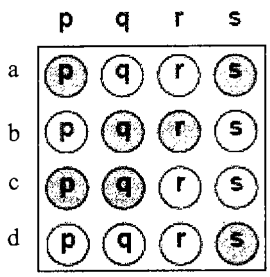
30. Length of direct common tangent is

a. $\sqrt{12}$ **b.** $4\sqrt{3}$ **c.** $2\sqrt{6}$ **d.** $3\sqrt{6}$

Matrix-Match Type

Solutions on page 2.91

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, , b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. Number of circles touching given three non-concurrent lines	**p.** 1
b. Number of circles touching $y = x$ at (2, 2) and also touching line $x + 2y - 4 = 0$	**q.** 2
c. Number of circles touching lines $x \pm y = 2$ and passing thorough the point (4, 3)	**r.** 4
d. Number of circle intersecting given three circles orthogonally	**s.** Infinite

2. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be an equation of circle.

Column I	Column II
a. If circle lie in first quadrant, then	**p.** $g < 0$
b. If circle lie above x-axis, then	**q.** $g > 0$
c. If circle lie on the left of y-axis, then	**r.** $g^2 - c < 0$
d. If circle touches positive x-axis and does not intersect y-axis, then	**s.** $c > 0$

3.

Column I	Column II
a. If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x - 3)^2 + (y - 4)^2 = 4$, which passes through (2, 3) and at the greatest distance from the centre of the circle, then $\|a + b\|$ is equal to	**p.** 6
b. Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of midpoint of OP is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $(g + f)$ is equal to	**q.** 3
c. The x-coordinates of the centre of the smallest circle which cuts the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally is	**r.** 2
d. If θ be the angle between two tangents which are drawn to the circles $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$ from the origin, then $2\sqrt{3}\tan\theta$ equals to	**s.** 1

4.

Column I	Column II
a. The length of the common chord of two circles of radii 3 and 4 units which intersect orthogonally is $\frac{k}{5}$, then k equals to	**p.** 1

b. The circumference of the circle $x^2 + y^2 + 4x + 12y + p = 0$ is bisected by the circle $x^2 + y^2 - 2x + 8y - q = 0$, then $p + q$ is equal to	**q.** 24
c. Number of distinct chords of the circle $2x(x - \sqrt{2}) + y(2y - 1) = 0$; chords are passing through the point $\left(\sqrt{2}, \frac{1}{2}\right)$ and are bisected on x-axis is	**r.** 32
d. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$, respectively, then the area of the rectangle is	**s.** 36

5. Let C_1 and C_2 be two circles whose equations are $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 2x = 0$. $P(\lambda_1, \lambda)$ is a variable point. Then match the following.

Column I	Column II
a. P lies inside C_1 but outside C_2	**p.** $\lambda \in (-\infty, -1) \cup (0, \infty)$
b. P lies inside C_2 but outside C_1	**q.** $\lambda \in (-\infty, -1) \cup (1, \infty)$
c. P lies outside C_1 and outside C_2	**r.** $\lambda \in (-1, 0)$
d. P does not lie inside C_2	**s.** $\lambda \in (0, 1)$

6.

Column I	Column II
a. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ touch each other then triplet (a_1, a_2, b) can be	**p.** $(2, 2, 2)$
b. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2y + b = 0$ touch each other then triplet (a_1, a_2, b) can be	**q.** $\left(1, 1, \frac{1}{2}\right)$
c. If the straight line $a_1x - by + b^2 = 0$ touches the circle $x^2 + y^2 = a_2x + by$, then triplet (a_1, a_2, b) can be	**r.** $(2, 1, 0)$
d. If the line $3x + 4y - 4 = 0$ touches the circle $(x - a_1)^2 + (y - a_2)^2 = b^2$, then triplet (a_1, a_2, b) can be	**s.** $\left(1, 1, \frac{3}{5}\right)$

Integer Type

Solutions on page 2.93

1. Let the lines $(y - 2) = m_1(x - 5)$ and $(y + 4) = m_2(x - 3)$ intersect at right angles at P (where m_1 and m_2 are parameters). If locus of P is $x^2 + y^2 + gx + fy + 7 = 0$, then the value of $|f + g|$ is

2. Consider the family of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$ passing through two fixed points A and B. Then the distance between the points A and B is

3. The number of points $P(x, y)$ lying inside or on the circle $x^2 + y^2 = 9$ and satisfying the equation $\tan^4 x + \cot^4 x + 2 = 4 \sin^2 y$, is

4. If real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2 = (14)^2$, then the minimum value of $\sqrt{x^2 + y^2}$ is

5. The line $3x + 6y = k$ intersect the curve $2x^2 + 2xy + 3y^2 = 1$ at points A and B. The circle on AB as diameter passes through the origin. Then the value of k^2 is

6. The sum of the slopes of the lines tangent to both circles $x^2 + y^2 = 1$ and $(x - 6)^2 + y^2 = 4$ is

7. A circle $x^2 + y^2 + 4x - 2\sqrt{2}\,y + c = 0$ is the director circle of circle S_1 and S_1 is the director circle of circle S_2 and so on. If the sum of radii of all these circles is 2, then the value of $c = k\sqrt{2}$, where value of k is

8. Two circles are externally tangent. Lines PAB and $PA'B'$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If $PA = AB = 4$, then the square of radius of circle is

9. The length of a common internal tangent to two circles is 7 and a common external tangent is 11. If the product of the radii of the two circles is p, then the value of $p/2$ is

10. Line segments AC and BD are diameters of circle of radius one. If $\angle BDC = 60°$, the length of line segment AB is

11. As shown in Fig. 2.76, three circles which have the same radius r, have centres at (0,0), (1,1) and (2,1). If they have a common tangent line, as shown, then the value of $10\sqrt{5}r$ is

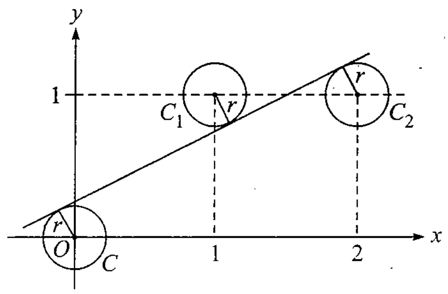


Fig. 2.76

12. The acute angle between the line $3x - 4y = 5$ and the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ is θ, then $9 \cos \theta$

13. If two perpendicular tangents can be drawn from the origin to the circle $x^2 - 6x + y^2 - 2py + 17 = 0$, then the value of $|p|$ is

14. Let $A(-4, 0)$ and $B(4, 0)$. If the number of points on the circle $x^2 + y^2 = 16$ such that the area of the triangle whose vertices are A, B and C is a positive integer, is N then the value of $[N/7]$ is, where N represents greatest integer function

15. If the circle $x^2 + y^2 + (3 + \sin\beta)x + (2\cos\alpha)y = 0$ and $x^2 + y^2 + (2\cos\alpha)x + 2cy = 0$ touches each other then the maximum value of 'c' is

16. Two circles C_1 and C_2 both passes through the points $A(1, 2)$ and $E(2, 1)$ and touch the line $4x - 2y = 9$ at B and D respectively. The possible coordinates of a point C such that the quadrilateral $ABCD$ is a parallelogram is (a, b) then the value of $|ab|$ is

17. Difference in values of radius of a circle whose centre is at the origin and which touches the circle $x^2 + y^2 - 6x - 8y + 21 = 0$ is

Archives

Solutions on page 2.96

Subjective Type

1. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5). **(IIT-JEE, 1978)**

2. Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2 y = 1$ as c tends to 1.

3. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C. Find the area of the quadrilateral $ABCD$. **(IIT-JEE, 1981)**

4. Find the equations of the circle passing through $(-4, 3)$ and touching the lines $x + y = 2$ and $x - y = 2$. **(IIT-JEE, 1982)**

5. Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the midpoints of the secants by the circle is $x^2 + y^2 = hx + ky$. **(IIT-JEE, 1983)**

6. The abscissa of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter. **(IIT-JEE, 1984)**

7. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. **(IIT-JEE, 1986)**

8. Let a given line L_1 intersect the x and y axes at P and Q, respectively. Let another line L_2, perpendicular to L_1, cut the x and y-axis at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. **(IIT-JEE, 1987)**

9. The circle $x^2 + y^2 - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcnetre of the triangle is $x + y - xy + k(x^2 + y^2)^{1/2} = 0$. Find k. **(IIT-JEE,1987)**

10. Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin. **(IIT-JEE, 1988)**

11. If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$, i = 1, 2, 3, 4 are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$. **(IIT-JEE, 1989)**

12. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle. **(IIT-JEE, 1990)**

13. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equation of the circles. **(IIT-JEE, 1991)**

14. Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$. **(IIT-JEE, 1992)**

15. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point. **(IIT-JEE, 1993)**

16. Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = -4$ and $x^2 + y^2 - 12x - 8y = -36$ touch each other. Also find equations common tangents touching the circles in the distinct points. **(IIT-JEE, 1993)**

17. Find the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$. **(IIT-JEE, 1996)**

18. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line from the point P intersects the curve at points Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle. **(IIT-JEE, 1997)**

19. Let C be any circle with centre $(0, \sqrt{2})$. Prove that at the most two rational points can be there on C (a rational point is a point both of whose coordinates are irrational numbers). **(IIT-JEE, 1997)**

20. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **(IIT-JEE, 1998)**

21. Let T_1, T_2 be two tangents drawn from $(-2, 0)$ onto the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time.

22. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. **(IIT-JEE, 2001)**

23. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **(IIT-JEE, 2003)**

24. Find the equation of circle touching the line $2x + 3y + 1 = 0$ at $(1, -1)$ and cutting orthogonally the circle having line segment joining $(0, 3)$ and $(-2, -1)$ as diameter. **(IIT-JEE, 2004)**

25. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C. (IIT-JEE, 2009)

Objective Type

Fill in the blanks

1. If A and B are points in the plane such that $\frac{PA}{PB} = k$ (constant) for all P on a given circle, then the value of k cannot be equal to____________. **(IIT-JEE, 1982)**

2. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are_____ and____. **(IIT-JEE, 1983)**

3. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of this circle is_________. **(IIT-JEE, 1984)**

4. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point (4, 5) with a pair of radii form a quadrilateral of area_____________. **(IIT-JEE, 1985)**

5. From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the midpoints of these chord is________________. **(IIT-JEE, 1985)**

6. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is_________. **(IIT-JEE, 1986)**

7. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. The equation of the locus of M is________. **(IIT-JEE, 1986)**

8. The area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact is___________. **(IIT-JEE, 1988)**

9. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that common chord is of maximum length and has a slope equal to $\frac{3}{4}$, then the coordinates of the centre of C_2 are____. **(IIT-JEE, 1988)**

10. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is,______________. **(IIT-JEE, 1989)**

11. If a circle passes through points of intersection of the coordinates axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is___. **(IIT-JEE, 1991)**

12. The equation of the locus of the midpoints of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is______________. **(IIT-JEE, 1993)**

13. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is________________. **(IIT-JEE, 1996)**

14. Two vertices of an equilateral triangle are (–1, 0) and (1, 0), and its third vertex lies above the x-axis. The equation of its circumcircle is___________. **(IIT-JEE, 1997)**

15. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to circle $x^2 + y^2 = 1$ pass through point____________________. **(IIT-JEE, 1997)**

True or false

1. No tangent can be drawn from the point (5/2, 1) to the circumcircle of the triangle with vertices $(1, \sqrt{3}), (1, -\sqrt{3}, (3, -\sqrt{3})$. **(IIT-JEE, 1985)**

2. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the coordinate axes. The one vertex of the square is **(IIT-JEE, 1980)**
   a. $(1 + \sqrt{2}, -2)$ **b.** $(1 - \sqrt{2}, -2)$
   c. $(1, -2 + \sqrt{2})$ **d.** None of these

2. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then, the equation of the circle through their points of intersection and the point (1, 1) is
   a. $x^2 + y^2 - 6x + 4 = 0$
   b. $x^2 + y^2 - 3x + 1 = 0$
   c. $x^2 + y^2 - 4y + 2 = 0$
   d. None of these **(IIT-JEE, 1980)**

3. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
   a. $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
   b. $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
   c. $4x^2 + 4y^2 - 17x - 10y + 25 = 0$
   d. None of these **(IIT-JEE, 1983)**

4. The locus of the midpoint of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is
   a. $x + y = 2$ **b.** $x^2 + y^2 = 1$
   c. $x^2 + y^2 = 2$ **d.** $x + y = 1$
   (IIT-JEE, 1984)

5. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its centre is **(IIT-JEE, 1988)**
   a. $2ax + 2by - (a^2 + b^2 + k^2) = 0$
   b. $2ax + 2by - (a^2 - b^2 + k^2) = 0$
   c. $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
   d. $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$

6. If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then
   a. $2 < r < 8$ **b.** $r < 2$
   c. $r = 2$ **d.** $r > 2$

7. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. Then the equation of this circle is
   a. $x^2 + y^2 + 2x - 2y = 62$
   b. $x^2 + y^2 + 2x - 2y = 47$
   c. $x^2 + y^2 - 2x + 2y = 47$
   d. $x^2 + y^2 - 2x + 2y = 62$

8. The centre of the circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is
   (IIT-JEE, 1992)
   a. $\left(\frac{3}{2}, \frac{1}{2}\right)$ **b.** $\left(\frac{1}{2}, \frac{3}{2}\right)$
   c. $\left(\frac{1}{2}, \frac{1}{2}\right)$ **d.** $\left(\frac{1}{2}, -2^{\frac{1}{2}}\right)$

9. The locus of the centre of a circle, which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis, is given by equation **(IIT-JEE, 1993)**
   a. $x^2 - 6x - 10y + 14 = 0$
   b. $x^2 - 10x - 6y + 14 = 0$
   c. $y^2 - 6x - 10y + 14 = 0$
   d. $y^2 - 10x - 6y + 14 = 0$

10. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Then equation of the locus of the point P is
    (IIT-JEE, 1996)
    a. $x^2 + y^2 + 4x - 6y + 4 = 0$
    b. $x^2 + y^2 + 4x - 6y - 9 = 0$
    c. $x^2 + y^2 + 4x - 6y - 4 = 0$
    d. $x^2 + y^2 + 4x - 6y + 9 = 0$

11. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 - px - qy = 0$ (where $pq \neq 0$) are bisected by the x-axis, then
    a. $p^2 = q^2$ **b.** $p^2 = 8q^2$
    c. $p^2 < 8q^2$ **d.** $p^2 > 8q^2$
    (IIT-JEE, 1999)

12. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is equal to
    a. $\frac{\pi}{2}$ **b.** $\frac{\pi}{3}$ **c.** $\frac{\pi}{4}$ **d.** $\frac{\pi}{6}$
    (IIT-JEE, 2000)

13. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$, $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
    (IIT-JEE, 2000)
    a. 2 or $-\frac{3}{2}$ **b.** -2 or $-\frac{3}{2}$
    c. 2 or $\frac{3}{2}$ **d.** -2 or $\frac{3}{2}$

14. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the triangle PAB as P moves on the circle is
    (IIT-JEE, 2001)
    a. a parabola **b.** a circle

c. an ellipse d. a pair of straight lines

15. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and PQ intersect at a point X on the circumference of the circle, then $2r$ equals **(IIT-JEE, 2001)**

a. $\sqrt{PQ.RS}$ b. $\frac{(PQ+RS)}{2}$

c. $\frac{2PQ \times RS}{PQ+RS}$ d. $\frac{\sqrt{(PQ^2+RS^2)}}{2}$

16. If the tangent at the point P on the circle $x^2+y^2+6x+6y=2$ meets a straight lines $5x-2y+6=0$ at a point Q on the y-axis, then the length of PQ is **(IIT-JEE, 2002)**

a. 4 b. $2\sqrt{5}$ c. 5 d. $3\sqrt{5}$

17. The centre of circle inscribed in square formed by the lines $x^2-8x+12=0$ and $y^2-14y+45=0$, is

a. (4, 7) b. (7, 4) c. (9, 4) d. (4, 9)

(IIT-JEE, 2003)

18. If one of the diameters of the circle $x^2+y^2-2x-6y+6=0$ is a chord to the circle with centre (2, 1), then the radius of the circle is **(IIT-JEE, 2004)**

a. $\sqrt{3}$ b. $\sqrt{2}$ c. 3 d. 2

19. A circle is given by $x^2+(y-1)^2=1$, another circle C touches it externally and also the x-axis, then the locus of its centre is

a. $\{(x,y): x^2=4y\} \cup \{(x,y): y \le 0\}$

b. $\{(x,y): x^2+(y-1)^2=4\} \cup \{(x,y): y \le 0\}$

c. $\{(x,y): x^2=y\} \cup \{(0,y): y \le 0\}$

d. $\{(x,y): x^2=4y\} \cup \{(0,y): y \le 0\}$

(IIT-JEE, 2005)

20. Tangents drawn from the point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

a. $x^2+y^2+4x-6y+19=0$

b. $x^2+y^2-4x-10y+19=0$

c. $x^2+y^2-2x+6y-20$

d. $x^2+y^2-6x-4y+19=0$

21. The circle passing through the point (– 1, 0) and touching the y–axis at (0, 2) also passes through the point

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$

(C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) (–4, 0) **(IIT-JEE, 2010)**

Multiple choice questions with one or more than one correct answer

1. The equations of the tangents drawn from the origin to the circle $x^2+y^2-2rx-2hy+h^2=0$, are

a. $x=0$

b. $y=0$

c. $(h^2-r^2)x-2rhy=0$

d. $(h^2-r^2)x+2rhy=0$

(IIT-JEE, 1988)

2. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then **(IIT-JEE, 1998)**

a. $x_1+x_2+x_3+x_4=0$

b. $y_1+y_2+y_3+y_4=0$

c. $x_1x_2x_3x_4=c^4$

d. $y_1y_2y_3y_4=c^4$

Comprehension Type

For Problems 1–3

Let $ABCD$ be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of square $ABCD$. L is a line through A.

1. If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

a. 0.75 b. 1.25 c. 1 d. 0.5

(IIT-JEE, 2006)

2. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

a. ellipse b. hyperbola

c. parabola d. pair of straight line

3. A line M through A is drawn parallel to BD. Point S moves such that its distance from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is **(IIT-JEE, 2006)**

a. 1/2 sq. units b. 2/3 sq. units

c. 1 sq. units d. 2 sq. units

For Problems 4–6

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x+y-6=0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. **(IIT-JEE, 2008)**

4. The equation of circle C is

a. $(x-2\sqrt{3})^2+(y-1)^2=1$

b. $(x-2\sqrt{3})^2+\left(y+\frac{1}{2}\right)^2=1$

c. $(x-\sqrt{3})^2+(y+1)^2=1$

d. $(x-\sqrt{3})^2+(y-1)^2=1$

5. Points E and F are given by

a. $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right), (\sqrt{3},0)$

b. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right), (\sqrt{3},0)$

c. $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right), \left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

d. $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

6. Equation of the sides QR, RP are

a. $y=\frac{2}{\sqrt{3}}x+1, y=-\frac{2}{\sqrt{3}}x-1$

b. $y=\frac{1}{\sqrt{3}}x, y=0$

c. $y=\frac{\sqrt{3}}{2}x+1, y=-\frac{\sqrt{3}}{2}x-1$

d. $y=\sqrt{3}x, y=0$

Assertion and reasoning

1. Tangents are drawn from the point (17, 7) to the circle $x^2+y^2=169$.

Statement 1: The tangents are mutually perpendicular.

Statement 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$. **(IIT-JEE, 2007)**

a. Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

b. Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

c. Statement 1 is true, Statement 2 is false.

d. Statement 1 is false, Statement 2 is true.

2. Consider

$L_1: 2x+3y+p-3=0$

$L_2: 2x+3y+p+3=0$,

where p is a real number, and $C: x^2+y^2+6x-10y+30=0$.

Statement 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

Statement 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

a. Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

b. Statement 1 is true, Statement 2 is true; Statement 2 is NOT a correct explanation for statement 1.

c. Statement 1 is true, Statement 2 is false

d. Statement 1 is false, statement 2 is true

(IIT-JEE, 2008)

Matrix-match

1. This question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q, and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a.	(p)	(q)	(r)	(s)
b.	(p)	(q)	(r)	(s)
c.	(p)	(q)	(r)	(s)
d.	(p)	(q)	(r)	(s)

(IIT-JEE, 2007)

	Column I		Column II
a.	Two intersecting circles	**p.**	have a common tangent
b.	Two mutually external circles	**q.**	have a common normal
c.	Two circles, one strictly inside the other	**r.**	do not have a common tangent
d.	Two branches of a hyperbola	**s.**	do not have a common normal

Integer type

1. The straight line $2x-3y=1$ divides the circular region $x^2+y^2\le 6$ into two parts. If

$S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right),\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\}$.

then the number of point(s) in S lying inside the smaller part is **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1.

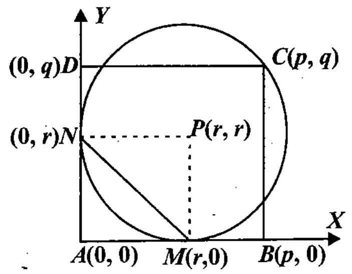


Fig. 2.77

Let us take AB and AD as coordinate axes

If r be the radius of circle, then its centre is $P(r, r)$.

Equation of circle is $(x - r)^2 + (y - r)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2rx - 2ry + r^2 = 0$

Let the coordinates of $C \equiv (p, q)$

Equation of MN is $x + y = r$

Its distance from C is 5 units $\Rightarrow \frac{|p+q-r|}{\sqrt{2}} = 5$

$\Rightarrow \quad (p + q - r)^2 = 50$

Since (p, q) lie on the circle,

$p^2 + q^2 - 2rp - 2rq + r^2 = 0$

$\Rightarrow \quad (p + q - r)^2 - 2pq = 0$

$\Rightarrow \quad 50 - 2pq = 0 \Rightarrow pq = 25$

Area of rectangle = 25 sq. units

2.

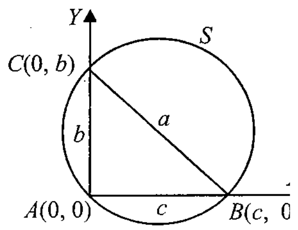


Fig. 2.78

Take AB as the x-axis and AC as the y-axis

Let $\quad BC = a$

Centre of S is $\left(\frac{c}{2}, \frac{b}{2}\right)$ and radius is $\frac{a}{2}$

If a circle S' touches the rays AB and AC, its centre must be (r, r) and its radius must be r, for some $r > 0$.

Circle S and S' touch each other if

$$\sqrt{\left(r-\frac{c}{2}\right)^2 + \left(r - \frac{b}{2}\right)^2} = \frac{a}{2} \pm r$$

Squaring both sides and using the fact that $a^2 = b^2 + c^2$, we get

$$r = b + c \pm a$$

$\Rightarrow \quad r_1 = b + c - a,\ r_2 = b + c + a$

$\Rightarrow \quad r_1 \cdot r_2 = (b + c)^2 - a^2 = 2bc$ (as $a^2 = b^2 + c^2$)

$= 4$ area (ΔABC)

3. The given circles are

$C_1 : (x - 1)^2 + (y - 1)^2 = 1,\ r_1 = 2$

and $C_2 : (x - 8)^2 + (y - 1)^2 = 4,\ r_2 = 2$

The line $y - 2x - a = 0$ will lie between these circles if centre of the circles lie on opposite sides of the line,

i.e. $(1 - 2 - a)(1 - 16 - a) < 0$

$\Rightarrow \quad a \in (-15, -1)$

Line wouldn't touch or intersect the circles, if

$$\frac{|1-2-a|}{\sqrt{5}} > 1, \frac{|1-16-a|}{\sqrt{5}} > 2$$

$\Rightarrow \quad |1 + a| > \sqrt{5},\ |15 + a| > 2\sqrt{5}$

$\Rightarrow a > \sqrt{5} - 1$ or $a < \sqrt{5} - 1$, $a > 2\sqrt{5} - 15$

or $a < -2\sqrt{5} - 15$

Hence, common values of 'a' are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

4. The given circle is:

$x^2 + y^2 - 2ax - 2by + 2 = 0$

or $\quad (x - a)^2 + (y - b)^2 = a^2 + b^2 - 2$

it's director circle is

$$(x - a)^2 + (y - b)^2 = 2(a^2 + b^2 - 2)$$

Given that tangents drawn from the origin to the circle are orthogonal, it implies that director circle of the circle must pass through the origin,

$\Rightarrow \quad a^2 + b^2 = 2(a^2 + b^2 - 2)$

$\Rightarrow \quad a^2 + b^2 = 4$

Thus, the locus of the centre of the given circle is,

$$x^2 + y^2 = 4$$

5. If 'd' be the common difference of A.P., then radius of the smallest circle is $1 - 2d$. If the given line $y - x - 1 = 0$ cuts the smallest circle in real and distinct points, then it will definitely cut the remaining circles in real and distinct points.

$\Rightarrow \quad \frac{|0-0-1|}{\sqrt{2}} < (1 - 2d)$

$\Rightarrow \quad 1 - 2d > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad d < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

Hence, $\quad d \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right)$

6. Equation of line PQ is

$$(y - 1) = m(x - 4)$$

or $\quad y - mx + 4m - 1 = 0$

For the required 'm', we have to make sure that the line PQ meets the circle, with diameter AB, at real and distinct points.

Equation of circle having AB as diameter is

$$x^2 + y^2 - 2x - 3 = 0$$

$$\Rightarrow \quad \frac{|0 - m + 4m - 1|}{\sqrt{1 + m^2}} < 2$$

$$\Rightarrow \quad 5m^2 - 6m - 3 > 0$$

$$\Rightarrow \quad m \in \left(\frac{3 - 2\sqrt{6}}{5}, \frac{3 + 2\sqrt{6}}{5}\right)$$

7. Equation of the circle having the ends of diameter at $(1, -3)$ and $(4, 1)$ is

$(x - 1)(x - 4) + (y + 3)(y - 1) = 0$

$\Rightarrow \quad x^2 + y^2 - 5x + 2y + 1 = 0$

Other circle will be $x^2 + y^2 - 5x + 2y + 1 + \lambda(x + y - 1) = 0$

It passes through $(1, 2)$

$$\Rightarrow \quad \lambda = -\frac{5}{2}$$

$\Rightarrow$ circle is; $x^2 + y^2 - \frac{15}{2}x - \frac{y}{2} + \frac{7}{2} = 0$

8. Let radius $= r$

$\therefore$ From figure $\quad \sqrt{\alpha^2 + a^2} = b + r \qquad$ (i)

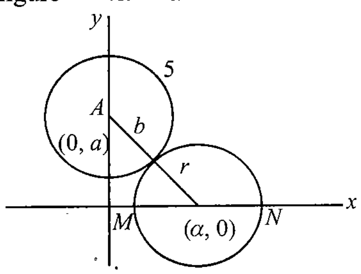


Fig. 2.79

Consider a point $P(0, k)$ on the y-axis

$M(\alpha - r, 0)$ and $N(\alpha + r, 0)$

Now, slope of $\quad MP = \frac{-k}{\alpha - r}$, slope $NP = \frac{-k}{\alpha + r}$

If $\quad \angle MPN = \theta$

$$\tan\theta = \left|\frac{\frac{k}{\alpha - r} - \frac{k}{\alpha + r}}{1 + \frac{k^2}{\alpha^2 - r^2}}\right|$$

$$= \left|\frac{2kr}{\alpha^2 - r^2 + k^2}\right|$$

According to the given condition, θ is a constant for any choice α

$$\frac{2kr}{\alpha^2 - r^2 + k^2} = \text{constant}$$

i.e., $\quad \frac{r}{\alpha^2 - r^2 + k^2} = \text{constant}$

i.e., $\quad \frac{\sqrt{\alpha^2 + a^2} - b}{\alpha^2 - \left(\sqrt{\alpha^2 + a^2} - b^2\right) + k^2} = \text{constant}$

(From Eq. (1))

i.e., $\quad \frac{\sqrt{\alpha^2 + a^2} - b}{2b\sqrt{\alpha^2 + a^2} - a^2 - b^2 + k^2} = \text{constant}$

$$\frac{\sqrt{\alpha^2 + a^2} - b}{\sqrt{\alpha^2 + a^2} - \lambda} = \text{constant}$$

where $\quad \left\{\frac{a^2 + b^2 - k^2}{2b}\right\} = \lambda$

which is possible only, if $\lambda = b$

$$\frac{a^2 + b^2 - k^2}{2b} = b \Rightarrow k = \pm\sqrt{a^2 - b^2}$$

$\therefore \quad P \equiv \left(0, \pm\sqrt{a^2 - b^2}\right)$

9. $S(x, 2) = 0$ given two identical solutions $x = 1$

$\Rightarrow$ line $y = 2$ is a tangent to the circle $S(x, y) = 0$ at the point $(1, 2)$ and $S(1, y) = 0$ gives two distinct solutions $y = 0, 2$.

$\Rightarrow$ Line $x = 1$ cuts the circle $S(x, y) = 0$ at points $(1, 0)$ and $(1, 2)$.

$\therefore A(1, 2)$ and $B(1, 0)$ are diametrically opposite points.

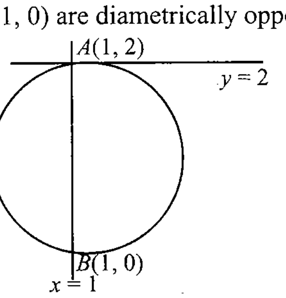


Fig. 2.80

$\therefore$ Equation of the circle is $(x - 1)^2 + y(y - 2) = 0$

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

10. $x^2 - y^2 + 2y - 1 = 0$

$$(x - y + 1)(x + y - 1) = 0$$

The centre of family of circles touching the above lines will lie on the angle bisectors of the above lines.

Equations of the angle bisectors of the above lines are given by

$$\frac{x - y + 1}{\sqrt{2}} = \pm\frac{x + y - 1}{\sqrt{2}}$$

$\Rightarrow x = 0$ and $y = 1$

Case 1:

Let $A(h, 1)$ be a point on the line $y = 1$, $AP = \frac{h}{\sqrt{2}}$

Equation of circle touching the given line is $(x - h)^2 + (y - 1)^2 = \frac{h^2}{2}$

i.e., $x^2 + y^2 - 2hx - 2y + \frac{h^2}{2} = 0$

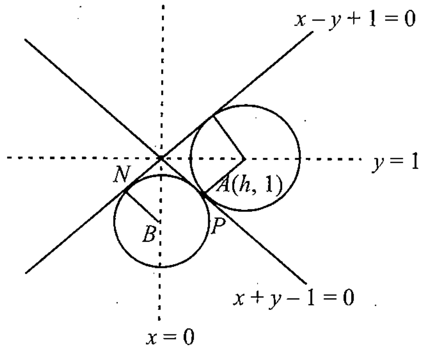


Fig. 2.81

Case 2:

Let $B(0, k)$ be a point on the line $x = 0$.

Now, $\qquad BN = \frac{k}{\sqrt{2}}$

Equation of circle touching the given lines is

$$(x - 0)^2 + (y - k)^2 = \frac{k^2}{2}$$

or $\qquad x^2 + y^2 - 2ky + \frac{k^2}{2} = 0$

11. AB subtends the greatest angle at C, so the line $x - y + 1 = 0$ touches the circle at C and hence AB is the diameter.

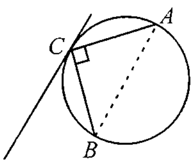


Fig. 2.82

Family of circle touching line $x - y + 1 = 0$ at point $(1, 2)$ is $(x - 1)^2 + (y - 2)^2 + \lambda(x - y + 1) = 0$

Its radius $= \sqrt{\left(\frac{\lambda - 2}{2}\right)^2 + \left(\frac{\lambda + 4}{2}\right)^2 - (5 + \lambda)}$

$= \sqrt{2}$

$\Rightarrow \qquad \lambda = \pm 2$

Therefore, The equations of circle are $x^2 + y^2 - 64 + 7 = 0$

and $\quad x^2 + y^2 - 4x - 2y + 3 = 0.$

12.

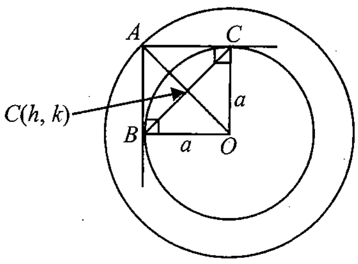


Fig. 2.83

Clearly, $x^2 + y^2 = 2a^2$ is director circle of the circle $x^2 + y^2 = a^2$

Hence, in the diagram $ABCO$ is a square and circumcentre $P(h, k)$ of ΔABC is midpoint of OA

Hence, $\sqrt{h^2 + k^2} = \frac{\sqrt{2}a}{2}$

or locus is $x^2 + y^2 = \frac{a^2}{2}$

13. Circle of minimum radius that touches the given circle $x^2 + y^2 = 4$ will be possible only if (4, 3) is the end point of the diameter of the required circle. Let the centre of the required circle be O' and point of contact of circles B.

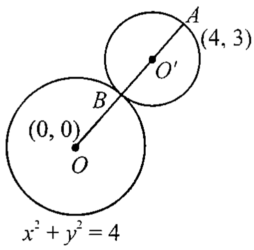


Fig. 2.84

Now $\qquad OA = 5$ and $OB = 2$

$\qquad BA = 5 - 2 = 3$

$\Rightarrow$ Point B divides OA in the ratio 2 : 3 internally

$\Rightarrow$ Point $\qquad B = \left(\frac{8}{5}, \frac{6}{5}\right)$

$\Rightarrow$ Point O' is $\left(\frac{14}{5}, \frac{21}{10}\right)$.

Hence, the required circle is $\left(x - \frac{14}{5}\right)^2 + \left(y - \frac{21}{10}\right)^2 = \frac{9}{4}$

14. Let the point of intersection of tangents at A and C is $P = (x, y)$.

Since $\qquad AB = AO = BO = r$

$\Rightarrow \qquad \angle AOB = 60°$

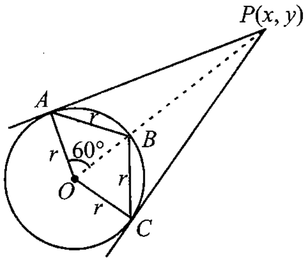


Fig. 2.85

$\Rightarrow \qquad \frac{PA}{r} = \tan 60° = \sqrt{3}$

$\Rightarrow \qquad \sqrt{x^2 + y^2 - r^2} = \sqrt{3}r$

$\Rightarrow x^2 + y^2 = 4r^2$ is the locus of the point P.

15. Angle between $3x + y = 0$ and the line joining $(2, -1)$ to $(0, 0)$ is

$$\theta = \tan^{-1}\left|\frac{-3 + \frac{1}{2}}{1 + (-3)\left(-\frac{1}{2}\right)}\right|$$

$$= \tan^{-1}|-1| = \frac{\pi}{4}$$

$\Rightarrow$ Other tangent is perpendicular to $3x + y = 0$.

$\Rightarrow$ Equation of the other tangent is $x - 3y = 0$.

16.

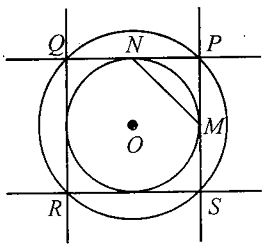


Fig. 2.86

From the diagram, $OMPN$ is square and P lies on director circle.

$$MN = OP$$

= radius of the direction of given circle

$= \sqrt{2}\,|\sqrt{2}\,b| = 2b$

17. Radius of given circle $= \sqrt{4 + 2 - c} = \sqrt{6 - c} = a$ (say)

Radius of circle $\qquad S_1 = \frac{a}{\sqrt{2}}$

Radius of circle $\qquad S_2 = \frac{a}{2}$ and so on.

$\therefore \quad a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \cdots + \infty = 2$

$\Rightarrow \qquad a\left[\frac{1}{1 - 1/\sqrt{2}}\right] = 2 \Rightarrow a\left(\frac{\sqrt{2}}{\sqrt{2} - 1}\right) = 2.$

$\Rightarrow \qquad a = 2 - \sqrt{2} \Rightarrow \sqrt{6 - c} = 2 - \sqrt{2}$

$\Rightarrow \qquad 6 - c = 4 + 2 - 4\sqrt{2} \Rightarrow c = 4\sqrt{2}$

18.

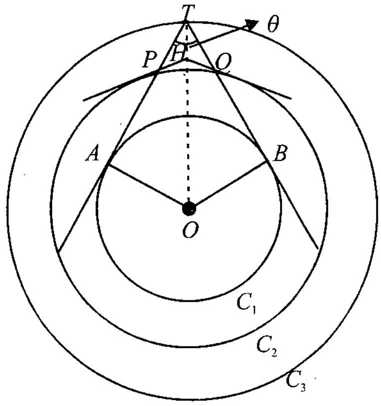


Fig. 2.87

Let C_1 be a circle of radius r.

So radius of $\qquad C_2 = \sqrt{2}\,r$ and that of $C_3 = 2r$.

In ΔAOT, $\qquad \sin\frac{\theta}{2} = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow \qquad \angle ATB = \frac{\pi}{3}$

$\Rightarrow$ In ΔAOP, $\quad \sin\angle APO = \frac{r}{r\sqrt{2}} \Rightarrow \angle APO = \frac{\pi}{4}$

In ΔOPT, $\qquad \angle POT = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

In ΔOPH,

$$\angle OPH = \frac{\pi}{2},$$

$$\angle PHO = \pi - \left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \frac{5\pi}{12}$$

$\Rightarrow \qquad \angle PHQ = \frac{5\pi}{6}$. As $\angle POQ = \frac{\pi}{6}$

$OPHQ$ is a cyclic quadrilateral. Hence, locus of H is a circle.

19.

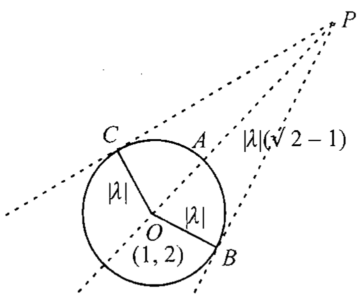


Fig. 2.88

Since line $y = x + c$ is normal to the given circle

$\Rightarrow$ $c = 1$

$\Rightarrow$ Equation of line is

$$y = x + 1 \quad \text{(i)}$$

Also, radius of the circle = $|\lambda|$

Given $AP = |\lambda|(\sqrt{2} - 1)$

$\Rightarrow$ $OP = \sqrt{2}|\lambda|$

$\Rightarrow$ $PC = |\lambda|$

$\Rightarrow$ Area of quadrilateral $OBPC$

$$= 2 \times \frac{1}{2}|\lambda|^2 = 36 \Rightarrow \lambda = \pm 6$$

20. Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

This is orthogonal to given circle

$\Rightarrow$ $2g \cdot 0 + 2f \cdot 0 = -1 + c \Rightarrow c = 1$

and $2 \cdot 4 \cdot g + 2 \cdot 4 \cdot f = -33 + c = -32$

$\Rightarrow$ $g + f = -4$

Hence, radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{g^2 + (g+4)^2 - 1}$$

i.e., $r = \sqrt{2g^2 + 8g + 15}$

$$= \sqrt{2(g+2)^2 + 7}$$

For minimum $r, g + 2 = 0$

$\Rightarrow$ $g = -2 \Rightarrow$ the centre is (2, 2).

21.

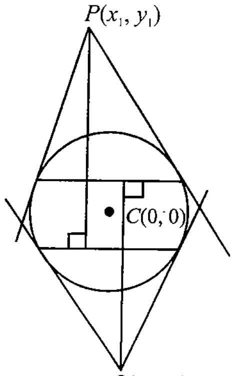


Fig. 2.89

Let the circle be $x^2 + y^2 = a^2$. Then the chord of contact of $Q(x_2, y_2)$ w.r.t. the circle is

$$xx_2 + yy_2 = a^2 \quad \text{(i)}$$

Its distance from $P(x_1, y_1)$ is $\dfrac{|a^2 - (x_1x_2 + y_1y_2)|}{\sqrt{x_2^2 + y_2^2}}$

Similarly, distance of Q from the chord of contact of P is $\dfrac{|a^2 - (x_1x_2 + y_1y_2)|}{\sqrt{x_1^2 + y_1^2}}$.

Hence, ratio of the lengths of perpendiculars $= \dfrac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}$

$$= \frac{\text{distance of the point } P \text{ from the centre of the circle}}{\text{distance of the point } Q \text{ from the centre of the circle}}$$

22. The given circles are $(x-1)^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 16$

The points $(a + 1, \sqrt{3}a)$ lie on the line

$\Rightarrow$ $x = a + 1, y = \sqrt{3}a$

$\Rightarrow$ $y = \sqrt{3}(x - 1)$ [eliminating a]

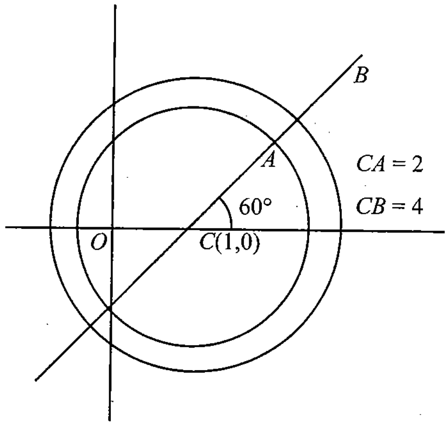


Fig. 2.90

This line makes an angle 60° with the +ve direction of the x-axis.

Hence, we have

$$A \equiv (1 + 2\cos 60°, 2\sin 60°) \equiv (2, \sqrt{3})$$

and $B \equiv (1 + 4\cos 60°, 4\sin 60°)(3, 2\sqrt{3})$

Hence, there is no point on the line segment AB whose abscissa is an integer since abscissa of A is 2 and that of B is 3.

23.

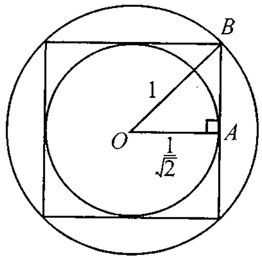


Fig. 2.91

The director circle $x^2 + y^2 = 2a^2$ must intersect the square formed by the lines $|x| + |y| = 1$ in eight points.

for which $\frac{1}{\sqrt{2}} < \sqrt{2}\,a < 1$ or $\frac{1}{2} < a < \frac{1}{\sqrt{2}}$

24. Obviously angle bisectors are $x = 2$ and $y = 0$

Now centre cannot lies on $y = 0$ because their chord of contact from origin will always be parallel to y-axis. So let the centre $(2, \alpha)$, then equation of circle will be

$$(x-2)^2 + (y-\alpha)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 - 4x - 2\alpha y - 21\alpha^2 = 0$$

Now chord of contact is

$$-4\frac{x}{2} - 2\alpha\frac{y}{2} + \alpha^2 - 21 = 0 \Rightarrow 2x + \alpha y - \alpha^2 + 21 = 0$$

Now, $\qquad -\frac{2}{\alpha} = 1 \quad \Rightarrow \alpha = -2$

So equation of circle is $(x-2)^2 + (y+2)^2 = 5^2$.

25. Equation of chord of contact AB is $5x - 5y = 5$

Solving it with $x^2 + y^2 = 5$ we get,

$$x^2 + (x-1)^2 = 5$$

$$\Rightarrow \quad x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow \quad x = -1, 2 \text{ and } y = -2, 1$$

So, A and B are $(-1, -2)$ and $(2, 1)$.

Let $\qquad P \equiv (\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$

Now as circumcentre of the triangle PAB is origin, orthocentre would be $h = x_1 + x_2 + x_3$ and $k = y_1 + y_2 + y_3$ (as centroid divides line joining orthocentre and circumcentre in 2 : 1).

i.e. $\qquad h = \sqrt{5}\cos\theta + 1$

and $\qquad k = \sqrt{5}\sin\theta - 1$

So the required locus is

$$(x-1)^2 + (y+1)^2 = 5.$$

26.

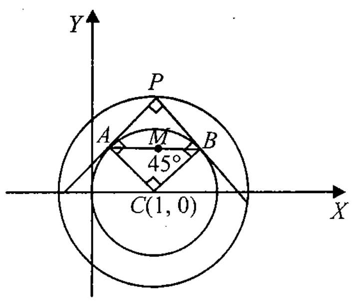


Fig. 2.92

The two circles are

$$(x-1)^2 + y^2 = 1 \qquad \text{(i)}$$

$$(x-1)^2 + y^2 = 2 \qquad \text{(ii)}$$

So the second one is the director circle of the first circle.

So $\angle APB = \frac{\pi}{2}$

$$\Rightarrow \quad \angle ACB = \frac{\pi}{2}$$

Now, circumcentre of the right angled triangle CAB would lie on the midpoint of AB.

So, let that point be $\qquad M \equiv (h, k)$

Now, $\qquad CM = CB \sin 45° = \frac{1}{\sqrt{2}}$

So, $\qquad (h-1)^2 + k^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x-1)^2 + y^2 = \frac{1}{2}$.

27.

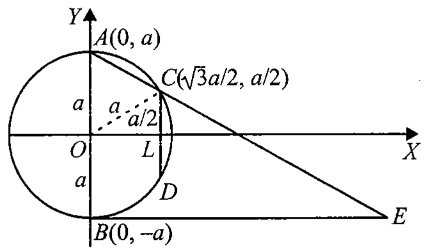


Fig. 2.93

Let origin be taken at the centre of the circle and y-axis along the diameter AB.

Then, equation of the circle is $x^2 + y^2 = a^2 \qquad$ (i)

and coordinates of A and B are $(0, a)$ and $(0, -a)$, respectively.

If CD is parallel to AB such that $2CD = AB$, then $CD = \frac{1}{2}AB = a$

Obviously, CD is bisected by x-axis say at L, then $CL = a/2$.

In ΔOLC, $\qquad OL = \sqrt{(OC^2 - CL^2)}$

$$= \sqrt{(a^2 - a^2/4)} = \sqrt{3}a/2$$

$\Rightarrow$ The coordinates of the point C are $(\sqrt{3}a/2, a/2)$.

$\Rightarrow$ Equation of line AC is $y - a = \dfrac{a/2 - a}{\sqrt{3}a/2 - 0}(x - 0)$ or

$$y = -\frac{1}{\sqrt{3}}x + a \qquad \text{(ii)}$$

Equation of the tangent to the circle at the point $B\ (0, -a)$ on it is

$$x.0 + y.(-a) = a \text{ or } y = -a \qquad \text{(iii)}$$

Solving Eqs. (ii) and (iii), their point of intersection is $E(2\sqrt{3}\ a, -a)$.

$$\Rightarrow \quad AE = \sqrt{[(2\sqrt{3}a - 0)^2 + (-a-a)^2]}$$

$$= 4a = 2AB$$

28. Equation of tangent to the circle at the point (h, k) is

$$hx + ky = a^2 \qquad \text{(i)}$$

Pair of lines $y = a$ and $y = -a$ is $y^2 - a^2 = 0$

Pair of lines OR and OQ is given by making $y^2 - a^2 = 0$ homogeneous with the help of line (i)

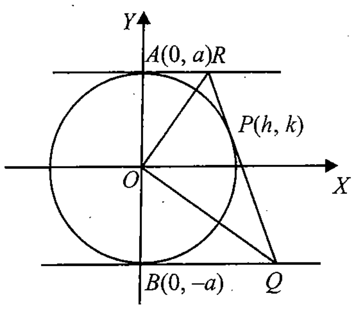


Fig. 2.94

$\therefore \quad y^2 - a^2\left(\frac{hx+ky}{a^2}\right)^2 = 0$

or $\quad a^2y^2 - (hx+ky)^2 = 0$

now, coefficient of x^2 + coefficient of y^2

$= -h^2 - k^2 + a^2$

$= 0$ (as (h, k) lies on the circle)

Hence, lines OR and OQ are perpendicular.

29.

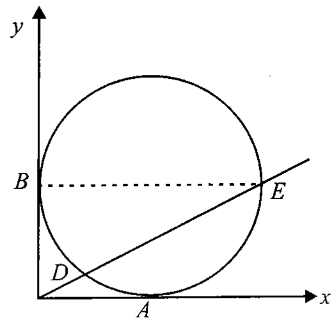


Fig. 2.95

The equation of the circle is $(x-1)^2 + (y-1)^2 = 1$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y + 1 = 0 \qquad$ (i)

Let the equation of the variable straight line be $y = mx$ (ii)

Solving Eqs. (i) and (ii), we get

$(1+m^2)x^2 - 2x(1+m) + 1 = 0$

$\therefore$ Length $\quad DE = \sqrt{\frac{8m}{1+m^2}}$

Area of ΔDEB,

$A = \frac{1}{2}DE \times$ distance of B from DE

$\therefore \quad A^2 = \frac{1}{4}\cdot\frac{8m}{1+m^2}\cdot\frac{1}{1+m^2}$

$= \frac{2m}{(1+m^2)^2}$

$\Rightarrow \quad A = \frac{\sqrt{2m}}{1+m^2}$

$\frac{dA}{dm} = \frac{1-3m^2}{\sqrt{m}(1+m^2)^2} = 0$

$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$

$\frac{d^2A}{dm^2} < 0$ if $m = \frac{1}{\sqrt{3}}$

$\therefore$ Area is maximum for $m = \frac{1}{\sqrt{3}}$.

Objective Type

1. a. If there are more than one rational points on the circumference of the circle $x^2 + y^2 - 2\pi x - 2ey + c = 0$ (as (π, e) is the centre), then e will be a rational multiple of π, which is not possible. Thus, the number of rational points on the circumference of the circle is at most one.

2. a. Point of intersection of diagonals lie on circumcircle i.e. (1, 1), since $(y - 2x + 1) + \lambda(2y - x - 1) = 0$

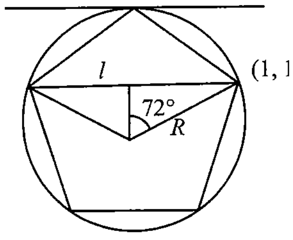


Fig. 2.96

$l = 2R \sin 72°$

$R = \frac{\sin 36°}{2 \sin 72°} = \cos 72°$

$\Rightarrow$ Locus is $(x-1)^2 + (y-1)^2 = \cos^2 72°$

$\Rightarrow x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$

3. c. Let the equation of the chord OA of the circle

$x^2 + y^2 - 2x + 4y = 0 \qquad$ (i)

be $\qquad y = mx \qquad$ (ii)

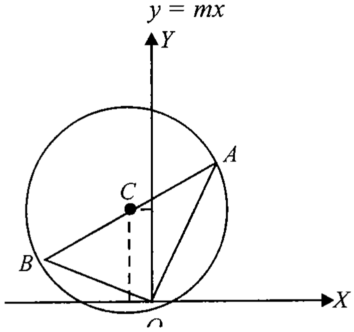


Fig. 2.97

Solving (i) and (ii), we get

$\Rightarrow \quad x^2 + m^2x^2 - 2x + 4mx = 0$

$\Rightarrow (1+m^2)x^2 - (2-4m)x = 0$

$$\Rightarrow \quad x = 0 \text{ and } x = \frac{2-4m}{1+m^2}$$

Hence, the points of intersection are

$(O, 0)$ and $A\left(\frac{2-4m}{1+m^2}, \frac{m(2-4m)}{1+m^2}\right)$

$$\Rightarrow \quad OA^2 = \left(\frac{2-4m}{1+m^2}\right)^2 (1+m^2)$$

$$= \frac{(2-4m)^2}{1+m^2}$$

Since OAB is an isosceles right-angled triangle $OA^2 = \frac{1}{2} AB^2$

where AB is a diameter of the given circle $OA^2 = 10$

$$\Rightarrow \quad \frac{(2-4m)^2}{1+m^2} = 10$$

$$\Rightarrow \quad 4 - 16m + 16m^2 = 10(1+m)^2$$

$$\Rightarrow \quad 3m^2 - 8m - 3 = 0$$

$$\Rightarrow \quad m = 3 \text{ or } -\frac{1}{3}$$

Hence, the required equations are $y = 3x$ or $x + 3y = 0$.

4. b. $y = mx$ be chord.

Then point of intersection are given by

$x^2(1+m^2) - x(3+4m) - 4 = 0$

$$\therefore \quad x_1 + x_2 = \frac{3+4m}{1+m^2} \text{ and } x_1x_2 = \frac{-4}{1+m^2}$$

Since (0, 0) divides chord in the ratio 1 : 4

$$\therefore \quad x_2 = -4x_1$$

$$\therefore \quad -3x_1 = \frac{3+4m}{1+m^2} \text{ and } 4x_1^2 = -\frac{-4}{1+m^2}$$

$$\therefore \quad 9 + 9m^2 = 9 + 16m^2 + 24m$$

i.e. $m = 0, -\frac{24}{7}$

Therefore, the lines are $y = 0$ and $y + 24x = 0$.

5. a.

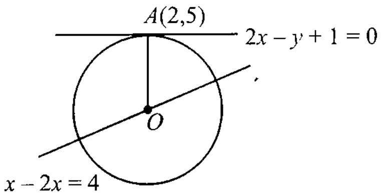


Fig. 2.98

$2x - y + 1 = 0$ is tangent

Slope of line $OA = -\frac{1}{2}$

Equation of OA, $(y-5) = -\frac{1}{2}(x-2)$

or $x + 2y = 12$

$\therefore$ Intersection with $x - 2y = 4$ will give coordinates of centre which are (8, 2)

$$\therefore \quad r = OA = \sqrt{(8-2)^2 + (2-5)^2} = 3\sqrt{5}$$

6. d.

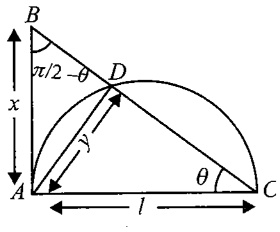


Fig. 2.99

ΔABC and ΔDBA are similar

$$\Rightarrow \quad l \cdot x = y\sqrt{l^2 + x^2}$$

$$\Rightarrow \quad l^2x^2 = y^2(l^2 + x^2)$$

$$\Rightarrow \quad l^2(x^2 - y^2) = x^2y^2$$

$$\Rightarrow \quad l = \frac{xy}{\sqrt{x^2 - y^2}} = \frac{AB.AD}{\sqrt{AB^2 - AD^2}}$$

7. a.

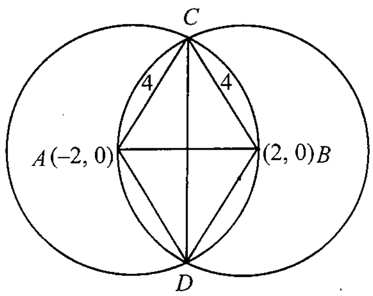


Fig. 2.100

Circles with centre (2, 0) and (– 2, 0) each with radius 4.

$\Rightarrow$ y-axis is their common chord.

ΔABC is equilateral. Hence, area of $ADBC$ is $\frac{2.\sqrt{3}}{4}(4)^2 = 8\sqrt{3}$.

8. a.

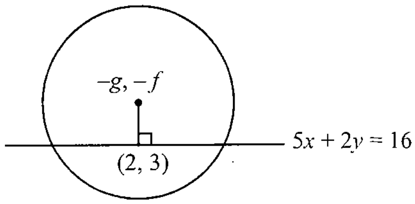


Fig. 2.101

Slope of the given line $= -\frac{5}{2}$

$\Rightarrow \quad \left(\frac{5}{2}\right)\left(\frac{3+f}{2+g}\right) = -1$

$\Rightarrow \quad 15 + 5f = 4 + 2g$

$\Rightarrow$ Locus is $2x - 5y + 11 = 0$

9. b.

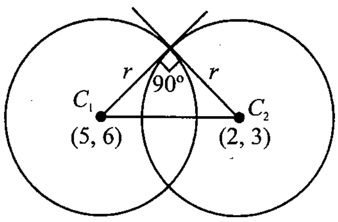


Fig. 2.102

$$2r^2 = 3^2 + 3^2 = 18 \Rightarrow r^2 = 9$$

$\Rightarrow \quad r = 3$

10. d.

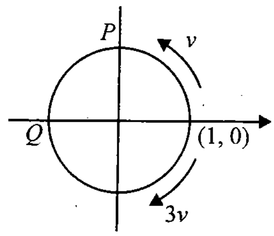


Fig. 2.103

The particle which moves clockwise is moving three times as fast as the particle moving anticlockwise.
This means the clockwise particle travels (3/4)th of the way around the circle, the anticlockwise particle will travel (1/4)th of the way around the circle and so the second particle will meet at $P(0, 1)$.

Using the same logic they will meet at $Q(-1, 0)$ when they meet the second time.

11. d.

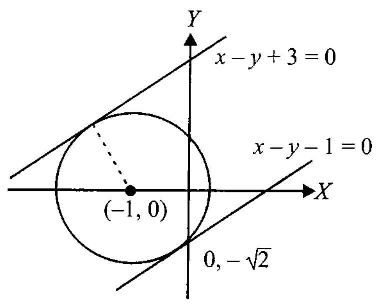


Fig. 2.104

$$x^2 + y^2 + 2x - 1 = 0$$

Centre $(-1, 0)$ and radius $= \sqrt{2}$

Line $x - y + c = 0$ must be tangent to the circle.

$\Rightarrow \quad \left|\frac{-1+c}{\sqrt{2}}\right| = \sqrt{2}$

$\Rightarrow \quad |c - 1| = 2$

$\Rightarrow \quad c - 1 = \pm 2$

$\Rightarrow \quad c = 3$ or -1

$\Rightarrow \quad c = 1$ ($\because$ for $c = 3$ there will be infinite points common lying inside circle)

12. b.

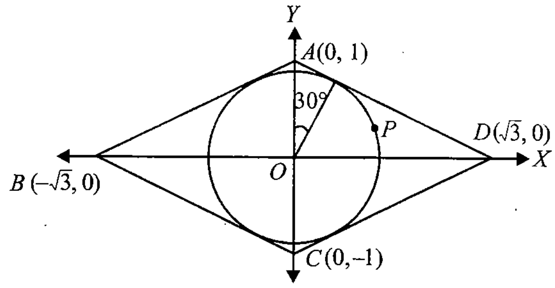


Fig. 2.105

$$OA = 1$$

$$r = OA \cos 30° = \frac{\sqrt{3}}{2}$$

Equation of circle is $x^2 + y^2 = 3/4$

$PA^2 + PB^2 + PC^2 + PD^2$

$= x_1^2 + (y_1 - 1)^2 + (x_1 + \sqrt{3})^2 + y_1^2 + x_1^2 + (y_1 + 1)^2 + (x_1 - \sqrt{3})^2 + y_1^2$

$= 4x_1^2 + 4y_1^2 + 8 = 4(x_1^2 + y_1^2) + 8$

$= 4 \times \frac{3}{4} + 8$

$= 11$

13. d.

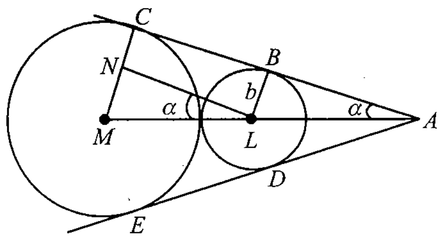


Fig. 2.106

From $\Delta\, MLN$

$$\sin\alpha = \frac{a-b}{a+b}$$

$\therefore \quad a = \sin^{-1}\left(\frac{a-b}{a+b}\right)$

$\therefore$ Angle between AB and AD

$$= 2\alpha = \sin^{-1}\left(\frac{a-b}{a+b}\right)$$

14. a.

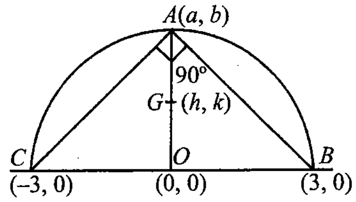


Fig. 2.107

Let $A(a, b)$ and $G(h, k)$

Now A, G, O are collinear with $AG : GO \equiv 2 : 1$

$$\Rightarrow \quad h = \frac{2.0 + a}{3}$$

$$\Rightarrow \quad a = 3h \text{ and similarly } b = 3k$$

Now (a, b) lies on the circle $x^2 + y^2 = 9$

Therefore, locus of (h, k) is $x^2 + y^2 = 1$.

15. a.

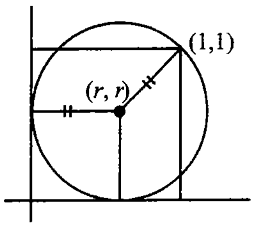


Fig. 2.108

From Fig. 2.108,

$$2(1 - r)^2 = r^2$$

$$\Rightarrow \quad \sqrt{2}(1 - r) = r$$

$$\Rightarrow \quad r(\sqrt{2} + 1) = \sqrt{2}$$

$$\Rightarrow \quad r = \frac{\sqrt{2}}{\sqrt{2} + 1} = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$$

16. b. $r = 1; L = \sqrt{3}$

Area of quadrilateral $= \sqrt{3}$

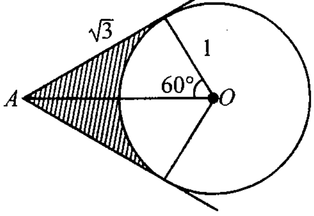


Fig. 2.109

$$\text{Sector} = \frac{1}{2}.1.\frac{2\pi}{3} = \frac{\pi}{3}$$

$$\text{Shaded region} = \sqrt{3} - \frac{\pi}{3}$$

17. d.

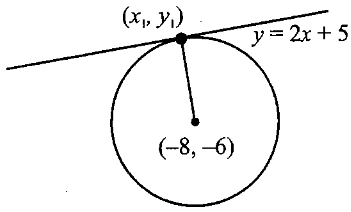


Fig. 2.110

From the figure, we have

$$y_1 = 2x_1 + 5 \text{ and } \frac{y_1 + 6}{x_1 + 8} \times 2 = -1$$

$$\Rightarrow \quad x_1 = -6 \text{ and } y_1 = -7$$

18. c.

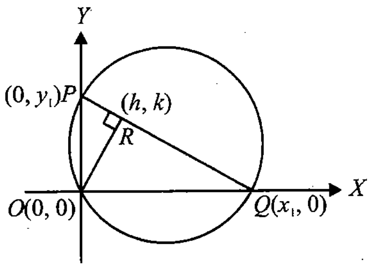


Fig. 2.111

Equation of line PQ is $y - k = -\frac{h}{k}(x - h)$

or $\quad hx + ky = h^2 + k^2$

$\Rightarrow$ Points $Q\left(\frac{h^2 + k^2}{h}, 0\right)$ and $P\left(0, \frac{h^2 + k^2}{k}\right)$

Also $\quad 2a = \sqrt{x_1^2 + y_1^2}$

$$\Rightarrow \quad x_1^2 + y_1^2 = 4a^2$$

Eliminating x_1 and y_1 we have

$$(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$$

19. c.

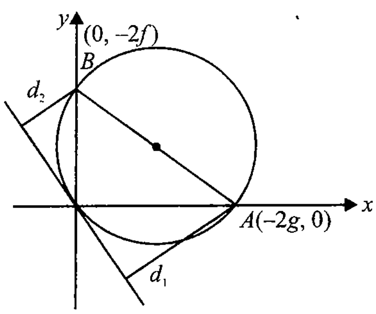


Fig. 2.112

Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$

Tangent at the origin is

$$gx + fy = 0$$

$$d_1 = \frac{2g^2}{\sqrt{g^2 + f^2}} \text{ and } d^2 = \frac{2f^2}{\sqrt{g^2 + f^2}}$$

$\Rightarrow$ $$d_1 + d_2 = 2\sqrt{g^2 + f^2}$$

= diameter of the circle

20. a.

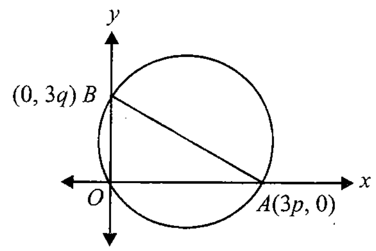


Fig. 2.113

Let the centroid of triangle OAB is (p, q).

Hence, points A and B are $(3p, 3q)$.

But diameter of triangle $AB = 6k$

Hence, $\sqrt{9p^2 + 9q^2} = 6k$

Therefore, locus of (p, q) is $x^2 + y^2 = 4k^2$.

21. a.

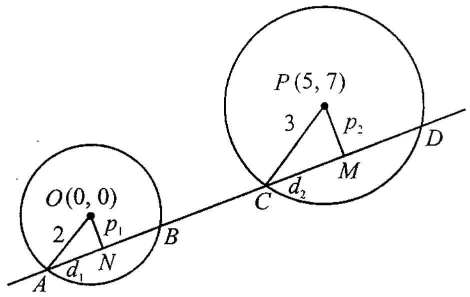


Fig. 2.114

Let equation of line be $y = x + c$ or $y - x = c$ (i)

Perpendicular from (0, 0) on line (i) is $\left|\frac{-c}{\sqrt{2}}\right| = \frac{c}{\sqrt{2}}$

In ΔAON, $\sqrt{2^2 - \left(\frac{c}{\sqrt{2}}\right)^2} = AN$

and in ΔCPM, $\sqrt{3^2 - \left(\frac{2-c}{\sqrt{2}}\right)^2} = CM$.

Given $AN = CM \Rightarrow 4 - \frac{c^2}{2} = 9 - \frac{(2-c)^2}{2}$

$\Rightarrow$ $$c = -\frac{3}{2}$$

Therefore, Equation of line $y = x - \frac{3}{2}$ or $2x - 2y - 3 = 0$

22. a.

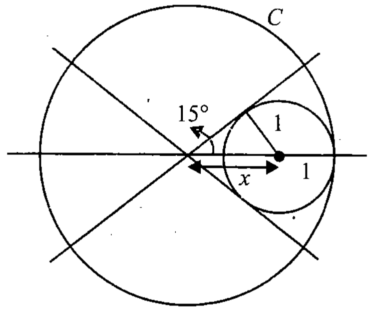


Fig. 2.115

$$\operatorname{cosec} 15° = \frac{x}{1}$$

$\Rightarrow$ $$x = \operatorname{cosec} 15°$$

$\Rightarrow$ $$R = x + 1 = 1 + \operatorname{cosec} 15°$$

$$= 1 + \frac{2\sqrt{2}}{\sqrt{3} - 1}$$

$$= 1 + \frac{4}{\sqrt{6} - \sqrt{2}}$$

$$= 1 + \sqrt{6} + \sqrt{2}$$

23. c.

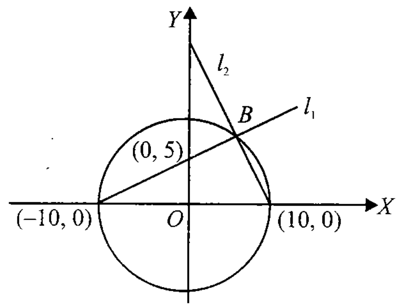


Fig. 2.116

Slope of $l_1 = \frac{1}{2}$

Slope of $l_2 = -2$

Equation of l_2, $y = -2(x - 10)$

$\Rightarrow$ $y + 2x = 20$

Hence, $t = 20$

24. a. $AB = \sqrt{a^2 + b^2}$

Hence, $D = \sqrt{b^2 + a^2}$ (i)

Now, $\frac{d}{2} = \frac{\Delta}{s} = \frac{ab}{2s}$ (where s is semi-perimeter)

$\therefore$ $$\frac{d}{2} = \frac{ab}{a + b + \sqrt{a^2 + b^2}}$$

or $$d = \frac{2ab}{a+b+\sqrt{a^2+b^2}} \quad \text{(ii)}$$

From Eqs. (i) and (ii)

$$d + D = \frac{\sqrt{a^2+b^2}\left[(a+b)+\sqrt{a^2+b^2}\right]+2ab}{a+b+\sqrt{a^2+b^2}}$$

$$= \frac{(a+b)^2+(a+b)\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} = a+b$$

25. c.

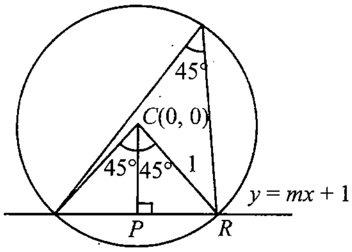


Fig. 2.117

Given circle is $x^2+y^2=1$.

$C(0, 0)$ and radius = 1 and chord is $y = mx + 1$

$$\cos 45° = \frac{CP}{CR}$$

CP = perpendicular distance from (0, 0) to chord $y = mx + 1$

$$CP = \frac{1}{\sqrt{m^2+1}} \quad (CR = \text{radius} = 1)$$

$$\Rightarrow \quad \cos 45° = \frac{1/\sqrt{m^2+1}}{1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{m^2+1}}$$

$$\Rightarrow \quad m^2+1 = 2$$

$$\Rightarrow \quad m = \pm 1$$

26. d.

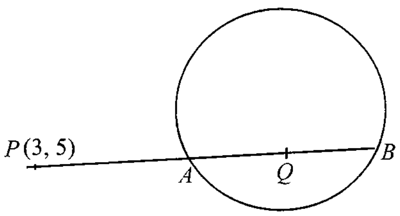


Fig. 2.118

$$2PQ = PA + PB$$

$$\Rightarrow \quad PQ - PA = PB - PQ$$

$$\Rightarrow \quad AQ = QB$$

$\Rightarrow Q$ is midpoint of AB.

Let Q has coordinates (h, k).

Then equation of chord AB is given by $T = S_1$

or $$hx + ky - 4 = h^2 + k^2 - 4$$

This variable chord passes through the point $P(3, 5)$.

$$\Rightarrow \quad 3h + 5k = h^2 + k^2$$

$$\Rightarrow \quad x^2 + y^2 - 3x - 5y = 0$$

which is required locus.

27. b.

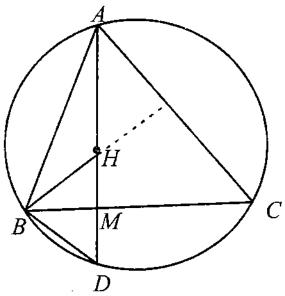


Fig. 2.119

Let orthocenter be $H(5, 8)$.

Now, $\angle HBM = \pi/2 - C$

Also, $\angle DBC = \angle DAC = \pi/2 - C$

Hence, ΔBMH and ΔBMD are congruent.

$$\Rightarrow \quad HM = MD$$

$\Rightarrow D$ is image of H in the line $x - y = 0$ which is $D(8, 5)$.

Thus, equation of circumcircle is

$$(x-2)^2 + (y-3)^2 = (8-2)^2 + (5-3)^2$$

i.e. $x^2 + y^2 - 4x - 6y - 27 = 0$

28. b. The given point is an interior point

$$\Rightarrow \quad \left(-5+\frac{r}{\sqrt{2}}\right)^2 + \left(-3+\frac{r}{\sqrt{2}}\right)^2 - 16 < 0$$

$$\Rightarrow \quad r^2 - 8\sqrt{2}\,r + 18 < 0$$

$$\Rightarrow \quad 4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14} \quad \text{(i)}$$

The point is on the major segment

$\Rightarrow$ The centre and the point are on the same side of the line $x + y = 2$

$$\Rightarrow \quad -5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2}} - 2 < 0$$

$$\Rightarrow \quad r < 5\sqrt{2} \quad \text{(ii)}$$

From Eqs. (i) and (ii): $4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$

29. a. Let $x = a, x = b, y = c, y = d$ be the sides of the square.

The length of each diagonal of the square is equal to the diameter of the circle, i.e., $2\sqrt{98}$.

Let l be the length of each side of the square. Then, $2l^2 = (\text{Diagonal})^2$

$\Rightarrow \quad l = 14$

Therefore, each side of the square is at a distance 7 from the centre (1, −2) of the given circle. This implies that $a = -6, b = 8, c = -9, d = 5$

Hence, the vertices of the square are (−6, −9), (−6, 5), (8, −9), (8, 5).

30. b.

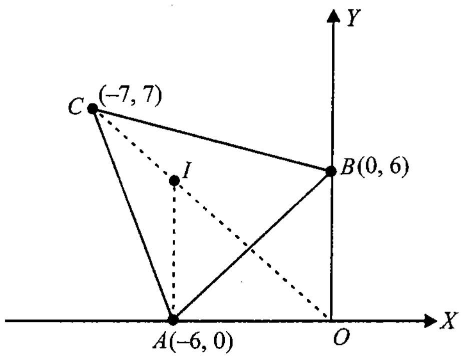


Fig. 2.120

The triangle is evidently isosceles and therefore the median through C is the angle bisector of $\angle C$.

The equation of the angle bisector is $y = -x$ and in centre $I = (-a, a)$ where a is positive.

Equation of AC is $y - 0 = -7(x + 6)$ or $7x + y + 42 = 0$ and equation of AB is $x - y + 6 = 0$.

The length of the perpendicular from I to AB and AC are equal.

$\therefore \left|\dfrac{-7a + a + 42}{\sqrt{50}}\right| = \left|\dfrac{-a - a + 6}{\sqrt{2}}\right|$

$\therefore \quad a = \dfrac{9}{2} \; (\because a > 0)$

$\therefore$ Centre is $\left(-\dfrac{9}{2}, \dfrac{9}{2}\right)$ and radius $= \dfrac{3}{\sqrt{2}}$

$\therefore$ The equation of the circle is $\left(x + \dfrac{9}{2}\right)^2 + \left(y - \dfrac{9}{2}\right)^2 = \dfrac{9}{2}$

$\therefore \quad x^2 + y^2 + 9x - 9y + 36 = 0$

31. b.

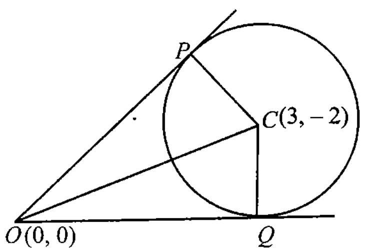


Fig. 2.121

Clearly, $OPCQ$ is cyclic quadrilateral, then circumcircle of ΔOPQ passes through the point C.

For this circle, OC is diameter, then centre is midpoint of OC which is $\left(\dfrac{3}{2}, -1\right)$.

32. d.

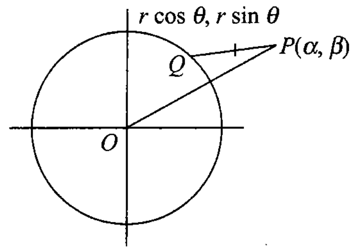


Fig. 2.122

$$2h = \alpha + r\cos\theta$$

$$2k = \beta + r\sin\theta$$

$\Rightarrow \quad (2h - \alpha)^2 + (2k - \beta)^2 = r^2$

$$\left(h - \frac{\alpha}{2}\right)^2 + \left(k - \frac{\beta}{2}\right)^2 = \left(\frac{r}{2}\right)^2$$

locus is $\left(x - \dfrac{\alpha}{2}\right)^2 + \left(y - \dfrac{\beta}{2}\right)^2 = \left(\dfrac{r}{2}\right)^2$

which is a circle with centre as midpoint of OP and radius $r/2$

33. a.

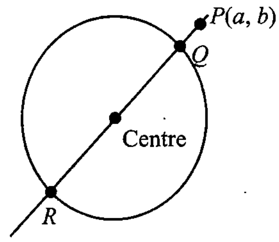


Fig. 2.123

The given circle is $(x + 1)^2 = (y + 2)^2 = 9$, which has radius = 3.

The points on the circle which are nearest and farthest to the point $P(a, b)$ are Q and R, respectively.

Thus, the circle centred at Q having radius PQ will be the smallest circle while the circle centred at R having radius PR will be the largest required circle.

Hence, difference between their radii $= PR - PQ = QR = 6$

34. a.

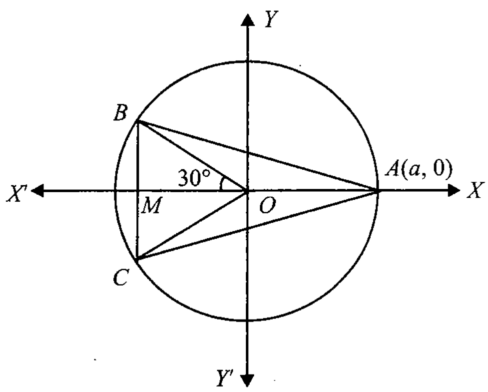


Fig. 2.124

Since $\angle B = \angle C = 75°$

$\Rightarrow \angle BAC = 30°$

$\Rightarrow \angle BOC = 60°$

$\Rightarrow$ B has coordinates $(-a\cos 30°, a\sin 30°)$

or $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$ and those of C are $\left(-\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

35. c.

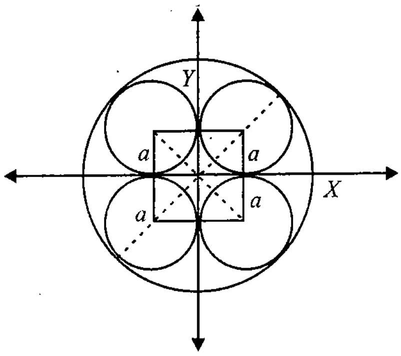


Fig. 2.125

The four circles are as shown in the Fig. 2.125.

The smallest circle touching all of them has the radius $= \sqrt{2}\,a - a$ and the greatest circle touching all of them has the radius $= \sqrt{2}\,a + a$.

36. a. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC, and let $P(h, k)$ be any point on the locus.

Then, $PA^2 + PB^2 + PC^2 = c$ (constant)

$$\Rightarrow \sum_{i=1}^{3} (h - x_i)^2 + (k - y_i)^2 = c$$

$$\Rightarrow h^2 + k^2 - \frac{2h}{3}(x_1 + x_2 + x_3) - \frac{2k}{3}(y_1 + y_2 + y_3) + \sum_{i=1}^{3} (x_i^2 + y_i^2) - c = 0$$

So, locus of (h, k) is

$$x^2 + y^2 - \frac{2x}{3}(x_1 + x_2 + x_3) - \frac{2y}{3}(y_1 + y_2 + y_3) + \lambda = 0,$$

where $\lambda = \sum_{i=1}^{3} (x_i^2 + y_i^2) - c = 0$ constant

37. c. Equation of any circles passing through (1, 0) and (5, 0) is

$$y^2 + (x - 1)(x - 5) + \lambda y = 0$$

i.e. $x^2 + y^2 + \lambda y - 6x + 5 = 0$

If $\angle ACB$ is maximum, then this circle must touch the y-axis at $(0, h)$. Putting $x = 0$ in the equation of circle, we get $y^2 + \lambda y + 5 = 0$. It should have $y = h$ as it is a repeated root.

$\Rightarrow h^2 = 5$ and $\lambda = -2h$

$\Rightarrow |h| = \sqrt{5}$

38. b.

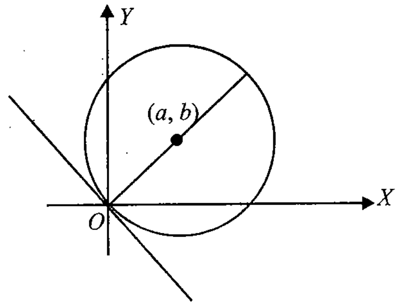


Fig. 2.126

Obviously, the slope of the tangent will be $-\left(\frac{1}{b/a}\right)$, i.e., $-\frac{a}{b}$.

Hence, the equation of the tangent is $y = -\frac{a}{b}x$, i.e., $by + ax = 0$.

39. c.

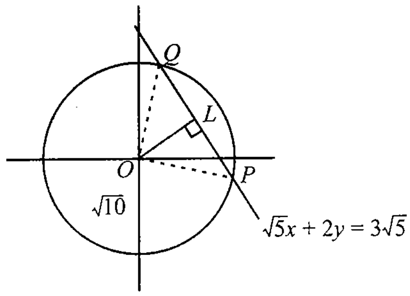


Fig. 2.127

Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is

$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle $= \sqrt{10} = OQ = OP$

$$PQ = 2QL = 2\sqrt{OQ^2 - OL^2}$$

$$= 2\sqrt{10 - 5} = 2\sqrt{5}$$

Thus, area of $\Delta OPQ = \frac{1}{2} \times PQ \times OL$

$$= \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5$$

40. c. If $(a, 0)$ is the centre C and P is $(2, -2)$, then $\angle COP = 45°$

Since the equation of OP is $x + y = 0$

$\therefore OP = 2\sqrt{2} = CP$. Hence, $OC = 4$

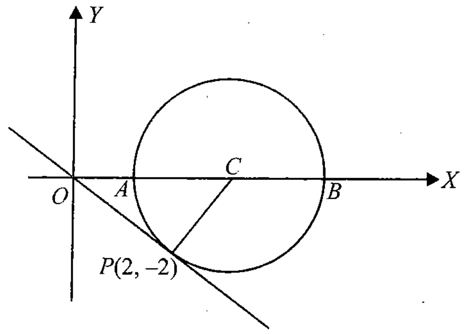


Fig. 2.128

The point on the circle with the greatest x coordinates is B.

$$\alpha = OB = OC + CB = 4 + 2\sqrt{2}$$

41. c.

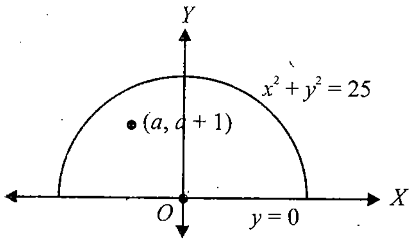


Fig. 2.129

$y = \sqrt{25 - x^2}$, $y = 0$ bound the semicircle above the x-axis.

$$\therefore \quad a + 1 > 0 \quad \text{(i)}$$

and $\quad a^2 + (a+1)^2 - 25 < 0 \Rightarrow 2a^2 + 2a - 24 < 0$

$$\Rightarrow \quad a^2 + a - 12 < 0$$

$$\Rightarrow \quad -4 < a < 3 \quad \text{(ii)}$$

From Eqs.(i) and (ii)

$$-1 < a < 3$$

42. c. For $x^2 + y^2 = 9$, the centre = (0, 0) and the radius = 3

For $x^2 + y^2 - 8x - 6y + n^2 = 0$,

the centre = (4, 3) and the radius = $\sqrt{4^2 + 3^2 - n^2}$

$\therefore\ 4^2 + 3^2 - n^2 > 0$ or $n^2 < 5^2$ or $-5 < n < 5$

Circles should cut to have exactly two common tangents.

So, $\quad r_1 + r_2 > d$ (distance between centres)

$$\therefore \quad 3 + \sqrt{25 - n^2} > \sqrt{4^2 + 3^2}$$

or $\quad \sqrt{25 - n^2} > 2$

or $\quad 25 - n^2 > 4$

or $\quad 25 - n^2 > 4$

$$\therefore \quad n^2 < 21 \text{ or } -\sqrt{21} < n < \sqrt{21}$$

Therefore, common values of n should satisfy $-\sqrt{21} < n < \sqrt{21}$.

But $n \in Z$. So, $n = -4, -3, \ldots, 4$

43. b.

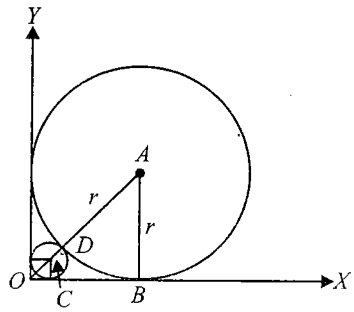


Fig. 2.130

From the diagram $\sin 45° = \dfrac{AB}{OA}$

$$= \frac{r}{OC + CD + DA}$$

$$= \frac{r}{\sqrt{2} + 1 + r}$$

$$\Rightarrow \quad \sqrt{2} + 1 + r = \sqrt{2}\, r$$

$$\Rightarrow \quad r = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = 3 + 2\sqrt{2}$$

44. a.

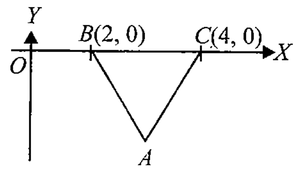


Fig. 2.131

Clearly, $\quad A \equiv (3, -\sqrt{3})$.

Centroid of triangle ABC is $\left(3, -\frac{1}{\sqrt{3}}\right)$, thus equation of incircle is

$$(x-3)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$\Rightarrow \quad x^2 + y^2 - 6x + \frac{2y}{\sqrt{3}} + 9 = 0$$

45. d. $x^2 + 2ax + c = (x-2)^2$

$$\Rightarrow \quad -2a = 4, c = 4$$

$$\Rightarrow \quad a = -2, c = 4,$$

$$y^2 + 2by + c = (y - 2)(y - 3)$$

$\Rightarrow$ $-2b = 5, c = 6$

$\Rightarrow$ $b = -\frac{5}{2}, c = 6$ clearly the data are not consistent.

46. d.

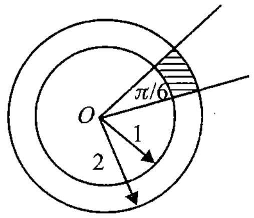


Fig. 2.132

The angle θ between the lines represented by $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ is given by

$$\theta = \tan^{-1}\frac{\sqrt{h^2 - ab}}{|a + b|}$$

$$= \tan^{-1}\frac{2\sqrt{2^2 - 3}}{\sqrt{3} + \sqrt{3}} = \frac{1}{\sqrt{3}}$$

Gives $\theta = \frac{\pi}{6}$

Hence, the shaded area $= \frac{\pi/6}{2\pi} \times \pi(2^2 - 1^2) = \frac{\pi}{4}$

47. c. Centre of circle is (1, 0) and radius is 1. Line will touch the circle if $|\cos\theta - 2| = 1 \Rightarrow \cos\theta = 1, 3$.

Thus, $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$

48. b.

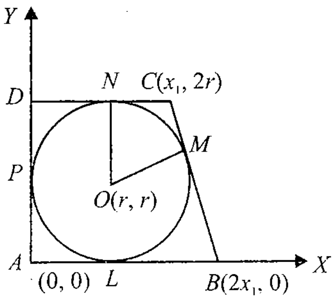


Fig. 2.133

Here $\frac{1}{2} \times 3x_1 \times 2r = 18$

$\Rightarrow$ $x_1 \times r = 6$ (i)

Equation of BC is: $y = -\frac{2r}{x_1}(x - 2x_1)$

BC is tangent to the circle $(x - r)^2 + (y - r)^2 = r^2$

$\therefore$ Perpendicular distance of BC from centre = radius

$\Rightarrow$ $\dfrac{\left|r + \frac{2r}{x_1}(r - 2x_1)\right|}{\sqrt{1 + \frac{4r^2}{x_1^2}}} = r$

$\Rightarrow$ $\frac{2r^2}{x_1} - 3r = r\sqrt{1 + 4\frac{r^2}{x_1^2}}$

$\Rightarrow$ $(2r - 3x_1)^2 = x_1^2 + 4r^2$

$\Rightarrow$ $r \cdot x_1 = \frac{2}{3}x_1^2 \Rightarrow 3r = 2x_1$ (ii)

From Eqs. (i) and (ii), $r = 2$ units

49. a.

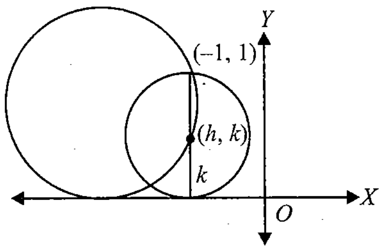


Fig. 2.134

From the figure, $k \geq \frac{1}{2}$

50. a. Curve passing through point of intersection of S and S' is

$\Rightarrow$ $S + \lambda S' = 0$

$\Rightarrow x^2(\sin^2\theta + \lambda\cos^2\theta) + y^2(\cos^2\theta + \lambda\sin^2\theta) + 2xy(h + \lambda h') + x(32 + 16\lambda) + y(16 + 32\lambda) + 19(1 + \lambda) = 0$

for this equation to be a circle

$$\sin^2\theta + \lambda\cos^2\theta = \cos^2\theta + \lambda\sin^2\theta \Rightarrow \lambda = 1$$

and $h + \lambda h' = 0 \Rightarrow h + h' = 0$

51. a.

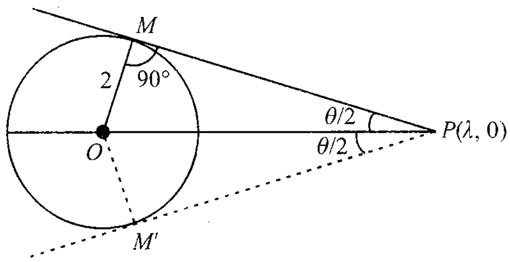


Fig. 1.235

We have $\frac{\pi}{2} < \theta < \frac{2\pi}{3}$, i.e., $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{3}$

$\Rightarrow$ $\frac{1}{\sqrt{2}} < \sin\frac{\theta}{2} < \frac{\sqrt{3}}{2}$

But, $\sin\frac{\theta}{2} = \frac{2}{\lambda} \Rightarrow \frac{1}{\sqrt{2}} < \frac{2}{\lambda} < \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\frac{4}{\sqrt{3}} < \lambda < 2\sqrt{2}$

52. a.

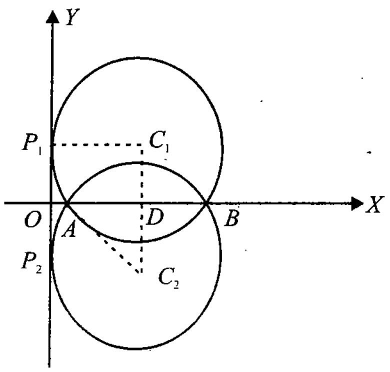


Fig. 2.136

Let $A \equiv (1, 0)$, $B \equiv (3, 0)$ and C_1, C_2 be the centre of circles passing through A, B and touching the y-axis at P_1 and P_2. If r be the radius of circle (here radius of both circles will be same), $C_1A = C_2A = r = OD = 2$ and $C_1 \equiv (2, h)$

where $h^2 = AC_1^2 - AD^2 = 4 - 1 = 3$

$\Rightarrow$ $C_1 \equiv (2, \sqrt{3})$, $C_2 \equiv (2, -\sqrt{3})$

If $\angle C_1AC_2 = \theta$

$\Rightarrow$ $\cos\theta = \frac{AC_1^2 + AC_2^2 - C_1C_2^2}{2AC_1 \cdot AC_2} = \frac{1}{2}$

53. c. Chord with midpoint (h, k) is

$$hx + ky = h^2 + k^2 \quad \text{(i)}$$

Chord of contact of (x_1, y_1) is

$$xx_1 + yy_1 = 2 \quad \text{(ii)}$$

Comparing, we get

$$x_1 = \frac{2h}{h^2 + k^2} \text{ and } y_1 = \frac{2k}{h^2 + k^2}$$

(x_1, y_1) lies on $3x + 4y = 10 \Rightarrow 6h + 8k = 10(h^2 + k^2)$

$\therefore$ Locus of (h, k) is $x^2 + y^2 - \frac{3}{5}x - \frac{4}{5}y = 0$

which is circle with centre $P\left(\frac{3}{10}, \frac{4}{10}\right)$

$\therefore$ $OP = \frac{1}{2}$

54. a. a, b, c are in A.P., so $ax + by + c = 0$ represents a family of lines passing through the point $(1, -2)$. So, the family of circles (concentric) will be given by $x^2 + y^2 - 2x + 4y + c = 0$. It intersects given circle orthogonally.

$\Rightarrow 2(-1 \times -2) + (2 \times -2) = -1 + c \Rightarrow c = -3$

55. d.

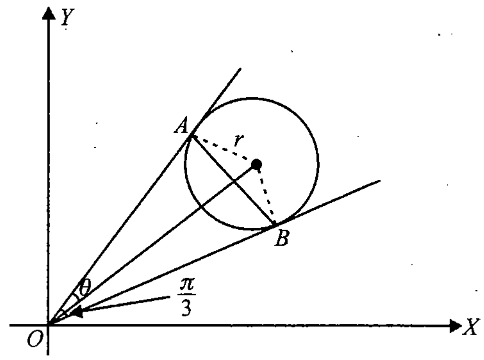


Fig. 2.137

Here $\tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3}$

$\Rightarrow$ $\theta = \pi/6$

Area of $\Delta OAB = \frac{1}{2}(r\cot\theta)^2(\sin 2\theta)$

$= \frac{1}{2}(r\sqrt{3})^2\frac{\sqrt{3}}{2}$

56. a. Let P be $(1 + \sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$ and C is $(1, 0)$. Circumcentre of triangle ABC is midpoint of PC.

$\Rightarrow$ $2h = 1 + \sqrt{2}\cos\theta + 1$

and $2k = \sqrt{2}\sin\theta$

$\Rightarrow$ $[2(h-1)]^2 + (2k)^2 = 2$

$\Rightarrow$ $2(h-1)^2 + k^2 - 1 = 0$

$\Rightarrow$ $2x^2 + 2y^2 - 4x + 1 = 0$

57. a.

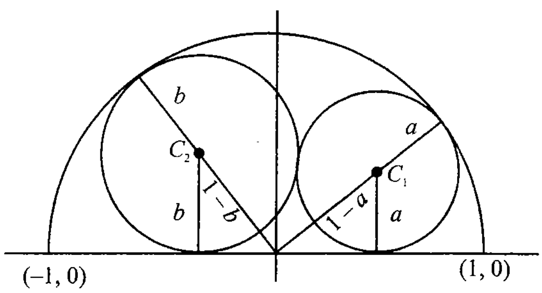


Fig. 2.138

Let centre of the circles be C_1 and C_2.

$\Rightarrow C_1$ is $(\sqrt{1-2a}, a)$ and C_2 is $(\sqrt{1-2b}, b)$

Now $C_1C_2 = a + b = a + \frac{1}{2}$

$$\Rightarrow \quad 1 - 2a + \left(a - \frac{1}{2}\right)^2 = \left(a + \frac{1}{2}\right)^2$$

$$\Rightarrow \quad a = \frac{1}{4}$$

58. a. Since ΔPOQ and ΔAOQ are congruent

Hence, $\angle POQ = \angle QOA = \theta$

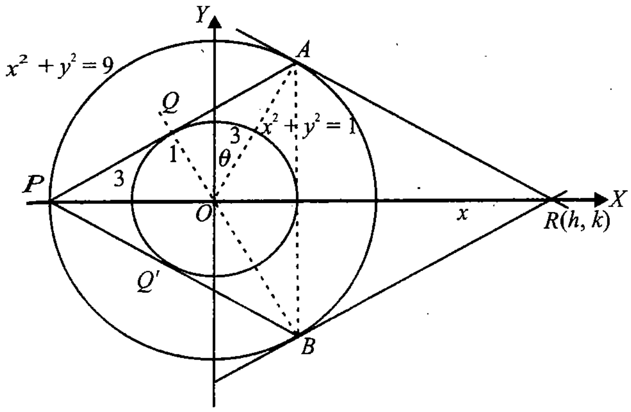


Fig. 2.139

$$\cos\theta = \frac{1}{3}, \text{ since } \angle POR = 180°$$

$$\Rightarrow \quad \angle AOR = \pi - 2\theta$$

Now in triangle AOR, $\angle AOR = \pi - 2\theta$ and $AO = 3$ unit

$$\Rightarrow \quad \cos(\pi - 2\theta) = \frac{OA}{OR} = \frac{3}{\sqrt{h^2 + k^2}}$$

$$\Rightarrow \quad \sqrt{h^2 + k^2} = \frac{27}{7}$$

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{27}{7}\right)^2$$

59. b.

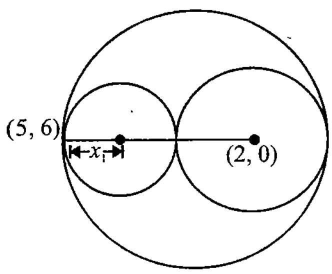


Fig. 2.140

Given circle is $(x - 2)^2 + y^2 = 4$

Centre is (2, 0) and radius = 2

Therefore, distance between (2, 0) and (5, 6) is $\sqrt{9 + 36} = 3\sqrt{5}$

$$\Rightarrow \quad r_1 = \frac{3\sqrt{5} - 2}{2}$$

and
$$r_2 = \frac{3\sqrt{5} + 2}{2}$$

$$= r_1 r_2 = \frac{41}{4}$$

60. c.
$$x^2 + y^2 - 12x + 35 = 0 \quad \text{(i)}$$
$$x^2 + y^2 + 4x + 3 = 0 \quad \text{(ii)}$$

Equation of radical axis of circles (i) and (ii) is

$$-16x + 32 = 0 \Rightarrow x = 2$$

It intersects the line joining the centres, i.e. $y = 0$ at the point (2, 0)

$\therefore$ Required radius $= \sqrt{4 - 24 + 35} = \sqrt{15}$ (length of tangent from (2, 0)

61. d.

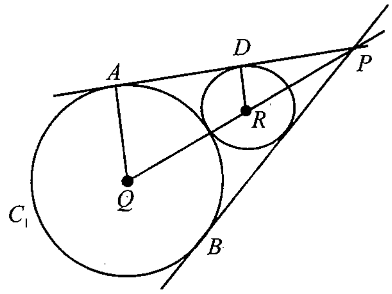


Fig. 2.141

$$AQ = 3 + 2\sqrt{2}$$
$$PQ = 3\sqrt{2} + 4$$

Let r be required radius

$$\therefore \quad 3\sqrt{2} + 4 = 3 + 2\sqrt{2} + r + r\sqrt{2}$$
$$(\because \angle RPD = \tfrac{\pi}{4})$$
$$\sqrt{2} + 1 = r(1 + \sqrt{2}) \Rightarrow r = 1$$

62. a.

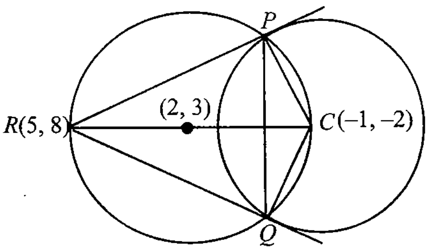


Fig. 2.142

Let C be the centre of the given circle.

Then, circumcircle of the ΔRPQ passes through C.

$\therefore$ (2, 3) is the midpoint of RC.

$\therefore$ Coordinates of C are (−1, −2).

$\therefore$ Equation of the circle is $x^2 + y^2 + 2x + 4y - 20 = 0$.

63. c.

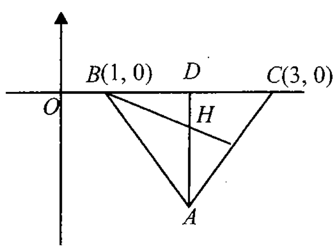


Fig. 2.143

Radical centre of the circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore$ $$D = (2, 0)$$

$$DH = -BD \tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

$\therefore$ Coordinates of H are $\left(2, -\frac{1}{\sqrt{3}}\right)$.

64. b. Let $(\alpha, 3 - \alpha)$ be any point on $x + y = 3$.

$\therefore$ Equation of chord of contact is $\alpha x + (3 - \alpha)y = 9$

i.e., $\alpha(x - y) + 3y - 9 = 0$

$\therefore$ The chord passes though the point (3, 3) for all values of α.

65. d. Given circle $(x - 1)^2 + (y + 2)^2 = 16$

Its director circle is $(x - 1)^2 + (y + 2)^2 = 32$

$\Rightarrow$ $$OS = 4\sqrt{2}$$

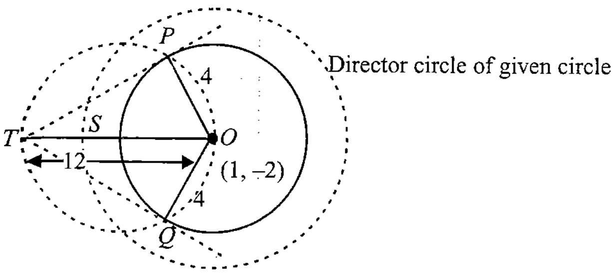


Fig. 2.144

Therefore, required distance, $TS = OT - SO = 12 - 4\sqrt{2}$

66. c. Equation of the two circles be $(x - r)^2 + (y - r)^2 = r^2$ i.e. $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where $r = r_1$ and r_2. Condition of orthogonality gives

$$2r_1r_2 + 2r_1r_2 = r_1^2 + r_2^2 \Rightarrow 4r_1r_2 = r_1^2 + r_2^2$$

Circle passes through (a, b)

$\Rightarrow$ $a^2 + b^2 - 2ra - 2rb + r^2 = 0$

i.e. $r^2 - 2r(a + b) + a^2 + b^2 = 0$

$$r_1 + r_2 = 2(a + b) \text{ and } r_1r_2 = a^2 + b^2$$

$\therefore$ $$4(a^2 + b^2) = 4(a + b)^2 - 2(a^2 + b^2)$$

i.e. $$a^2 - 4ab + b^2 = 0$$

67. c. $$C_1 = (-1, -4)\ ;\ C_2 = (2, 5)\ ;$$

$$r_1 = \sqrt{1 + 16 + 23} = 2\sqrt{10};$$

$$r_2 = \sqrt{4 + 25 + 19} = \sqrt{10};$$

$$C_1C_2 = \sqrt{9 + 18} = 3\sqrt{10}$$

$\Rightarrow$ $$C_1C_2 = r_1 + r_2.$$

Hence, circles touch externally.

68. a.

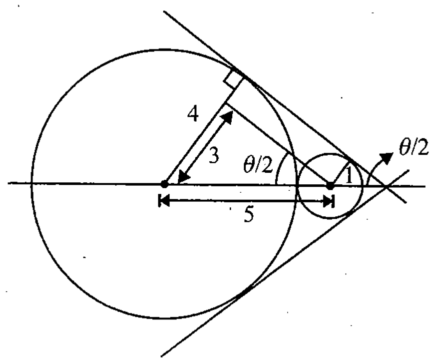


Fig. 2.145

$$\sin \frac{\theta}{2} = \frac{3}{5}$$

and $$\cos \frac{\theta}{2} = \frac{4}{5}$$

$\therefore$ $$\sin \theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

69. c.

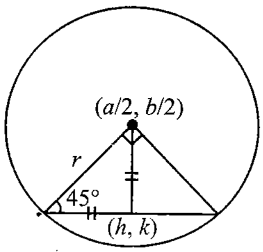


Fig. 2.146

$$r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$\sin 45° = \frac{\sqrt{\left(h - \frac{a}{2}\right)^2 + \left(h - \frac{b}{2}\right)^2}}{\frac{\sqrt{a^2 + b^2}}{2}}$$

$\Rightarrow$ $$\frac{1}{2} = 4\left[\frac{\frac{(2h - a)^2}{4} + \frac{(2k - b)^2}{4}}{a^2 + b^2}\right]$$

Simplify to get locus $x^2 + y^2 - ax - by - \frac{a^2 + b^2}{8} = 0$.

70. a.

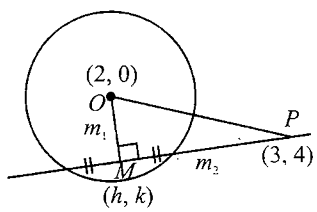


Fig. 2.147

$\Rightarrow$ $$m_1m_2 = -1$$

$$\Rightarrow \quad \left(\frac{4-k}{3-h}\right)\left(\frac{k-0}{h-2}\right) = -1$$

Hence, locus is $x^2 + y^2 - 5x - 4y + 6 = 0$

71. c. Locus of the centre of the circle cutting $S_1 = 0$ and $S_2 = 0$ orthogonally is the radical axis between $S_1 = 0$ and $S_2 = 0$, i.e., $S_1 - S_2 = 0$ or $9x - 10y + 11 = 0$.

72. b. For given $r_1 = \sqrt{10}, C_1(1, 0)$

and $\qquad r_2 = \sqrt{5}, C_2(0, 2)$

$$d = C_1C_2 = \sqrt{5}$$

If θ is the angle between the circle, then

$$\cos\theta = \frac{|d^2 - r_1^2 - r_2^2|}{2r_1r_2}$$

$$= \frac{|5 - 10 - 5|}{2\sqrt{10}\sqrt{5}}$$

$$= \frac{1}{\sqrt{2}}$$

Hence, $\qquad \theta = \frac{\pi}{4}$

73. a.

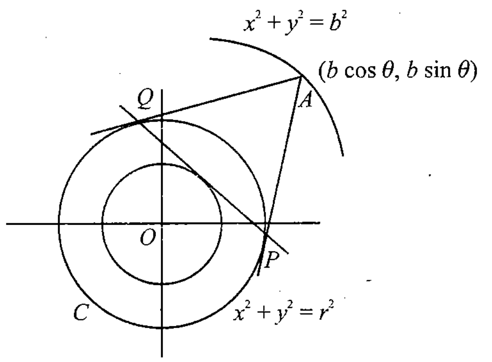


Fig. 2.148

Chord of contact of the point A w.r.t. $x^2 + y^2 = r^2$ is

$$xb\cos\theta + yb\sin\theta = r^2 \qquad \text{(i)}$$

This must be a tangent to the circle $x^2 + y^2 = a^2$

$$\Rightarrow \left[\frac{r^2}{\sqrt{b^2\cos^2\theta + b^2\sin^2\theta}}\right] = a \Rightarrow r^2 = ab$$

Hence, equation of circle is $x^2 + y^2 = ab$.

74. a. Locus of point of intersection of tangents chord of contact of (x_1, y_1) w.r.t.

$$x^2 + y^2 = 1 \text{ is } xx_1 + yy_1 = 1 \ (AB) \qquad \text{(i)}$$

AB is also common chord between two circles

$\therefore -1 + (\lambda + 6)x - (8 - 2\lambda)y + 3 = 0$

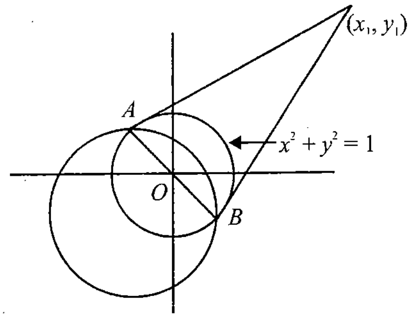


Fig. 2.149

$$\Rightarrow (\lambda + 6)x - (8 - 2\lambda)y + 2 = 0 \qquad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$\frac{x_1}{\lambda + 6} = \frac{y_1}{2\lambda - 8} = \frac{-1}{2}$$

Eliminate $\lambda \Rightarrow 2x - y + 10 = 0$ which is required locus.

75. c.

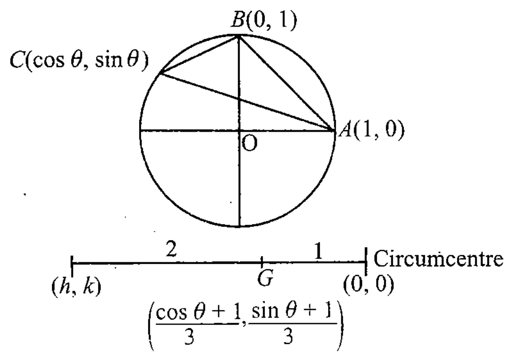


Fig. 2.150

Let $C(\cos\theta, \sin\theta)$; $H(h, k)$ is the orthocentre of the ΔABC.

Since circumcentre of the triangle is $(0, 0)$, for orthocentre $h = 1 + \cos\theta$ and $k = 1 + \sin\theta$.

Eliminating θ, $(x - 1)^2 + (y - 1)^2 = 1$

$$\therefore \quad x^2 + y^2 - 2x - 2y + 1 = 0$$

76. d.

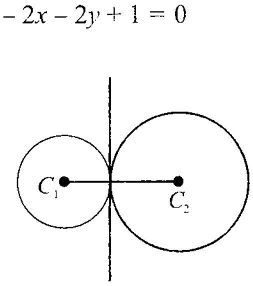


Fig. 2.151

$$C_1C_2 = r_1 + r_2$$

$$C_1 = (0, 0);\ C_2 = (3\sqrt{3}, 3)$$

and $\qquad r_1 = 2, r_2 = 4$

$\Longrightarrow$ Circles touch each other externally.

Equation of common tangent is $\sqrt{3}x + y - 4 = 0$ (i)

Comparing it with $x\cos\theta + y\sin\theta = 2$, we get

$$\theta = \frac{\pi}{6}$$

77. a.

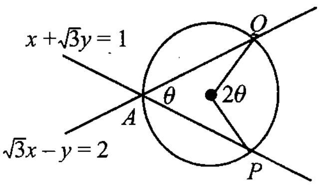


Fig. 2.152

Let the point of intersection of two lines is A.

$\therefore$ The angle subtended by PQ on centre C

$= 2 \times$ the angle subtended by PQ on point A.

For $x + \sqrt{3}y = 1$, $m_1 = \frac{-1}{\sqrt{3}}$ and for $\sqrt{3}x - y = 2$, $m_2 = \sqrt{3}$

$\because \qquad m_1 \times m_2 = \frac{-1}{\sqrt{3}} \times \sqrt{3} = -1,$

$\therefore \qquad \angle A = 90°$

$\therefore$ The angle subtended by arc PQ at its centre

$$= 2 \times 90° = 180°$$

78. b. Clearly (0, 0) lies on director circle of the given circle.

Now, equation of director circle is

$$(x+g)^2 + (y+f)^2 = 2(g^2 + f^2 - c)$$

If (0, 0) lies on it, then

$$g^2 + f^2 = 2(g^2 + f^2 - c)$$

$$\Rightarrow \quad g^2 + f^2 = 2c$$

79. c. Let the second circle be $x^2 + y^2 + 2gx + 2fy = 0$.

But $y = x$ touches the circle.

Hence, $x^2 + x^2 + 2gx + 2fx = 0$ has equal roots, i.e., $f + g = 0$

Therefore, the equation of the common chord is $2(g-3)x + 2(-g-4)y + 7 = 0$

or $(-6x - 8y + 7) + g(2x - 2y) = 0$, which passes through the point of intersection of

$-6x - 8y + 7 = 0$ and $2x - 2y = 0$ which is (1/2, 1/2).

80. b. Let the coordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. Then, the chords of contact of tangents drawn from A, B and C are

$xx_1 + yy_1 = a^2$, $xx_2 + yy_2 = a^2$ and $xx_3 + yy_3 = a^2$, respectively. These three lines will be concurrent, if

$$\begin{vmatrix} x_1 & y_1 & -a^2 \\ x_2 & y_2 & -a^2 \\ x_3 & y_3 & -a^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad -a^2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ Points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

81. c. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points and $x^2 + y^2 = a^2$ be the given circle. Then, the chord of contact of tangents drawn from P to the given circle is $xx_1 + yy_1 = a^2$.

It will pass through $Q(x_2, y_2)$, if

$x_1x_2 + y_1y_2 = a^2$ (i)

Now, $\quad l_1 = \sqrt{x_1^2 + y_1^2 - a^2},$

$\qquad l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$

and $\quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\qquad = \sqrt{(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2(x_1x_2 + y_1y_2)}$

$\therefore \quad PQ = \sqrt{[(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2a^2]}$

[Using Eq. (i)]

$\Rightarrow \quad PQ = \sqrt{(x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2)}$

$\Rightarrow \quad PQ = \sqrt{l_1^2 + l_2^2}$

82. a. Centres are (10, 0) and (– 15, 0)

and radii are $\qquad r_1 = 6; r_2 = 9$

Also $\qquad d = 25$

$$r_1 + r_2 < d$$

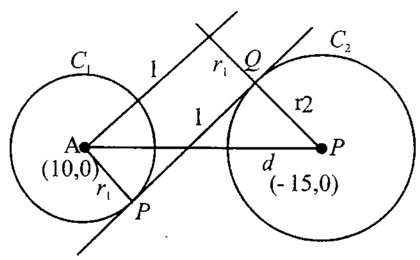


Fig. 2.153

$\Rightarrow$ circles are neither intersecting nor touching

$$PQ = \sqrt{d^2 - (r_1 + r_2)^2}$$

$= \sqrt{625 - 225}$

$= 20$

83. b. If (α, β) is the centre

then $(\alpha - 1)^2 + (\beta - 3)^2 = (\alpha - 3)^2 + (\beta - 1)^2$ (i)

and $\frac{\beta - 3}{\alpha - 1} \cdot \frac{\beta - 1}{\alpha - 3} = -1$

or $(\alpha - 1)(\alpha - 3) + (\beta - 1)(\beta - 3) = 0$ (ii)

(i) $\Rightarrow 4\alpha - 4\beta = 0 \quad \therefore \alpha = \beta$

(ii) $\Rightarrow 2(\alpha - 1)(\alpha - 3) = 0 \quad \therefore \alpha = 1, 3$

$\therefore \quad (\alpha, \beta) = (1, 1), (3, 3).$

84. b. The centre of $x^2 + y^2 - 4x - 4y = 0$ is (2, 2).

It is $ax + by = 2$

$\therefore \quad 2a + 2b = 2$ or $a + b = 1$

$ax + by = 2$ touches $x^2 + y^2 = 1$.

So, $1 = \left|\frac{-2}{\sqrt{a^2 + b^2}}\right|$

$\therefore \quad a^2 + b^2 = 4$ or $a^2 + (1 - a)^2 = 4$

or $2a^2 - 2a - 3 = 0$

$\therefore \quad a = \frac{2 \pm \sqrt{4 + 24}}{4} = \frac{1 \pm \sqrt{7}}{2}$

$\therefore \quad b = 1 - a = 1 - \frac{1 \pm \sqrt{7}}{2} = \frac{1 \mp \sqrt{7}}{2}$

85. c. Equation of any circles through (0, 1) and (0, 6) is

$x^2 + (y - 1)(y - 6) + \lambda x = 0$

$\Rightarrow \quad x^2 + y^2 + \lambda x - 7y + 6 = 0$

If it touches x-axis, then $x^2 + \lambda x + 6 = 0$ should have equal roots

$\Rightarrow \quad \lambda^2 = 24 \Rightarrow \lambda = \pm\sqrt{24}$

Radius of these circles $= \sqrt{6 + \frac{49}{4} - 6} = \frac{7}{2}$ units.

That means we can draw two circles but radius of both circles is $\frac{7}{2}$.

86. b. Let the tangent be of form $\frac{x}{x_1} + \frac{y}{y_1} = 1$ and area of Δ formed by it with coordinate axes is

$\frac{1}{2}x_1y_1 = a^2$ (i)

Again, $y_1x + x_1y - x_1y_1 = 0$

Applying conditions of tangency

$\frac{|-x_1y_1|}{\sqrt{x_1^2 + y_1^2}} = a$ or $(x_1^2 + y_1^2) = \frac{x_1^2y_1^2}{a^2}$ (ii)

From Eqs. (i) and (ii), we get x_1, y_1, which gives equation of tangent as $x \pm y = \pm a\sqrt{2}$.

87. c. The equation of the line $y = x$ in distance form is $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r$, where $\theta = \frac{\pi}{4}$.

For point P, $r = 6\sqrt{2}$. Therefore, coordinates of P are given by $\frac{x}{\cos\frac{\pi}{4}} = \frac{y}{\sin\frac{\pi}{4}} = 6\sqrt{2} \Rightarrow x = 6, y = 6$.

Since $P(6, 6)$ lies on $x^2 + y^2 + 2gx + 2fy + c = 0$, therefore

$72 + 12(g + f) + c = 0$ (i)

Since $y = x$ touches the circle, therefore the equation $2x^2 + 2x(g + f) + c = 0$ has equal roots

$\Rightarrow \quad 4(g + f)^2 = 8c$

$\Rightarrow \quad (g + f)^2 = 2c$ (ii)

From (i), we get

$[12(g + f)]^2 = [-(c + 72)]^2$

$\Rightarrow \quad 144(g + f)^2 = (c + 72)^2$

$\Rightarrow \quad 144(2c) = (c + 72)^2$

$\Rightarrow \quad (c - 72)^2 = 0 \Rightarrow c = 72$

88. a.

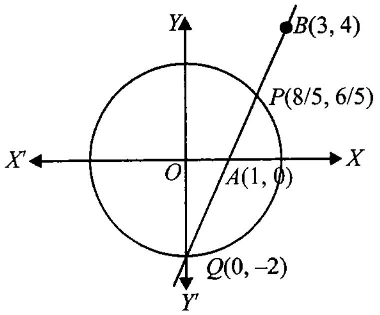


Fig. 2.154

The equation of the line joining $A(1, 0)$ and $B(3, 4)$ is $y = 2x - 2$.

This cuts the circle $x^2 + y^2 = 4$ at $Q(0, -2)$ and $P\left(\frac{8}{5}, \frac{6}{5}\right)$.

We have $BQ = 3\sqrt{5}$, $QA = \sqrt{5}$, $BP = \frac{7}{\sqrt{5}}$ and $PA = \frac{3}{\sqrt{5}}$

$\therefore \quad \alpha = \frac{BP}{PA} = \frac{7/\sqrt{5}}{3/\sqrt{5}} = \frac{7}{3}$

and $\beta = \frac{BQ}{QA} = \frac{3\sqrt{5}}{-\sqrt{5}} = -3$

$\therefore \alpha, \beta$ are roots of the equation $x^2 - x(\alpha + \beta) + \alpha\beta = 0$

i.e., $x^2 - x\left(\frac{7}{3} - 3\right) + \frac{7}{3}(-3) = 0$

or $3x^2 + 2x - 21 = 0$

89. a.

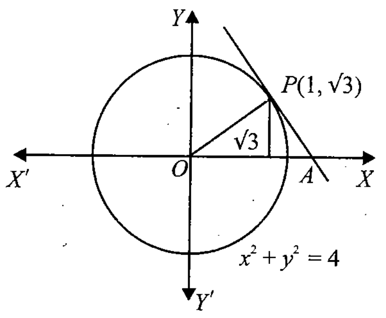


Fig. 2.155

The equations of the tangent and normal to $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ are $x + \sqrt{3}y = 4$ and $y = \sqrt{3}x$

The tangent meets x-axis at (4, 0).

Therefore, area of $\Delta OAP = \frac{1}{2}(4)\sqrt{3} = 2\sqrt{3}$ sq. units

90. d. Any point on the line $7x + y + 3 = 0$ is $Q(t, -3 - 7t)$, $t \in R$

Now $P(h, k)$ is image of point Q in the line $x - y + 1 = 0$

Then, $$\frac{h-t}{1} = \frac{k-(-3-7t)}{-1}$$

$$= -\frac{2(t-(-3-7t)+1)}{1+1}$$

$$= -8t - 4$$

$\Rightarrow$ $(h, k) \equiv (-7t - 4, t + 1)$

This point lies on the circle $x^2 + y^2 = 9$

$\Rightarrow$ $(-7t - 4)^2 + (t + 1)^2 = 9$

$\Rightarrow$ $50t^2 + 58t + 8 = 0$

$\Rightarrow$ $25t^2 + 29t + 4 = 0$

$\Rightarrow$ $(25t + 4)(t + 1) = 0$

$\Rightarrow$ $t = -4/25, t = -1$

$\Rightarrow$ $(h, k) \equiv \left(-\frac{72}{25}, \frac{21}{25}\right)$ or $(3, 0)$

91. c. Let the coordinates be $A(a, 0)$ and $B(-a, 0)$ and let the straight line be $y = mx + c$. Then,

$$\frac{mx + c}{\sqrt{1+m^2}} + \frac{-mx + c}{\sqrt{1+m^2}} = 2k$$

$\Rightarrow$ $c = k\sqrt{1+m^2}$

So, the straight line is $y = mx + k\sqrt{1+m^2}$.

Clearly, it touches the circle $x^2 + y^2 = k^2$ of radius k.

92. a. The midpoint is the intersection of the chord and perpendicular line to it from the centre (3, –1).

The equation of perpendicular line is $5x + 2y - 13 = 0$.

Solving this with the given line, we get the point (1, 4).

93. d. The locus is the radical axis which is perpendicular to the line joining the centres of the circles.

94. b. In an equilateral triangle, circumcentre and in centre are coincident.

$\therefore$ Incentre $= (-g, -f)$

(1, 1) lies on the circle

$\Rightarrow$ $1^2 + 1^2 + 2g + 2f + c = 0$

$\Rightarrow$ $c = -2(g + f + 1)$

Also, in an equilateral triangle,

Circumradius = 2 × inradius

Therefore, inradius $= \frac{1}{2} \times \sqrt{g^2 + f^2 - c}$

it is continuous single word.

$\therefore$ The equation of the incircle is

$$(x+g)^2 + (y+f)^2 = \frac{1}{4}(g^2 + f^2 - c)$$

$$= \frac{1}{4}(g^2 + f^2 + 2(g + f + 1))$$

95. a.

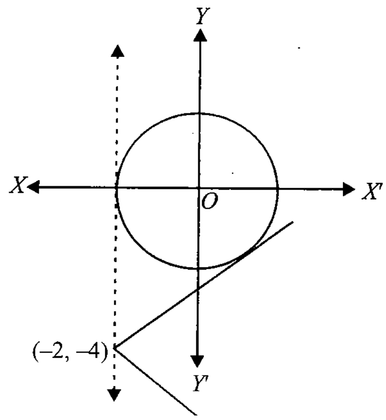


Fig. 2.156

Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{1+m^2}$ if it passes through (–2, –4) then $(2m - 4)^2 = 4(1 + m^2)$

$\Rightarrow$ $4m^2 + 16 - 16m = 4 + 4m^2$

$\Rightarrow$ $m = \infty, m = \frac{3}{4}$.

Hence, slope of reflected ray is $\frac{3}{4}$.

Thus, equation of incident ray is $(y + 4) = -\frac{3}{4}(x + 2)$, i.e., $4y + 3x + 22 = 0$.

96. a. Any point on line $x + y = 25$ is $P \equiv (a, 25 - a)$, $a \in R$

Equation of chord AB is $T = 0$,

i.e., $$xa + y(25 - a) = 9 \quad \text{(i)}$$

If midpoint of chord AB is $C(h, k)$, then equation of chord AB is

$$T = S_1, \text{ i.e., } xh + yk = h^2 + k^2 \quad \text{(ii)}$$

Comparing the ratio of coefficients of Eqs. (i) and (ii), we get

$$\frac{a}{h} = \frac{25 - a}{k} = \frac{9}{h^2 + k^2}$$

$$\Rightarrow \quad \frac{a + 25 - a}{h + k} = \frac{9}{h^2 + k^2}$$

Thus, locus of 'C' is $25(x^2 + y^2) = 9(x + y)$.

97. a.

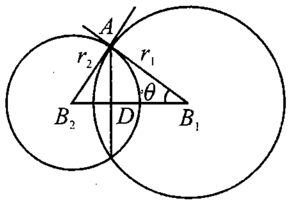


Fig. 2.157

Let $$\angle AB_1B_2 = \theta$$

$$\Rightarrow \quad AD = r_1 \sin \theta$$

and $$AD = r_2 \cos \theta$$

$$\Rightarrow \quad AD^2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) = 1$$

$$\Rightarrow \quad AD = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

Thus, length of common chord $= \dfrac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$

98. b.

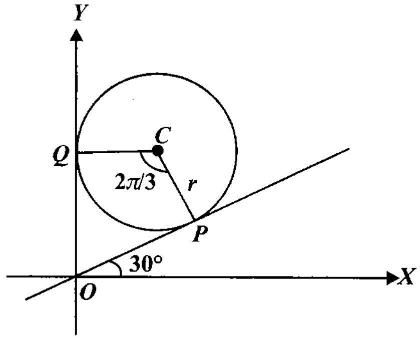


Fig. 2.158

$$\tan 30° = \frac{r}{OP}$$

$$\Rightarrow \quad OP = r\sqrt{3}$$

Also $$r^2 \frac{1}{2} \frac{2\pi}{3} = 3\pi$$

$$\therefore \quad r = 3$$

$$\therefore \quad OP = 3\sqrt{3}$$

99. c.

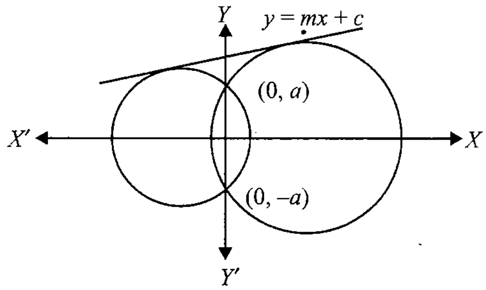


Fig. 2.159

Equation of family circles through $(0, a)$ and $(0, -a)$ is

$$[x^2 + (y - a)(y + a)] + \lambda x = 0, \lambda \in R$$

$$\Rightarrow \quad x^2 + y^2 + \lambda x - a^2 = 0$$

and $$\sqrt{\left(\frac{\lambda}{2}\right)^2 + a^2} = \frac{-\frac{m\lambda}{2} + c}{\sqrt{1 + m^2}}$$

$$\Rightarrow \quad (1 + m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \left(\frac{m\lambda}{2} - c\right)^2$$

$$\Rightarrow \quad (1 + m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \frac{m^2\lambda^2}{4} - mc\lambda + c^2$$

$$\Rightarrow \lambda^2 + 4mc\lambda + 4a^2(1 + m^2) - 4c^2 = 0$$

$$\therefore \quad \lambda_1\lambda_2 = 4[a^2(1 + m^2) - c^2]$$

$$\Rightarrow \quad g_1g_2 = [a^2(1 + m^2) - c^2]$$

and $$g_1g_2 + f_1f_2 = \frac{c_1 + c_2}{2}$$

$$\Rightarrow \quad a^2(1 + m^2) - c^2 = -a^2$$

Hence, $$c^2 = a^2(2 + m^2)$$

100. a. Distance of given line from the centre of the circle is $|p|$.

Now line subtends right angle at the centre.

Hence, $$\text{radius} = \sqrt{2}\,|p|$$

$$\Rightarrow \quad a = \sqrt{2}\,|p|$$

$$\Rightarrow \qquad a^2 = 2p^2$$

101.c. Let the midpoint of the chord be $P(h, k)$.

Then $CP = \sqrt{h^2 + k^2}$, where C is centre of the circle.

Since chord subtends right angle at the centre.

$$\text{Radius} = \sqrt{2}\sqrt{h^2 + k^2}$$

$$\Rightarrow \qquad 2 = \sqrt{2}\sqrt{h^2 + k^2}$$

$\Rightarrow$ locus of P is $\quad x^2 + y^2 = 2$

102.a. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points and $x^2 + y^2 = a^2$ be the circle.

The chord of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

If it passes through $Q(x_2, y_2)$, then

$$x_1x_2 + y_1y_2 = a^2 \qquad \text{(i)}$$

The equation of the circle on PQ as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \qquad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

This circle will cut the given circle orthogonally, if

$$0(x_1 + x_2) + 0(y_1 + y_2) = -a^2 + x_1x_2 + y_1y_2$$

$\Rightarrow \qquad x_1x_2 + y_1y_2 - a^2 = 0$, which is true by Eq. (i).

103. a.

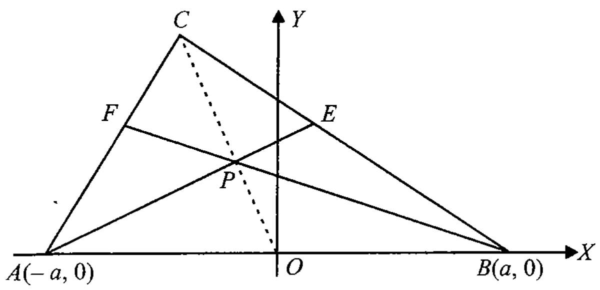


Fig. 2.160

Let A and B be $(-a, 0)$ and $(a, 0)$. Also let P be (h, k).

Then by geometry, we know $\dfrac{CP}{PO} = \dfrac{CF}{FA} + \dfrac{CE}{EB}$

$$\therefore \qquad \frac{CP}{PO} = 1$$

If $C(\alpha, \beta)$ lies on $x^2 + y^2 + 2gx + 2fy + c = 0$, then $\alpha = 2h$ and $\beta = 2k$

$\Rightarrow 4(h^2 + k^2 + gh + fk) + c = 0$

$\therefore$ Locus of $P(h, k)$ is $x^2 + y^2 + gx + fy + \frac{c}{4} = 0$ which is a circle of radius $= \sqrt{\left(\frac{g}{2}\right)^2 + \left(\frac{f}{2}\right)^2 - \frac{c}{4}}$

$$= \frac{1}{2}\sqrt{g^2 + f^2 - c}$$

$$= \frac{r}{2}$$

104. b. The slope of the chord is $m = -\frac{8}{y}$

$$\Rightarrow \qquad y = \pm 1, \pm 2, \pm 4, \pm 8$$

But $(8, y)$ must also lie inside the circle $x^2 + y^2 = 125$

$\Rightarrow y$ can be equal to $\pm 1, \pm 2, \pm 4 \Rightarrow 6$ values.

105. b.

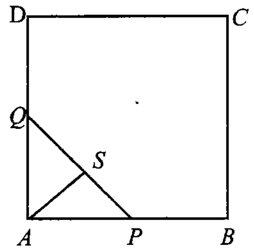


Fig. 2.161

Let S be the midpoint of PQ.

Since $\angle PAQ = \frac{\pi}{2}$, we get $AS = SP = SQ = \frac{1}{2}$

$\Rightarrow S$ lies on the quarter circle of radius $\frac{1}{2}$ with centre at A.

Similarly S can also lie on quarter circle of radius $\frac{1}{2}$ with centre at B, C or D.

$$\Rightarrow \qquad \text{area } A = 1 - \frac{\pi}{4}$$

106.b. The line $2y = gx + \alpha$ should pass through $(-g, -g)$, so $-2g = -g^2 + \alpha \Rightarrow \alpha = g^2 - 2g = (g - 1)^2 - 1 \geq -1$.

107.c. Let $\sum_{i=1}^{6} x_i = \alpha$ and $\sum_{i=1}^{6} y_i = \beta$

Let O be the orthocentre of the triangle made by (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$\Rightarrow O$ is $(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \equiv (\alpha_1, \beta_1)$

Similarly let G be the centroid of the triangle made by other three points

$$\Rightarrow \quad G \text{ is } \left(\frac{x_4 + x_5 + x_6}{3}, \frac{y_4 + y_5 + y_6}{3}\right)$$

$\Rightarrow \quad G \text{ is } \left(\frac{\alpha-\alpha_1}{3}, \frac{\beta-\beta_1}{3}\right)$

The point dividing OG is the ratio 3 : 1 is $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right) \equiv (2, 1)$

$$\Rightarrow \qquad h + k = 3$$

108. a. Let the centre be $(0, \alpha)$ equation of circle $x^2 + (y - \alpha)^2 = |a|^2$

$\therefore$ Equation of chord of contact for $P(h, k)$ is
$xh + yk - \alpha(y + k) + \alpha^2 - a^2 = 0$

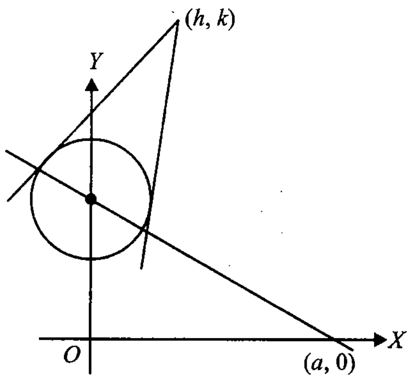


Fig. 2.162

It passes through $(a, 0)$

$\Rightarrow \qquad \alpha^2 - \alpha k + ah - a^2 = 0$

As α is real

$\Rightarrow \qquad k^2 - 4(ah - a^2) \geq 0$

109. b. Let A and B be the centres and r_1 and r_2 the radii of the two circles, then

$$A = \left(-\frac{1}{2}, -\frac{1}{2}\right), B \equiv \left(-\frac{1}{2}, \frac{1}{2}\right),$$

$$r_1 = \frac{1}{\sqrt{2}}, r_2 = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{r_1^2 + r_2^2 - AB^2}{2r_1r_2}$$

$$= \frac{\frac{1}{2} + \frac{1}{2} - 1}{2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = 0$$

$$\therefore \qquad \theta = \frac{\pi}{2}$$

$\therefore$ Required line is parallel to x-axis and since it passes through $(1, 2)$, therefore its equation will be $y = 2$.

110. a.

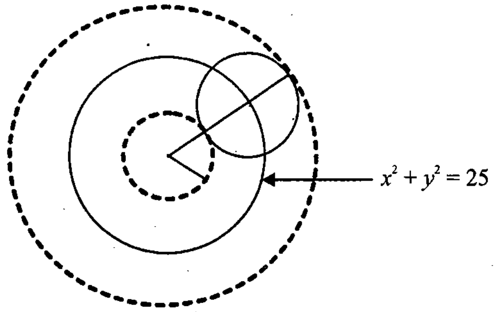


Fig. 2.163

Let (h, k) be any point in the set, then equation of circle is

$$(x - h)^2 + (y - k)^2 = 9$$

But $\quad (h, k)$ lies on $x^2 + y^2 = 25$,

then $\qquad h^2 + k^2 = 25$

$\therefore$ 2 ≤ Distance between the centres of two circles ≤ 8

$$4 \leq h^2 + k^2 \leq 64$$

Therefore, locus of (h, k) is $4 \leq x^2 + y^2 \leq 64$.

111. b. $\qquad OR = \dfrac{2 \text{ area of } \Delta OPQ}{PQ}$

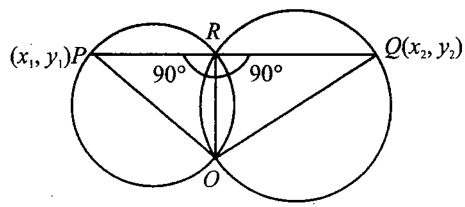


Fig. 2.164

$$= \frac{2 \cdot \left|\frac{1}{2}(x_1y_2 - x_2y_1)\right|}{PQ}$$

$$= \frac{|x_1y_2 - x_2y_1|}{PQ}$$

112. c. Substituting $y = mx$ in the equation of circle we get $x^2 + m^2x^2 + ax + bmx + c = 0$ (y/x denotes the slope of the tangent from the origin on the circle)

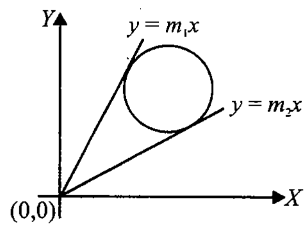


Fig. 2.165

Since line is touching the circle, we must have discriminant

$\Rightarrow \quad (a + bm)^2 - 4c(1 + m^2) = 0$

$\Rightarrow a^2 + b^2m^2 + 2abm - 4c - 4cm^2 = 0$

$\Rightarrow \quad m^2(b^2 - 4c) + 2abm + a^2 - 4c = 0$

This equation has two roots m_1 and m_2,

$$\Rightarrow \quad m_1 + m_2 = -\frac{2ab}{b^2 - 4c} = \frac{2ab}{4c - b^2}$$

113. c. Equation of radical axis (i.e. common chord) of the two circles is

$$10x + 4y - a - b = 0 \quad \text{(i)}$$

Centre of first circle is $H(-4, -4)$.

Since second circle bisects the circumference of the first circle, therefore, centre $H(-4, -4)$ of the first circle must lie on the common chord Eq. (i).

$\therefore \quad -40 - 16 - a - b = 0$

$\Rightarrow \quad a + b = -56$

114. d. Let the equation of circle be

$$x^2 + y^2 - 4 + k(2x + y - 1) = 0$$

where k is a real number

$$\text{Radius} = \sqrt{\frac{5k^2}{4} + 4 + k}$$

Radius is minimum when $k = -\frac{2}{5}$

$\therefore$ The required equation will be

$$5x^2 + 5y^2 - 4x - 2y - 18 = 0$$

115. b.

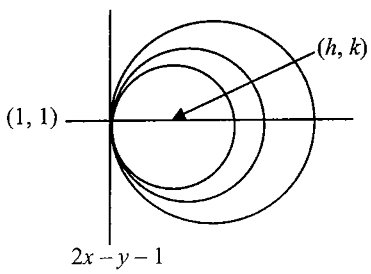


Fig. 2.166

Obviously, locus of centre is line perpendicular to the given line.

Hence, locus is $\frac{k-1}{h-1} = -\frac{1}{2}$ or $x + 2y = 0$.

116. a. Centre of the circle $x^2 + y^2 = 2x$ is (1, 0).

Common chord of the other two circles is

$$8x - 15y + 26 = 0$$

Distance from (1, 0) to $8x - 15y + 26 = 0$

$$= \frac{|8 + 26|}{\sqrt{15^2 + 8^2}} = 2$$

117. d. Equation of any circle though the points of intersection of given circles is

$$x^2 + y^2 - 4x - 2y - 8 + k(x^2 + y^2 - 2x - 4y - 8) = 0 \quad \text{(i)}$$

Since circle Eq. (i) passes through (–1, 4)

$\therefore \quad k = 1$

$\therefore$ Required circle is

$$x^2 + y^2 - 3x - 3y - 8 = 0$$

118. b. Given circles are

$$(x - 1)^2 + (y - 2)^2 = 1 \quad \text{(i)}$$

and $\quad (x - 7)^2 + (y - 10)^2 = 4 \quad$ (ii)

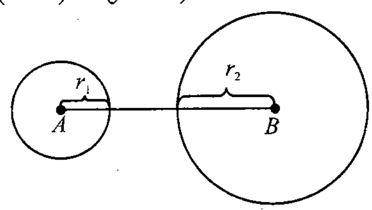


Fig. 2.167

Let $A \equiv (1, 2)$, $B \equiv (7, 10)$, $r_1 = 1$, $r_2 = 2$

$AB \equiv 10$, $r_1 + r_2 = 3$

$AB > r_1 + r_2$, hence the two circles are separated.

Radius of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$

For the two circle to touch each other

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

or $\quad 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$

or $\quad 100 = (3 + 0.7t)^2, [(0.1)t + 1]^2$

or $\quad 3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$

$\therefore \quad t = 10, t = 90 \quad [\because t > 0]$

The two circles will touch each other externally in 10 seconds and internally in 90 seconds.

119. a. The two normals are $x = 1$ and $y = 2$

Their point of intersection (1, 2) is the centre of the required circle

$$\text{Radius} = \frac{|3 + 8 - 6|}{5} = 1$$

$\therefore$ Required circle is

$$(x - 1)^2 + (y - 2)^2 = 1$$

i.e., $\quad x^2 + y^2 - 2x - 4y + 4 = 0$

Multiple Correct Answers Type

1. a., c. Equation of radical axis of the given circle is $x = 0$.

If one circle lies completely inside the other, centre of both circles should lie on the same side of radical axis and radical axis should not intersect the circles.

$\Rightarrow \qquad (-a_1)(-a_2) > 0$

$\Rightarrow a_1a_2 > 0$ and $y^2 + c = 0$ should have imaginary roots

$\Rightarrow \qquad c > 0.$

2. a, c, d. Coordinates of O are (5, 3) and radius = 2

Equation of tangent at $A(7, 3)$ is $7x + 3y - 5(x + 7) - 3(y + 3) + 30 = 0$

i.e., $\qquad 2x - 14 = 0$, i.e., $x = 7$

Equation of tangent at $B(5, 1)$ is $5x + y - 5(x + 5) - 3(y + 1) + 30 = 0$, i.e., $-2y + 2 = 0$, i.e. $y = 1$

$\therefore$ Coordinate of C are (7, 1)

$\therefore$ Area of $\qquad OACB = 4$

Equation of AB is $x - y = 4$ (radical axis)

Equation of the smallest circles is

$(x - 7)(x - 5) + (y - 3)(y - 1) = 0$

i.e., $\quad x^2 + y^2 - 12x - 4y + 38 = 0$

3. a., c. $\qquad 2gg' + 2ff' = c + c'$

$\Rightarrow \qquad 2 \times 1 \times 0 + 2\cdot k\cdot k = 6 + k$

$\Rightarrow \qquad 2k^2 - k - 6 = 0$

$\Rightarrow \qquad (2k + 3)(k - 2) = 0$

$\therefore \qquad k = 2, -\frac{3}{2}$

4. a., b. Let $O \equiv (0, 0)$ be the centre of the circle.

$\because$ Arc length $AB = \frac{\pi}{2} = \frac{1}{4}$ (circumference of the circle)

$\therefore \qquad \angle AOB = \frac{\pi}{2}$

$\therefore \qquad$ Slope of $OB = -\dfrac{1}{\text{slope of } OA}$

$\Rightarrow \qquad$ slope of $OB = -\frac{1}{1} = -1 \qquad$ (i)

Let $\qquad B \equiv \left(\alpha, \pm\sqrt{1 - \alpha^2}\right)$

$\therefore \qquad \pm\dfrac{\sqrt{1 - \alpha^2}}{\alpha} = -1 \qquad$ [from Eq. (i)]

$\therefore \quad B$ can be $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ but possible points are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

5. a., c., d.

$x^2 + y^2 + 8x - 10y - 40 = 0$

Centre of the circle is (– 4, 5)

Its radius = 9

Distance of the centre (– 4, 5) from the point (– 2, 3) is $\sqrt{4 + 4} = 2\sqrt{2}$

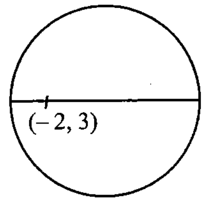


Fig. 2.168

$\therefore \qquad a = 2\sqrt{2} + 9$ and $b = -2\sqrt{2} + 9$

$\therefore \qquad a + b = 18$

$a - b = 4\sqrt{2}$

$a \cdot b = 81 - 8 = 73$

6. a., b. $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$

$\Rightarrow (y - m_1x)(y - m_2x) + \lambda(y - m_2x)(y - m_3x)$
$+ \mu(y - m_3x)(y - m_1x) = 0 \qquad$ (i)

Clearly Eq. (i) represents a curve passing through points of intersection of lines L_1, L_2 and L_3.

Equation (i) will represent a circle if coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$\therefore \qquad 1 + \lambda + \mu = m_1m_2 + \lambda m_2m_3 + \mu m_1m_3$

and $m_1(1 + \mu) + m_2(1 + \lambda) + m_3(\mu + \lambda) = 0$

7. a., b., c., d. Given circle is

$x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0 \qquad$ (i)

For Eq. (i) to represent a circle, $h = 0$

$\therefore$ Given circle is

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad$ (ii)

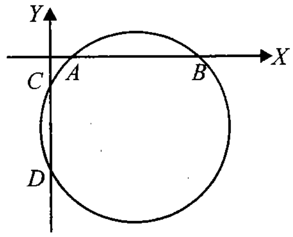


Fig. 2.169

For circle Eq. (ii) to pass thorough three quadrants only

i. $AB > 0 \therefore g^2 - c > 0$

ii. $CD > 0 \therefore g^2 - c > 0$

iii. Origin should lie outside circle Eq. (ii)

$\therefore \qquad c > 0$

Therefore, required conditions are $g^2 > c, f^2 > c, c > 0, h = 0$

8. a., c. The point from which the tangents drawn are at right angle lie on the director circle.

Equation of director circle is $x^2 + y^2 = 2 \times 16 = 32$

Putting $\qquad x = 2$, we get

$y^2 = 28$

$\Rightarrow \qquad y = \pm 2\sqrt{7}$

$\therefore$ The points can be $(2, 2\sqrt{7})$ or $(2, -2\sqrt{7})$.

9. b., d. Line pair is $(x-1)^2 - y^2 = 0$, i.e. $x + y - 1 = 0$, $x - y - 1 = 0$.

Let the centre be $(\alpha, 0)$, then its distance from $x + y - 1 = 0$ is

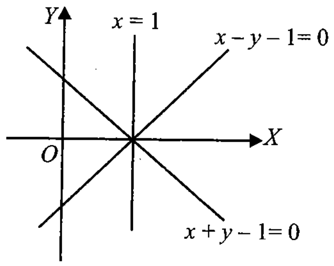


Fig. 2.170

$$\left|\frac{\alpha - 1}{\sqrt{2}}\right| = 2(\text{radius})$$

i.e. $\alpha = 1 \pm 2\sqrt{2}$

$\therefore$ Centre may be $(1 + 2\sqrt{2}, 0)$, $(1 - 2\sqrt{2}, 0)$

Now let the centre be $(1, \beta)$, then

$$\left|\frac{1 + \beta - 1}{\sqrt{2}}\right| = 2$$

$\Rightarrow$ $\beta = \pm 2\sqrt{2}$

$\therefore$ Centre may be $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$

10. a., d. Equation of the radical axis is

$$2\alpha x + 2y + 10 = 0$$

i.e. $\alpha x + y + 5 = 0$ (i)

Putting the value of y from Eq. (i) in the circle $x^2 + y^2 = 9$, we get

$$(1 + \alpha^2)x^2 + 10\alpha x + 16 = 0$$

$\because$ Radical axis is tangent

$\therefore$ $D = 0$

$\Rightarrow$ $36\alpha^2 - 64 = 0$

$\Rightarrow$ $\alpha = \pm\frac{4}{3}$

11. b., c.

$$x^2 + y^2 - 8x - 16y + 60 = 0 \quad \text{(i)}$$

Equation of chord of contact from $(-2, 0)$ is $-2x - 4(x - 2) - 8y + 60 = 0$

$$3x + 4y - 34 = 0 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii)

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

$\Rightarrow$ $16x^2 + 1156 - 204x + 9x^2 - 128x - 2176 + 192x + 960 = 0$

$\Rightarrow$ $5x^2 - 28x - 12 = 0$

$\Rightarrow$ $(x - 6)(5x + 2) = 0$

$\Rightarrow$ $x = 6, -\frac{2}{5}$

$\Rightarrow$ Points are $(6, 4)$, $\left(-\frac{2}{5}, \frac{44}{5}\right)$.

12. a., c. Equation of any tangent to the circle $x^2 + y^2 = 25$ is of the form

$$y = mx + 5\sqrt{1 + m^2}$$

(where m is the slope)

$\because$ It passes through $(-2, 11)$.

$\therefore$ $11 = -2m + 5\sqrt{1 + m^2}$

$\Rightarrow$ $(11 + 2m)^2 = 25(1 + m^2)$

$\Rightarrow$ $m = \frac{24}{7}, -\frac{4}{3}$

Therefore, equation of the tangents are

$$24x - 7y + 125 = 0$$

or $4x + 3y = 25$

13. a., d. Area of the quadrilateral $= \sqrt{c} \times \sqrt{9 + 25 - c} = 15$

$\therefore$ $c = 9, 25$

14. a., d.

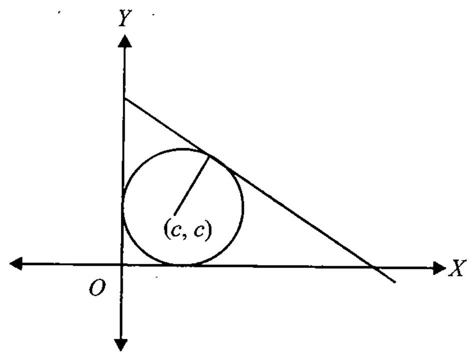


Fig. 2.171

We must have $\left|\dfrac{\frac{c}{3} + \frac{c}{4} - 1}{\sqrt{\frac{1}{3^2} + \frac{1}{4^2}}}\right| = c$

$\Rightarrow$ $c = 6, 1$

15. a.c. Since the given circle is $(x - 3)^2 + (y - 3)^2 = 9$ is touching both the axis, tangents from the origin are x-axis and y-axis or $y = 0$ and $x = 0$.

16. b. Since A, B, C, D are concyclic

$\therefore$ $OA \cdot OC = OB \cdot OD$

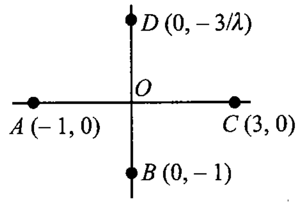


Fig. 2.172

$\Rightarrow \quad 1 \times 3 = 1 \times \left|\frac{3}{\lambda}\right|$

$\Rightarrow \quad \lambda = \pm 1$

But when $\lambda = -1$, B and D will not lie on the circle simultaneously

$\therefore \quad \lambda = 1$

17. a., b., c., d. Chords equidistance from the centre are equal.

18. b., d. Let the equation of the tangent be

$$x - 2y = k \qquad \text{(i)}$$

$\because$ Line Eq. (i) touches the circle

$\therefore$ Distance from centre to line Eq. (i)

= radius of the circle

$$\therefore \quad \frac{|2-2-k|}{\sqrt{5}} = \sqrt{4+1+15}$$

$$|k| = 10 \Rightarrow k = \pm 10$$

$\therefore$ The tangents can be $x - 2y \pm 10 = 0$

19. b., c. For given circle $S_1: x^2 + y^2 - 2x - 4y + 1 = 0$ and $S_2: x^2 + y^2 + 4x + 4y - 1 = 0$

$$C_1(1, 2), r_1 = 2 \text{ and } C_2(-2, -2), r_2 = 3$$

Now $\quad r_1 + r_2 = 5$ and $C_1C_2 = 5$

Hence, circles touch externally. Also comon tangent at point of contact is $S_1 - S_2 = 0$ or $3x + 4y - 1 = 0$

20. a., b., c., d.

$$r_1 = 5;\ r_2 = \sqrt{15};\ C_1C_2 = \sqrt{40}$$

$$\Rightarrow \quad r_1 + r_2 > C_1C_2 > r_1 - r_2$$

Hence, circles intersect in two distinct points.

There are two common tangents.

Also $\quad 2g_1g_2 + 2f_1f_2 = 2(1)(3) + 2(2)(-4) = -10$

and $\quad c_1 + c_2 = -20 + 10 = -10$

Thus, two circle are orthogonal.

Length of common chord is $\dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}} = 5\sqrt{\dfrac{3}{2}}$

Length of common tangent is $\sqrt{C_1C_2^2 - (r_1 - r_2)^2}$ $= 5\left(\frac{12}{5}\right)^{\frac{1}{4}}$.

21. b., d.

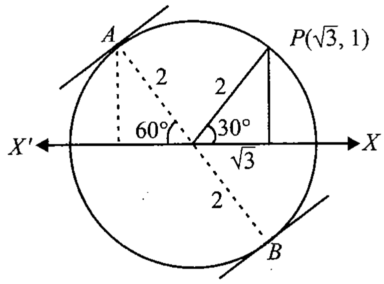


Fig. 2.173

Clearly, $A = (-2\cos 60°, 2\sin 60°)$ and $B = (2\cos 60°, -2\sin 60°)$

The tangent at A is $x(-2\cos 60°) + y(2\sin 60°) = 4$ and the tangent at B is $x(2\cos 60°) + y(-2\sin 60°) = 4$.

22. b., c.

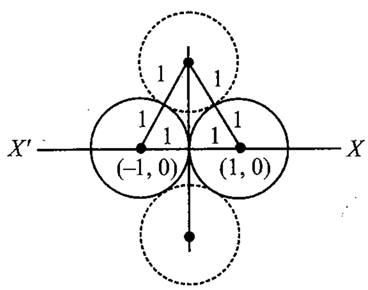


Fig. 2.174

The given circles are $x^2 + y^2 - 2x = 0$, $x > 0$, and $x^2 + y^2 + 2x = 0$, $x < 0$

From the above figure, the centres of the required circles will be $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

$\therefore$ The equations of the circles are $(x-0)^2 + (y \mp \sqrt{3})^2 = 1^2$

23. a., d. When two circle touch each other externally, then

$$r_1 + r_2 = \sqrt{\{0-(-a)\}^2 + \{0-(-1)\}^2}$$

$$\Rightarrow \quad 3 + a = \sqrt{a^2 + 1}$$

$$\Rightarrow \quad a = -\frac{4}{3}$$

When two circle touch each other internally, then

$$|r_1 - r_2| = \sqrt{\{0-(-a)\}^2 + \{0-(-1)\}^2}$$

$$\Rightarrow \quad |3 - a| = \sqrt{a^2 + 1}$$

$$\Rightarrow \quad a = \frac{4}{3}$$

24. b., c. Equation of pair of tangent by $SS' = T^2$ is

$$(ax + 0 - 1)^2 = (x^2 + y^2 - 1)(a^2 + 0 - 1)$$

or $(a^2 - 1)y^2 - x^2 + 2ax - a^2 = 0$

If θ be the angle between the tangents, then

$$\tan\theta = \frac{2\sqrt{H^2 - AB}}{A+B}$$

$$= \frac{2\sqrt{-(a^2-1)(-1)}}{a^2-2}$$

$$= \frac{2\sqrt{a^2-1}}{a^2-2}$$

If θ lies in II quadrant, then $\tan\theta < 0$

$\therefore \quad \frac{2\sqrt{a^2-1}}{a^2-2} < 0$

$\Rightarrow \quad a^2 - 1 > 0$

and $\quad a^2 - 2 < 0$

$\Rightarrow \quad |a| > 1$ and $|a| < 2$

$\Rightarrow \quad a \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$

25. b,. c.

Distance of line $x + y - 1 = 0$ from the centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$ is $\frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}} = \sqrt{2}$

Now distance of line in options (b) and (c) from the centre is also $\sqrt{2}$.

Hence, given lines are $x - y = 0$ and $x + 7y = 0$.

26. c., d.

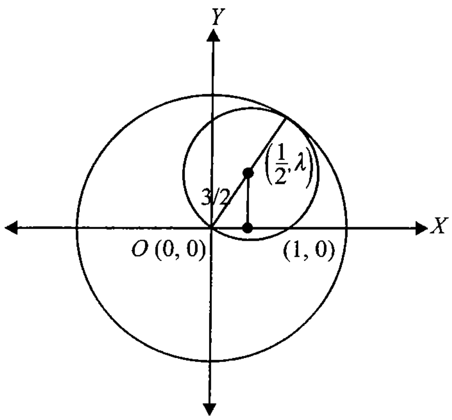


Fig. 2.175

From the diagram

$$\sqrt{\left(\frac{1}{2}\right)^2 + \lambda^2} = \frac{3}{2} \Rightarrow 1 = \pm\sqrt{2}$$

Hence, centres of the circle are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.

Reasoning Type

1. **d.** Statement 2 is true as the centre is equidistant from A and B, hence lies on the perpendicular bisector of AB.

   Statement 1 is false as the distance between the given points is 10 and hence any circle through A and B has radius more than or equal to 5, and hence there is no circle of radius 4 through A and B is possible.

2. **a.** We know that chords of contact of given circle generated by any point on given line passes through the fixed point, as they form family of straight lines, hence both the statements are true and statement 2 is the correct explanation of statement 1.

3. **b.** For circle $x^2 + y^2 = 144$, centre $C_1(0, 0)$ and radius $r_1 = 12$.

   For circle $x^2 + y^2 - 6x - 8y = 0$, centre $C_2 = (3, 4)$ and radius $r_2 = 5$.

   Now $C_1C_2 = 5$ and $r_1 - r_2 = 7$, thus $C_1C_2 < r_1 - r_2$, hence one circle is completely lying inside other without touching it, hence there is no common tangent. Therefore, statement 1 is true. Therefore, both the statements are true but statement 2 is not correct explanation of statement 1.

4. **b.** Centre of the circle $C(2, 1)$ and radius $r = 5$.

   Distance of $P(10, 7)$ from $C(2, 1)$ is 10 units, hence required distances are 5, 15, respectively. Therefore, Statement 1 is true. Statement 2 is true but not the correct explanation of statement 1, as the information is not sufficient to get distance said in Statement 1.

5. **d.** Given points are collinear, hence circle is not possible. Hence, statement 1 is false, however statement 2 is true.

6. **a.** Here $(O_1O_2)^2 = t^2 + (t^2 + 1)^2 = t^4 + 3t^2 + 1 \geq 0$

   $\Rightarrow \quad O_1O_2 \geq 1$ and $|r_1 - r_2| = 1$

   $\Rightarrow O_1O_2 \geq |r_1 - r_2|$ hence the two circles have at least one common tangent.

7. **a.** The centre of circle is (h, h) and radius $= h$

   $\Rightarrow$ The circle is touching the co-ordinate axes.

8. **b.** Circles $S_1: x^2 + y^2 - 4x - 6y - 8 = 0$ and $S_2: x^2 + y^2 - 2x - 3 = 0$

   $C_1(2, 3), r_1 = \sqrt{21}, C_2(1, 0), r_2 = 2$

   $C_1C_2 = \sqrt{10}, r_1 + r_2 = 2 + \sqrt{21}, r_2 - r_1 = \sqrt{21} - 2$

   Here $r_2 - r_1 < C_1C_2 < r_1 + r_2$. Hence, two circle intersect at two distinct points. Statement 2 is true, but does not explain statement 1.

9. **a.** Clearly $(\sqrt{2}, \sqrt{6})$ lies on $x^2 + y^2 = 8$, which is the director circle of $x^2 + y^2 = 4$.

$\Rightarrow$ Tangents PA and PB are perpendicular to each other.

$\therefore$ $(OAPB)$ is a square.

$\therefore$ Area of $OAPB = 4$.

10. **d.** Point of intersection of $x + 7 = 3$ and $x - y = 1$ is $(2, 1)$.

11. **d.** Statement 2 is correct (a known fact).

Using statement 2, x intercept made by $x^2 + y^2 - 2x + 6y + 5 = 0$ is $2\sqrt{(-1)^2 - 5}$ an imaginary number. Thus, $x^2 + y^2 - 2x + 6y + 5 = 0$ is away from x-axis. Hence, statement 1 is false.

12. **d.** Statement 2 is true as in any triangle in-circle and three ex-circles touches the three sides of the triangle. But Statement 1 is false as given lines are concurrent, hence triangle is not formed.

13. **a.** Given points are $A(1, 1)$, $B(2, 3)$ and $C(3, 5)$ which are collinear as slope AB = slope $BC = 2$. Hence, statement 2 is true.

Chord of contact are concurrent then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Hence, point (x_1, y_1), (x_1, y_1) and (x_3, y_3) are collinear.

14. **a.** $x^2 + y^2 - 2x - 2ay - 8 = 0$

$\Rightarrow (x^2 + y^2 - 2x - 8) - 2a(y) = 0$

$$S + \lambda L = 0$$

Solving circle $x^2 + y^2 - 2x - 8 = 0$ and line $y = 0$

$\therefore$ $x^2 - 4x + 2x - 8 = 0$

$\therefore$ $x = 4, x = -2$

So, $(4, 0)$, $(-2, 0)$ are the points of intersection which lie on x-axis.

15. **c.** Equation of chord of contact from $A(x_1, y_1)$ is

$$xx_1 + yy_1 - a^2 = 0$$

$$xx_2 + yy_2 - a^2 = 0$$

$$xx_3 + yy_3 - a^2 = 0$$

i.e., $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow A, B, C$ are collinear.

16. **a.**

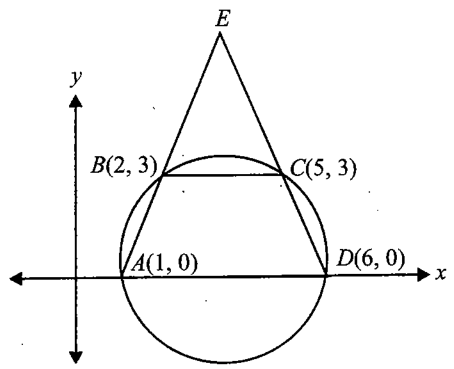


Fig. 2.176

From the figure, it is clear that $ABCD$ is isosceles trapezium as $AB = CD$. Also ΔEAD is isosceles $\Rightarrow EA \times EB = EC \times ED$

Hence, both the statements are correct and statement 1 is correct explanation of statement 1.

17. **c.** Statement 1 is true because common chord itself passes through origin.

Statement 2 is false (common chord is $x - y = 0$).

18. **a.** Common chord of two orthogonal circles subtend supplementary angles at the centre and so complemtary angles on the circumferences of the two circle.

$\therefore$ Both the statements are correct and statement 2 is the correct explanation of statement 1.

19. **a.** Since point lies inside the circle

$\Rightarrow$ $a^2 + a^2 - 4a - 2a - 8 < 0$

$\Rightarrow$ $a^2 - 3a - 4 < 0$

$\Rightarrow$ $-1 < a < 4$

20. **a.** We know that the radical axis of the circle is the locus of point from which length of tangents to given two circles is same, also it the locus of the centre of the circle which intersect the given two circles orthogonally.

Now radical axis of the given two circles is $2x + y - 4 = 0$. Any point on this line is $(t, 4 - 2t)$, $t \in R$.

Hence, both the statements are true and statement 2 is correct explanation of statement.

21. **d.** Since $S_1 = 0$ and $S_3 = 0$ has no radical axis

$\therefore$ Radical centre does not exist.

22. **d.** The statement 2 is well-known result, but if applied to the data given in statement 1 will yield $5x - 9y + 46 = 0$.

$\Rightarrow$ Statement 1 is false, statement 2 is true.

23. **c.** Statement 2 is false because line joining centres may not be parallel to common tangents.

Statement 1 can be proved easily by using distance between centres = sum of radii.

24. **a.** Two circles touch each other $C_1C_2 = |r_1 \pm r_2|$

$\Rightarrow$ $\sqrt{p^2 + q^2} = \sqrt{p^2 - r} = \sqrt{q^2 - r}$

$$\Rightarrow \quad p^2 + q^2 = p^2 - r + q^2 - r + 2\sqrt{(p^2 - r)(q^2 - r)}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{p^2} + \frac{1}{q^2}$$

Linked Comprehension Type

For Problems 1–3

1. d, 2. b, 3. a.

Sol. It is given that one of the diagonals of the square is parallel to the line $y = x$.

Also the length of the diagonal of the square is $4\sqrt{2}$.

Hence, the equation of the one of diagonals is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

Hence, $\quad x - 3 = y - 4 = \pm 2$

$\Rightarrow \quad x = 5, 1$ and $y = 6, 2$

Hence, two of the vertices are (1, 2) and (5, 6).

The other diagonal is parallel to the line $y = -x$, so that its equation is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{-\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

Hence, the two vertices on this diagonal are (1, 6) and (5, 2).

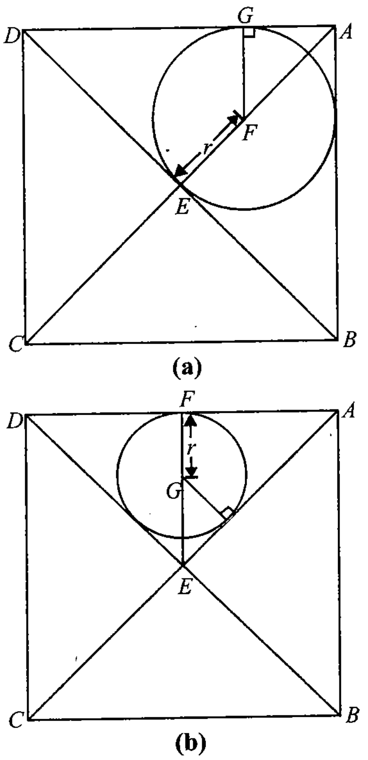


Fig. 2.177

$AB = 4, AC = 4\sqrt{2}$

$\Rightarrow \quad AE = 2\sqrt{2}$

In Fig. (a), $\quad EF + FA = AE$

$\Rightarrow \quad r + \sqrt{2}r = 2\sqrt{2}$

$\Rightarrow \quad r = \frac{2\sqrt{2}}{\sqrt{2}+1} = 2\sqrt{2}(\sqrt{2}-1)$

In Fig. (b), $\quad EG + GF = EF$

$\Rightarrow \quad \sqrt{2}\, r + r = 2$

$\Rightarrow \quad r = \frac{2}{\sqrt{2}+1} = 2(\sqrt{2}-1)$

For Problems 4–6

4. d., 5.c., 6. c.,

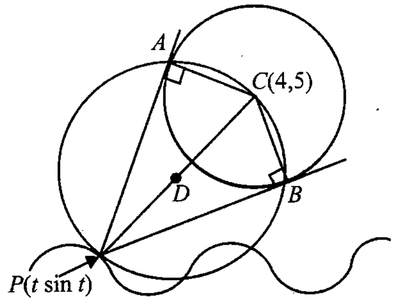


Fig. 2.178

Sol. Centre of the given circle is $C(4, 5)$. Points P, A, C, B are concyclic such that PC is diameter of the circle. Hence, centre D of the circumcircle of ΔABC is midpoint of PC, then we have

$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t - 5}{2}.$$

Eliminating t, we have $\quad k = \frac{\sin(2h-4)+5}{2}$

or $\quad y = \frac{\sin(2x-4)+5}{2}$

$\Rightarrow \quad f^{-1}(x) = \frac{\sin^{-1}(2x-5)+4}{2}$

Thus range of $y = \frac{\sin(2x-4)+5}{2}$ is [2, 3] and period is π.

Also $f(x) = 4 \Rightarrow \sin(2x - 4) = 3$ which has no real solutions.

For $f(x) = 1 \Rightarrow \sin(2x - 4) = -3$ which has no real solutions.

But range of $y = \frac{\sin^{-1}(2x-5)+4}{2}$ is $\left[-\frac{\pi}{4}+2, \frac{\pi}{4}+2\right]$

For Problems 7–9

7. c., 8. d., 9. c.

Sol. Equation of line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3)$$

or $\quad 2x + 3y - 27 = 0$

Also equation of circle with A and B as diameter end points is

$(x-3)(x-6)+(y-7)(y-5) = 0$

Now family of circle through A and B is

$(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)=0$ (i)

If circle belonging to this family touches the x-axis, then equation $(x-3)(x-6)+(0-7)(0-5)+\lambda(2x+3(0)-27) = 0$ has two equal roots, for which Discriminant $D = 0$, which gives two values of λ.

Equation of common chord of (i) and $x^2+y^2-4x-6y-3=0$ is radical axis, which is

$[(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)] - [x^2+y^2-4x-6y-3]=0$

or $(2\lambda-5)x+(3\lambda-6)y+(-27\lambda+56)=0$

or $(-5x-6y+56)+\lambda(2x+3y-27)=0$

This is family of lines which passes through the point of intersection of $-5x-6y+56=0$ and $2x+3y-27=0$ which is $(2, 23/3)$.

If circle (i) cuts $x^2+y^2 = 29$ orthogonally, then $0+0 = -29+56-27\lambda = 0 \Rightarrow \lambda = 1$

$\Rightarrow$ Required circle is $x^2+y^2-7x-9y+26 = 0$, centre is $(7/2, 9/2)$

For Problems 10–12

10. c., 11. a., 12.c.

Sol. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{(i)}$$

The line $lx+my+1=0$, will touch circle (i), if the length of $\perp$ from the centre $(-g, -f)$ of the circle on the line is equal to its radius,

i.e., $$\frac{|-gl-mf+1|}{\sqrt{(l^2+m^2)}} = \sqrt{(g^2+f^2-c)}$$

$$(gl+mf-1)^2 = (l^2+m^2)(g^2+f^2-c)$$

$\Rightarrow (c-f^2)l^2+(c-g^2)m^2-2gl-2fm+2gflm+1=0$ (ii)

But the given condition of tangency is

$$4l^2-5m^2+6l+1=0 \quad \text{(iii)}$$

$\therefore$ Comparing Eqs. (ii) and (iii), we get $c-f^2=4$, $c-g^2 = -5$, $-2g=6$, $-2f=0$, $2gf=0$.

Solving, we get $f=0$, $g=-3$, $c=4$

Substituting these values in Eq. (i), the equation of the circle is $x^2+y^2-6x+4=0$. Any point on the line $x+y-1=0$ is $(t, 1-t)$, $t \in R$.

Chord of contact generated by this point for the circle is $tx+y(1-t)-3(t+x)+4=0$ or $t(x-y-3)+(-3x+y+4)=0$, which are concurrent at point of intersection of the lines $x-y-3=0$ and $-3x+y+4=0$ for all values of t. Hence, lines are concurrent at $\left(\frac{1}{2}, -\frac{5}{2}\right)$.

Also point $(2, -3)$ lies outside the circle from which two tangents can be drawn.

For Problems 13–15

13. c., 14. a., 15. a.

Sol. 13. c. Given $QT = QA = 1$

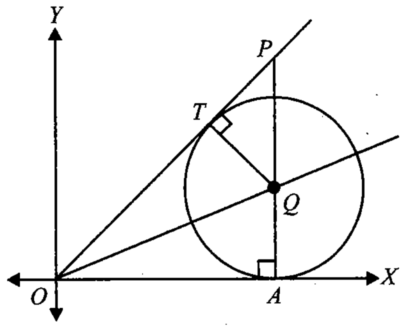


Fig. 1.179

Let $PQ = x$, then $PT = \sqrt{x^2-1}$

ΔTQP and ΔAPO are similar triangles

Then, $$OT = OA = \frac{x+1}{\sqrt{x^2-1}}$$

$$\Rightarrow 1+x+\frac{2(x+1)}{\sqrt{x^2-1}}+\sqrt{x^2-1} = 8$$

$$\Rightarrow x = \frac{5}{3}$$

14. a. $$AP = \frac{8}{3}, OP = \frac{16}{3}$$

Let $\angle AOP = 2\theta$, then $\sin 2\theta = \frac{1}{2}$

From ΔOAQ, $\tan\theta = \frac{1}{OA}$

$$\Rightarrow OA = \frac{1}{\tan\theta}$$

From $$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{1}{2}$$

$$\Rightarrow \tan\theta = 2-\sqrt{3}$$

Hence, $OA = 2+\sqrt{3}$

Hence, equation of circle is $(x-(2+\sqrt{3}))^2+(y-1)^2 = 1$

15. a. Equation of tangent OT is

$$\frac{x-0}{\cos 2\theta} = \frac{y-0}{\sin 2\theta}$$

$$\Rightarrow x-\sqrt{3}y = 0.$$

For Problems 16–18

16. b., 17.d., 18.b.

Sol. 16. b. $\because PQ = PR$, i.e., parallelogram $PQRS$ is a rhombus

$\therefore$ Midpoint of QR = midpoint of PS and $QR \perp PS$

$\therefore$ S is the mirror image of P w.r.t. QR

$\because$ $L \equiv 2x+y=6$

Let $P \equiv (k, 6-2k)$

$\because$ $\angle PQO = \angle PRO = \frac{\pi}{2}$

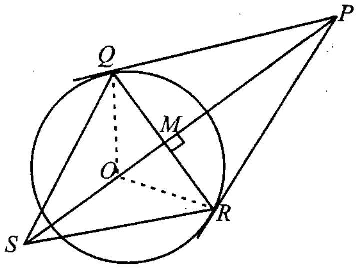


Fig. 1.180

$\therefore$ OP is diameter of circumcircle PQR, then centre is $\left(\frac{k}{2}, 3-k\right)$

$\therefore$ $x = \frac{k}{2} \Rightarrow k = 2x$ and $y = 3 - k$

$\therefore$ Required locus is $2x + y = 3$.

17. d. $P(6, 8)$

$\therefore$ Equation of QR(chord of contact) is $6x + 8y = 4$

$\Rightarrow$ $3x + 4y - 2 = 0$

$\therefore$ $PM = \frac{48}{5}$ and $PQ = \sqrt{96}$

$$QM = \sqrt{96 - \frac{48^2}{25}} = \sqrt{\frac{96}{25}}$$

$\therefore$ $QR = 2\sqrt{\frac{96}{25}}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2} \cdot PM \times QR = \frac{192\sqrt{6}}{25}$

$\because$ $PQRS$ is a rhombus

$\therefore$ Area of ΔQRS = Area of ΔPQR

$= \frac{192\sqrt{6}}{25}$ sq. units

18. b. As $P \equiv (3, 4)$

$\therefore$ Equation of QR is $3x + 4y = 4$ (i)

Let $S \equiv (x_1, y_1)$

$\because$ S is the mirror image of P w.r.t. Eq. (i)

Then $\frac{x_1 - 3}{3} = \frac{y_1 - 4}{4}$

$$= \frac{-2(3 \times 3 + 4 \times 4 - 4)}{3^2 + 4^2}$$

$$= -\frac{42}{25}$$

$\therefore$ $x_1 = -\frac{51}{25}, y_1 = -\frac{68}{25}$

$$S\left(-\frac{51}{25}, -\frac{68}{25}\right)$$

For Problems 19–21

19. a., 20. d., 21. a.

Sol. 19. a.

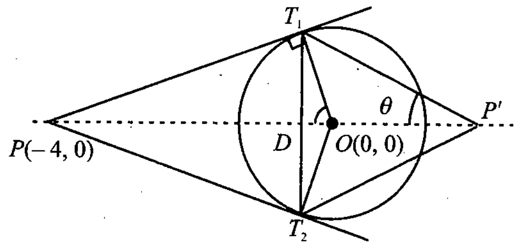


Fig. 2.181

$$PT_2 = PT_1 = \sqrt{(-4)^2 + 0^2 - 4} = 2\sqrt{3}$$

Circumcentre of triangle PT_1T_2 is midpoint of PO as

$$\angle PT_1O = \angle PT_2O = 90°$$

So, $\left(\frac{-4+0}{2}, \frac{0+0}{2}\right) = (-2, 0)$

20. d.

21. a. Let P' be a point on the circle

$$3\theta = \frac{\pi}{2}$$

$\Rightarrow$ $\theta = \frac{\pi}{6}$

Area of the rhombus $= (2\sqrt{3})(2\sqrt{3})\sin\frac{\pi}{3} = 6\sqrt{3}$

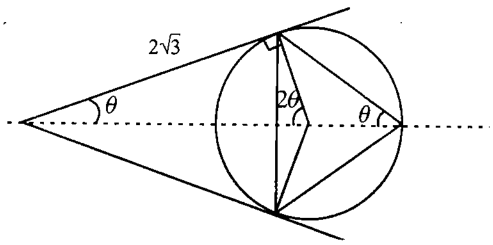


Fig. 2.182

For Problems 22–24

22. b., 23.a., 24. a.

Sol. 22. b.

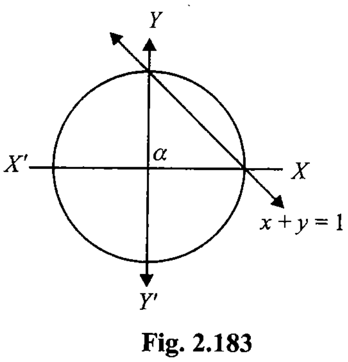


Fig. 2.183

23. a. Slope of chord = 1

Since the chord is $\frac{\pi}{3}$ chord

$\therefore$ Distance from the origin is $= \sqrt{3}$

Let the equation of the chord be $x - y + k = 0$

$\therefore \quad \left|\frac{k}{\sqrt{3}}\right| = \sqrt{3}$ i.e., $k = \pm\sqrt{6}$

24. **a.** Radius of the circle = 2

Distance from the origin $= 2\cos\frac{\pi}{3} = 1$.

For Problems 25–27

25. b., 26. c., 27. c.

Sol.

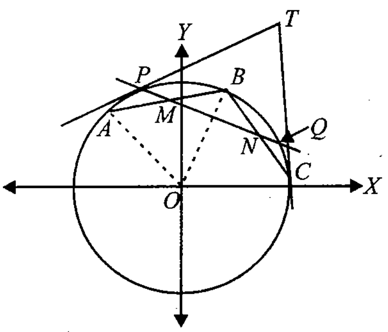


Fig. 2.184

25. **b.** From the Figure,

Since ΔOAB is equilateral triangle

$\therefore \quad \angle OAB = 60°$

26. **c.** Let T be the point of intersection of tangents.

Since $\quad \angle AOC = 120°$

$\Rightarrow$ Angle between tangents is 60°.

27. **c.** Locus of point of intersection of tangents at A and C is a circle whose centre is $O(0, 0)$ and radius is

$$OT = a \operatorname{cosec}\frac{\pi}{6} = 2a$$

$\Rightarrow$ So locus is $\quad x^2 + y^2 = 4a^2$.

For Problems 28–30

28. b., 29. b., 30. c.

Sol.

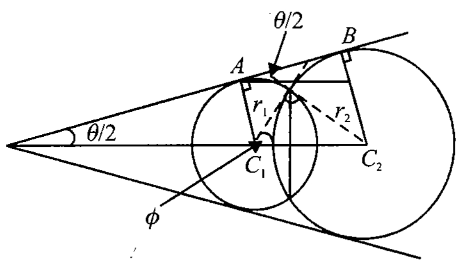


Fig. 2.185

We have $\quad \sin\phi = \frac{d}{r_1}$, $\cos\phi = \frac{d}{r_2}$, (where $2d$ = length of common chord)

$\Rightarrow \quad 1 = \frac{d^2}{r_1^2} + \frac{d^2}{r_2^2}$

$\Rightarrow \quad d = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$

$\Rightarrow \quad 2d = \frac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}} = \frac{24}{5}$, where $r_1 = 3$

$\Rightarrow \quad d = \frac{6r_2}{\sqrt{9 + r_2^2}} = \frac{24}{5}$

$\Rightarrow \quad r_2 = 4$

From the figure, $\quad \sin\frac{\theta}{2} = \frac{r_2 - r_1}{C_1C_2}$

where $\quad C_1^2C_2^2 = r_1^2 + r_1^2$

$\Rightarrow \quad C_1C_2 = 5$

$\therefore \quad \sin\frac{\theta}{2} = \frac{1}{5}$

$\Rightarrow \quad \cos\frac{\theta}{2} = \frac{\sqrt{24}}{5}$

$\Rightarrow \quad \sin\theta = 2\frac{1}{5}\cdot\frac{\sqrt{24}}{5}$

$= \frac{4\sqrt{6}}{25}$

$\Rightarrow \quad \theta = \sin^{-1}\frac{4\sqrt{6}}{25}$

$\Rightarrow$ Also $\quad AB = \sqrt{C_1C_2^2 - (r_1 - r_2)^2}$

$= \sqrt{25 - 1} = \sqrt{24}$

Matrix-Match Type

1. a → r; b → q; c → q; d → p

a.

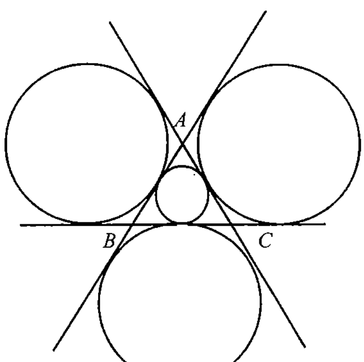


Fig. 2.186

b.

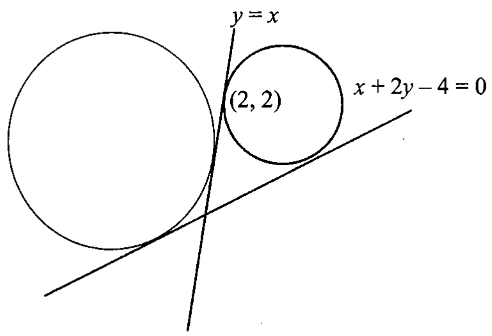


Fig. 2.187

c.

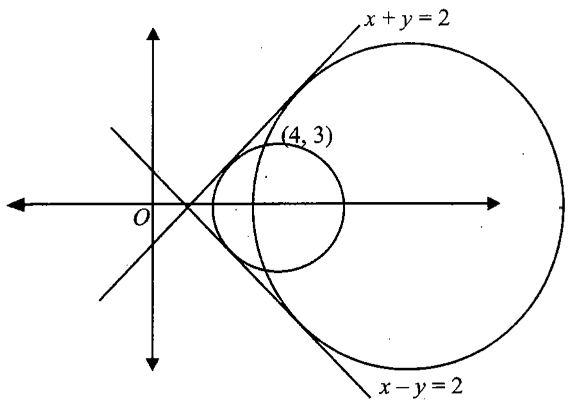


Fig. 2.188

d. Obviously one (see theory).

2. a → p, r, s; b → r, s; c → q, s; d → p, s

a. $(-g, -f)$ lies in first quadrant, then $g < 0$ and $f < 0$; also x-axis and y-axis must not cut the circle.

Solving circle and x-axis, we have $x^2 + 2gx + c = 0$, which must have imaginary roots; then $g^2 - c < 0$, then c must be positive. Also $f^2 - c < 0$.

b. If circle lies above x-axis then $x^2 + 2gx + c = 0$ must have imaginary roots, then $g^2 - c < 0$ and $c > 0$.

c. $(-g, -f)$ lies in third or fourth quadrant, then $g > 0$, Also y-axis must not cut the circle.

Solving circle and y-axis, we have $y^2 + 2gy + c = 0$, which must have imaginary roots, then $f^2 - c < 0$, then c must be positive.

d. $x^2 + 2gx + c = 0$ must have equal roots, then $g^2 = c$, hence $c > 0$.

Also $-g > 0 \Rightarrow g < 0$

3. a → r; b → s; c → q; d → p

a. Since (2, 3) lies inside circle, such chord is bisected at (2, 3), which has equation $y - 3 = -(x - 2)$

or $\quad x + y - 5 = 0 \Rightarrow a = b = 1$

b. Let P be the point (α, β), then $\alpha^2 + \beta^2 + 2\alpha + 2\beta = 0$

Midpoint of OP is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

$\therefore$ Locus of $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ is

$$4x^2 + 4y^2 + 4x + 4y = 0$$

i.e., $\quad x^2 + y^2 + x + y = 0$

$\therefore \quad 2g = 1, 2f = 1$

$\therefore \quad g + f = 1$

c. Centres of the circle are (1, 2), (5, − 6).

Equation of C_1C_2 is $y - 2 = -\frac{8}{4}(x - 1)$, i.e., $2x + y - 4 = 0$

Equation of radical axis is $8x - 16y - 56 = 0$

i.e., $\quad x - 2y - 7 = 0$

Points of intersection is (3, −2).

d. If θ is angle between tangents, then

$$\sin\frac{\theta}{2} = \frac{\text{radius}}{\text{distance between } (-3\sqrt{3}\tan\theta) \text{ and } (0, 0)}$$

$$= \frac{1}{2}$$

$$\Rightarrow \quad \frac{\theta}{2} = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{3} = 2\sqrt{3}\tan\theta = 6$$

4. a → q; b → s; c → p; d → r

a. Let length of common chord be $2a$, then

$$\sqrt{9 - a^2} + \sqrt{16 - a^2} = 5$$

$$\sqrt{16 - a^2} = 5 - \sqrt{9 - a^2}$$

$$16 - a^2 = 25 + 9 - a^2 - 10\sqrt{9 - a^2}$$

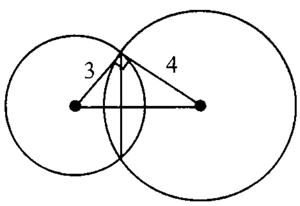


Fig. 2.189

$$10\sqrt{9 - a^2} = 18$$

$\Rightarrow \quad 100(9 - a^2) = 324$, i.e., $100a^2 = 576$

$$\therefore \quad a = \sqrt{\frac{576}{100}} = \frac{24}{10}$$

$$\therefore \quad 2a = \frac{24}{5} = \frac{k}{5} \Rightarrow k = 24$$

b. Equation of common chord is $6x + 4y + p + q = 0$

Common chord pass through centre $(-2, -6)$ of circle $x^2 + y^2 + 4x + 12y + p = 0$

$\therefore \quad p + q = 36$

c. Equation of the circle is $2x^2 + 2y^2 - 2\sqrt{2}\,x - y = 0$

Let $(\alpha, 0)$ be midpoint of a chord. Then, equation of the chord is

$$2\alpha x - \sqrt{2}(x + \alpha) - \frac{1}{2}(y + 0) = 2\alpha^2 - 2\sqrt{2}\alpha$$

Since it passes through the point $\left(\sqrt{2}, \frac{1}{2}\right)$

$\therefore \quad 2\sqrt{2}\alpha - \sqrt{2}(\sqrt{2} + \alpha) - \frac{1}{4} = 2\alpha^2 - 2\sqrt{2}\alpha$

i.e. $\quad 8\alpha^2 - 12\sqrt{2}\alpha + 9 = 0,$

i.e., $\quad (2\sqrt{2}\alpha - 3)^2 = 0$

i.e., $\quad \alpha = \frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}$

$\therefore$ Number of chords is 1.

d. Midpoint of $AB = (1, 4)$

$\therefore$ Equation perpendicular bisector of AB is $x = 1$

A diameter is $4y = x + 7$

$\therefore$ Centre of the circle is $(1, 2)$.

$\therefore$ Sides of the rectangle are 8 and 4.

$\therefore$ Area = 32

5. a → s, b → r, c → q, d → p

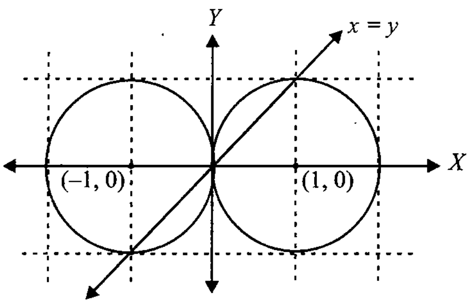


Fig. 2.190

(λ, λ) lies on the line $y = x$.

From the diagram $a \to s, b \to r, c \to q, d \to p$.

6. a → r; b → p, q; c → q, r; d → p, s

a. Radical axis of $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ is $(a_1 - a_2)x = 0$ or $x = 0$, it must touch both of the circle. Solving it with one of the circles we get $y^2 + b = 0 \Rightarrow b \le 0$

b. Radical axis of $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2y + b = 0$ is $a_1x - a_2y = 0$

Solving it with one the circles, we have $x^2 + (a_1/a_2)^2x^2 + 2a_1x + b = 0$.

This equation must have equal roots.

Hence, $4a_1^2 - 4b\left[1 + (a_1^2/a_2^2)\right] = 0$

$\Rightarrow a_1^2 - b\left[1 + (a_1^2/a_2^2)\right] = 0$

Options p and q satisfy this condition.

c. If the straight line $a_1x - by + b^2 = 0$ touches the circle $x^2 + y^2 = a_2x + by$

$$\Rightarrow \frac{\left|a_1\frac{a_2}{2} - b\frac{b}{2} + b^2\right|}{\sqrt{a_1^2 + b^2}} = \sqrt{\frac{a_2^2}{4} + \frac{b^2}{4}}$$

$$\Rightarrow \frac{\left|a_1a_2 + b^2\right|}{\sqrt{a_1^2 + b^2}} = \sqrt{a_2^2 + b^2}$$

$\Rightarrow a_1^2a_2^2 + 2b^2a_1a_2 + b^4 = a_1^2a_2^2 + a_1^2b^2 + a_2^2b^2 + b^4$

$\Rightarrow \quad b^2 = 0$ or $2a_1a_2 = a_1^2 + a_2^2$

$\Rightarrow$ Options (q) and (r)

d. Line $3x + 4y - 4 = 0$ touches the circle $(x - a_1)^2 + (y - a_2)^2 = b^2$,

$$\Rightarrow \frac{|3a_1 + 4a_2 - 4|}{5} = b$$

Integer Type

1. (6) Clearly locus of point of intersection of lines is $(x - 5)(x - 3) + (y - 2)(y + 4) = 0$

$\Rightarrow \quad x^2 + y^2 - 8x + 2y + 7 = 0$

Hence $|f + g| = |2 + (-8)| = 6$

2. (6) $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$

$\Rightarrow \quad (x^2 + y^2 - 2x - 8) - 2\lambda y = 0$, which is of the form of $S + \lambda L = 0$

All the circles passing through the point of intersection of circle $x^2 + y^2 - 2x - 8 = 0$ and $y = 0$

Solving we get $x^2 - 2x + 8 = 0$ or $(x - 4)(x + 2) = 0$

$\Rightarrow \quad A \equiv (4, 0)$ and $B \equiv (-2, 0)$

$\Rightarrow \quad$ Distance between A and B is 6

3. (8) We have $\tan^4 x + \cot^4 x + 2 = 4\sin^2 y$

$\Rightarrow \quad (\tan^2 x - \cot^2 x)^2 + 4 = 4\sin^2 y$

Now L.H.S. ≥ 4 and R.H.S. ≤ 4

$\therefore \quad \tan^2 x = 1$ and $\sin^2 y = 1 \quad \Rightarrow \quad \tan x = \pm 1$ and $\sin y = \pm 1$

But $-3 \le x \le 3$ and $-3 \le y \le 3$

$\therefore$ Acceptable values of x are $\pm\frac{\pi}{4}$ and $\pm\frac{3\pi}{4}$ and acceptable values of y are $\pm\frac{\pi}{2}$

Hence the number of points $P(x, y)$ are 8

4. (1) Let $x + 5 = 14\cos\theta$ and $y - 12 = 14\sin\theta$

$\therefore\quad x^2 + y^2 = (14\cos\theta - 5)^2 + (14\sin\theta + 12)^2$

$= 196 + 25 + 144 + 28(12\sin\theta - 5\cos\theta)$

$= 365 + 28(12\sin\theta - 5\cos\theta)$

$\therefore\quad \sqrt{x^2+y^2}\Big|_{\min} = \sqrt{365 - 28\times 13} = \sqrt{365-364} = 1$

5. (9) $3x + 6y = k$.

$\Rightarrow\quad \dfrac{3x+6y}{k} = 1$ (1)

Also $2x^2 + 2xy + 3y^2 - 1 = 0$ (2)

Now homogenising (2) with the help of (1), we gets

$\Rightarrow\quad 2x^2 + 2xy + 3y^2 - \left(\dfrac{3x+6y}{k}\right)^2 = 0$

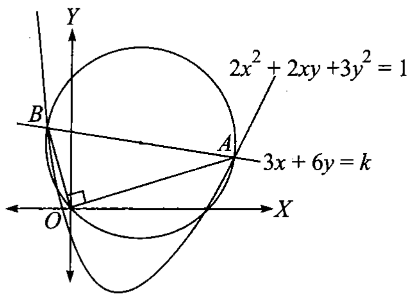


Fig. 2.191

$\Rightarrow\quad k^2(2x^2 + 2xy + 3y^2) - (3x + 6y)^2 = 0$ (3)

This is the equation of pair of straight lines OA and OB.

Now AB is diameter, then OA and OB are perpendicular.

$\Rightarrow$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow\quad (2k^2 - 9) + (3k^2 - 36) = 0$

$\Rightarrow\quad 5k^2 = 45$

$\Rightarrow\quad k^2 = 9$

6. (0) Since both the circles are symmetric about the x-axis, sum of slopes of tangent is 0.

7. (4) Radius of given circle $x^2 + y^2 + 4x - 2\sqrt{2}\,y + c = 0$ is $\sqrt{4+2-c} = \sqrt{6-c} = a$ (let)

Now radius of circle $S_1 = \dfrac{a}{\sqrt{2}}$,

radius of circle $S_2 = \dfrac{a}{2}$ and so on.

Now $a + \dfrac{a}{\sqrt{2}} + \dfrac{a}{2} + \ldots\infty = 2$ (given)

$\Rightarrow\quad a\left(\dfrac{1}{1-\dfrac{1}{\sqrt{2}}}\right) = 2$

$\Rightarrow\quad \dfrac{a\sqrt{2}}{\sqrt{2}-1} = 2$

$\Rightarrow\quad a = 2 - \sqrt{2} = \sqrt{6-c}$

$\Rightarrow\quad 4 + 2 - 4\sqrt{2} = 6 - c$

$\Rightarrow\quad c = 4\sqrt{2}$

8. (2)

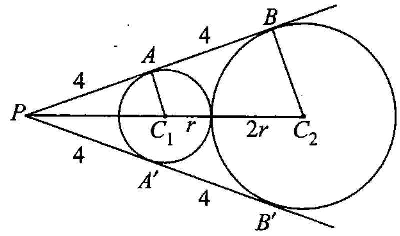


Fig. 2.192

$\dfrac{AC_1}{PA} = \dfrac{BC_2}{PB} \Rightarrow BC_2 = 2AC_1$

$PC_1 = \sqrt{16 + r^2}$

and $PC_2 = \sqrt{64 + 4r^2} = 2\sqrt{16 + r^2}$

$PC_2 - PC_1 = 3r$

$\Rightarrow\quad 2\sqrt{16+r^2} - \sqrt{16+r^2} = 3r$

$\Rightarrow\quad \sqrt{16+r^2} = 3r$

$\Rightarrow\quad 16 + r^2 = 9r^2$

$\Rightarrow\quad r^2 = 2$

9. (9)

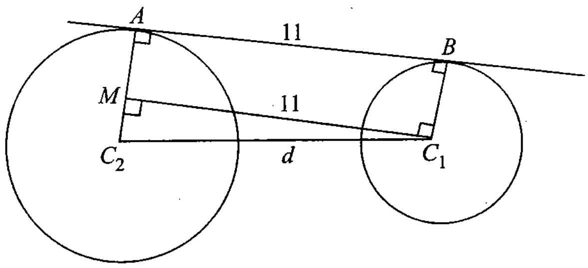


Fig. 2.193

From the figure $d^2 = 11^2 + (r_1 - r_2)^2$ (1)

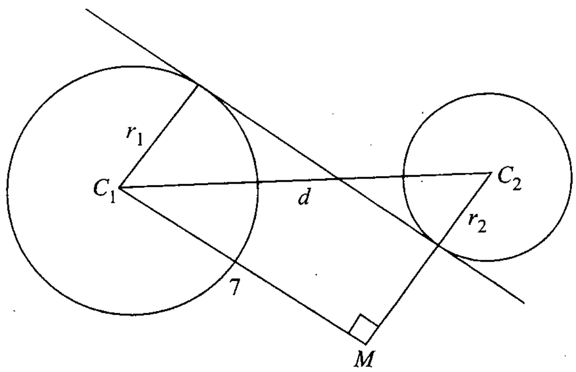


Fig. 2.194

From the figure

$d^2 = 7^2 + (r_1 + r_2)^2$ (2)

from (1) and (2), $4r_1r_2 = 72 \Rightarrow r_1r_2 = 18$

10. (1)

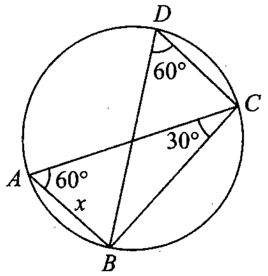


Fig. 2.2195

$\angle A = 60° = \angle D$

$AC = 2$ (given)

$\angle ABC = 90°$

$\Rightarrow x = 1$

11. (5)

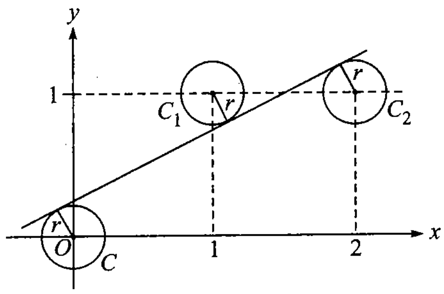


Fig. 2.196

Equation of line joining origin and centre of circle $C_2 \equiv (2, 1)$ is, $y = \frac{x}{2}$

$\Rightarrow x - 2y = 0$

let equation of common tangent is $x - 2y + c = 0$ (1)

$\therefore$ perpendicular distance from (0, 0) on this line = perpendicular distance from (1, 1)

$\Rightarrow \left|\frac{c}{\sqrt{5}}\right| = \left|\frac{c-1}{\sqrt{5}}\right|$

$\Rightarrow c = 1 - c \Rightarrow c = \frac{1}{2}$

Equation of common tangent is

$$x - 2y + \frac{1}{2} = 0 \text{ or } 2x - 4y + 1 = 0 \quad (2)$$

Perpendicular from (2, 1) on the line (2)

$r = \left|\frac{4-4+1}{\sqrt{20}}\right| = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

12. (3) θ is the angle between the tangent and the line

Circle with centre (2, –1) and $r = 3$

perpendicular from centre on $3x - 4y = 5$ is

$p = \left|\frac{6+4-5}{5}\right| = 1$

$\Rightarrow \sin(90 - \theta) = \frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{3}$

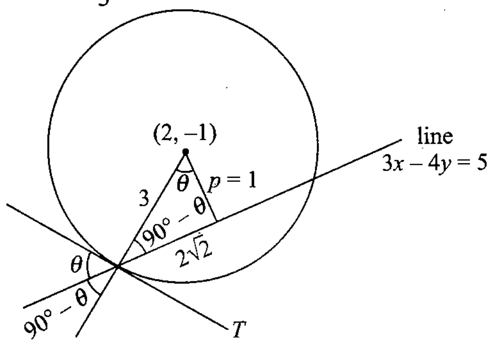


Fig. 2.197

13. (5) The equation of given circle is $x^2 + y^2 - 6x - 2py + 17 = 0$

Or $(x - 3)^2 + (y - p)^2 = (p^2 - 8)$ (1)

Also (0, 0) lies outside the circle.

Equation of director circle of $S = 0$ will be

$(x - 3)^2 + (y - p)^2 = 2(p^2 - 8)$ (2)

Tangents drawn from (0, 0) to circle (i) are perpendicular to each other

$\therefore$ (0, 0) must lie on director circle.

$\therefore$ $(0 - 3)^2 + (0 - p)^2 = 2(p^2 - 8)$

$\Rightarrow p^2 = 25$

$\Rightarrow p = \pm 5$

14. (8)

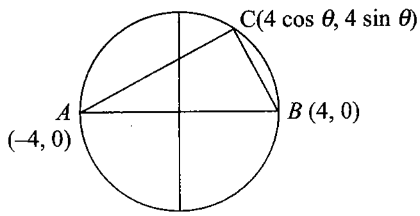


Fig. 2.198

Required area

$A = \frac{1}{2} \cdot 8 \cdot 4 \sin\theta = |16 \sin\theta|$

Now area is integer then the possible values of

$\sin\theta$ are $\frac{1}{16}, \frac{2}{16}, \ldots\ldots \frac{15}{16}$

i.e. 15 points in each quadrant

$\Rightarrow$ 60 + 2 more with $\sin\theta = 1$

$\Rightarrow N = 62$

15. **(1)** $x^2 + y^2 + (3 + \sin\beta)x + (2\cos\alpha)y = 0$ (1)

$x^2 + y^2 + (2\cos\alpha)x + 2cy = 0$ (2)

Since both the circles are passing through the origin (0, 0), equation of tangent at (0, 0) will be same tangent at (0, 0) to circle (1),

$(3 + \sin\beta)x + (2\cos\alpha)y = 0$ (3)

tangent at (0, 0) to circle (2),

$(2\cos\alpha)x + 2cy = 0$

$\therefore$ (1) and (2) must be identical

comparing (1) and (2)

$$\frac{3+\sin\beta}{2\cos\alpha} = \frac{2\cos\alpha}{2c}$$

$$\Rightarrow c = \frac{2\cos^2\alpha}{3+\sin\beta}$$

$\Rightarrow c_{max} = 1$ when $\sin\beta = -1$ and $\alpha = 0$

16. **(4)** The radical axis bisects the common tangent BD.

Hence M is the mid point of BD

Let $C(a, b)$

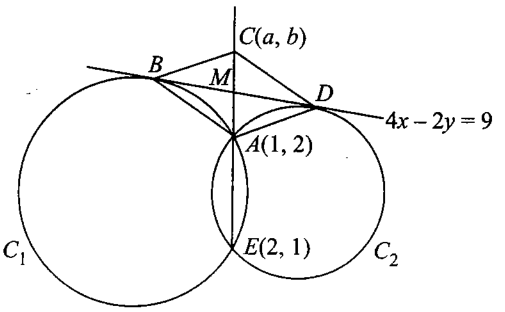


Fig. 2.199

Now $C(a, b)$ lies on common chord AE which is $y - 2 = -1(x - 1)$ or $x + y = 3$

$\therefore \quad a + b = 3$ (1)

Also M $\left(\frac{a+1}{2}, \frac{b+2}{2}\right)$ lies on $4x - 2y = 9$

$$\Rightarrow 4\left(\frac{a+1}{2}\right) - 2\left(\frac{b+2}{2}\right) = 9$$

$\Rightarrow 2a + 2 - b - 2 = 9$

$\Rightarrow 2a - b = 9$ ....(2)

Solving (1) and (2) $a = 4$ and $b = -1$

$\Rightarrow a + b = 3$

17. **(4)** Let r be the radius of required circle.

Now, if two circles touches each other then distance between their centres $= |r \pm 2| = 5$ (given)

$\therefore r = 3, 7$

Archives

Subjective Type

1. The given circle is

$$x^2 + y^2 - 2x - 4y - 20 = 0$$

whose centre is (1, 2) and radius = 5

Radius of required circle is also 5.

Let its centre be $C_2(\alpha, \beta)$.

Both the circles touch each other at $P(5, 5)$.

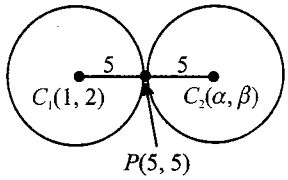


Fig. 2.200

It is clear from the figure that $P(5, 5)$ is the midpoint of C_1C_2.

Therefore, we should have

$$\frac{1+\alpha}{2} = 5 \text{ and } \frac{2+\beta}{2} = 5$$

$\Rightarrow \quad \alpha = 9$ and $\beta = 8$

Therefore, centre of required circle is (9, 8) and its equation is

$$(x - 9)^2 + (y - 8)^2 = 25$$

$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0$

2. The point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ is

$$\left(\frac{1-c^2}{2+c-3c^2}, \frac{c-1}{(2+c-3c^2)}\right)$$

If (h, k) is the centre of the required circle, then

$$h = \lim_{c\to 1} \frac{c-1}{2+c-3c^2}$$

$$= \lim_{c\to 1} \frac{(1-c)(1+c)}{(1-c)(2+3c)}$$

$$= \lim_{c\to 1} \frac{1+c}{2+3c} = \frac{2}{5}$$

and

$$k = \lim_{c\to 1} \frac{c-1}{5(2+c-3c^2)}$$

$$= \lim_{c\to 1} \frac{-1}{5(2+3c)}$$

$$= \frac{-1}{25}$$

Therefore, centre is $\left(\frac{2}{5}, -\frac{1}{25}\right)$.

Also circle passes through (2, 0)

$\therefore$ Radius $= \sqrt{\left(2-\frac{2}{5}\right)^2 + \left(\frac{1}{25}\right)^2} = \frac{\sqrt{1601}}{25}$

$\therefore$ Equation of required circle is

$$\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{1601}{625}$$

$\Rightarrow 25x^2 + 25y^2 - 20x + 2y - 60 = 0$

3. The equation of circle is

$x^2 + y^2 - 2x - 4y - 20 = 0$

Centre (1, 2) and radius $= \sqrt{1+4+20} = 5$

Equation of tangent at (1, 7) is

$x \cdot 1 + y \cdot 7 - (x+1) - 2(y+7) - 20 = 0$

$\Rightarrow \quad y - 7 = 0 \quad$ (i)

Similarly, equation of tangent at (4, –2) is

$4x - 2y - (x+4) - 2(y-2) - 20 = 0$

$\Rightarrow \quad 3x - 4y - 20 = 0 \quad$ (ii)

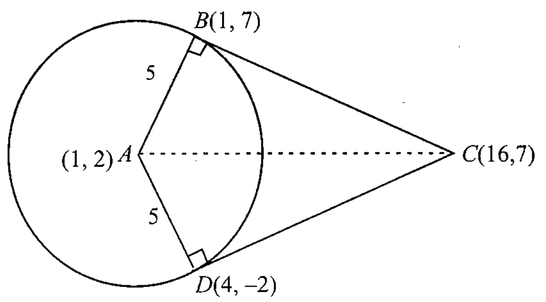


Fig. 2.201

For point C, solving Eqs. (i) and (ii), we get $x = 16$, $y = 7$

$\therefore C(16, 7)$

Now, clearly ar(quad. $BCDA$)

$= 2\text{ar}(\Delta ABC)$

$= 2 \times \frac{1}{2} \times AB \times BC$

$= AB \times BC$

where AB = radius of circle = 5

and BC = length of tangent from C to circle

$= \sqrt{16^2 + 7^2 - 32 - 28 - 20}$

$= 15$

$\therefore$ ar(quad $ABCD$) $= 5 \times 15 = 75$ sq. units

4. See the solution of Example 2.22.

5. From the diagram,

$CM \perp CP$

$\Rightarrow \left(\frac{y_1}{x_1}\right)\left(\frac{y_1 - k}{x_1 - h}\right) = -1$

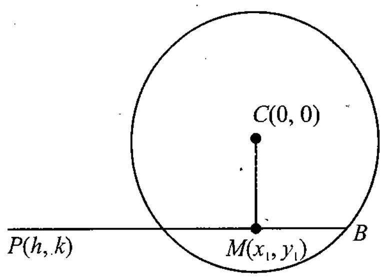


Fig. 2.202

$\Rightarrow$ Required locus is

$x^2 + y^2 = hx + ky$

6. See the solution of Example 2.13.

7. Let equation of tangent PAB be $5x + 12y - 10 = 0$ and that of PXY be $5x - 12y - 40 = 0$.

Now let centre of circles C_1 and C_2 be $C(h, k)$.

Let CM ^ PAB, then CM = radius of $C_1 = 3$

Also C_2 makes on intercept of length 8 units on PAB $\Rightarrow AM = 4$

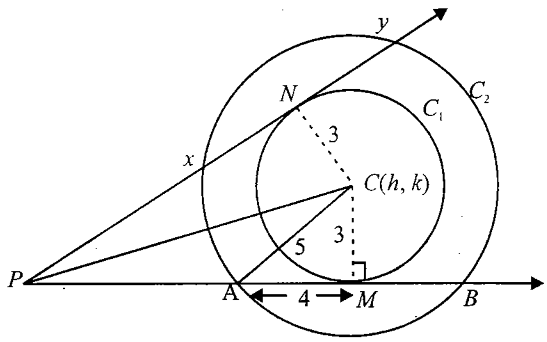


Fig. 2.203

Then in ΔAMC, we get $AC = \sqrt{4^2 + 3^2} = 5$

$\therefore$ Radius of C_2 is = 5 units

Also as $\quad 5x + 12y - 10 = 0 \quad$ (i)

and $\quad 5x - 12y - 40 = 0 \quad$ (ii)

are tangents to C_1, length of perpendicular from C to AB = 3 units

$\therefore$ We get $\frac{5h + 12k - 10}{13} = \pm 3$

$\Rightarrow \quad 5h + 12k - 49 = 0 \quad$ (i)

or $\quad 5h + 12k + 29 = 0 \quad$ (ii)

Similarly, $\frac{5h - 12k - 40}{13} = \pm 3$

$\Rightarrow \quad 5h - 12k - 79 = 0 \quad$ (iii) or $\qquad 5h - 12k - 1 = 0 \qquad$ (iv)

As C lies in first quadrant, therefore

h, k are + ve.

∴ Equation (ii) is not possible.

Solving (i) and (iii) we get $h = 64/5$, $k = -5/4$ which is also not possible.

Now solving (i) and (iv), we get $h = 5$, $k = 2$.

Thus, centre for C_2 is (5, 2) and radius 5.

Hence, equation of C_2 is $(x-5)^2 + (y-2)^2 = 5^2$

8. From the figure in ΔPQS

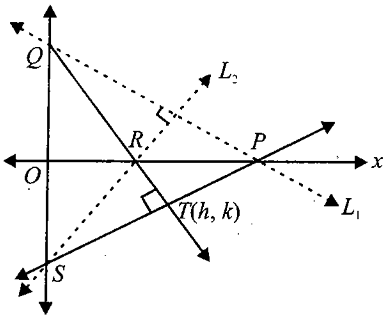


Fig. 2.204

$PQ \perp SR$, $PR \perp QS$

then we have $QT \perp PS$ (as altitudes of triangle are concurrent)

⇒ point Q, R, T and S are concyclic

Hence, locus of T is a circle which passes through the origin.

9. See problem no. 4 in concept application exercise 2.6.

10. The equation of the given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

Let $P(h, k)$ be the foot of perpendicular drawn from origin to the chord LM of circle S.

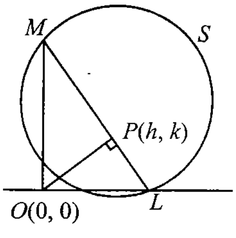


Fig. 2.205

$$\text{Slope } OP = \frac{k}{h}$$

$$\therefore \quad \text{Slope of } LM = -\frac{h}{k}$$

∴ Equation of LM is: $y - k = -\frac{h}{k}(x - h)$

$$\Rightarrow \quad ky - k^2 = -kx + h^2$$

$$\Rightarrow \quad hx + ky = h^2 + k^2 \qquad \text{(ii)}$$

Now the combined equations of lines joining the points of intersection of Eqs. (i) and (ii) to origin can be obtained by making Eq. (i) homogeneous with the help of Eqs. (ii) as follows:

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{hx + ky}{h^2 + k^2}\right) + c\left(\frac{hx + ky}{h^2 + k^2}\right)^2 = 0$$

$\Rightarrow (h^2 + k^2)^2 (x^2 + y^2) + (h^2 + k^2)(2gx + 2fy)(hx + ky) + c(hx + ky)^2 = 0 \qquad$ (iii)

Above represents the combined equation of OL and OM.

But $\qquad \angle LOM = 90°$

∴ From Eq. (iii), we must have

Coeff. of x^2 + coeff. of $y^2 = 0$

$\Rightarrow (h^2 + k^2)^2 + 2gh(h^2 + k^2) + ch^2 + (h^2 + k^2)^2 + 2fk(h^2 + k^2) + ck^2 = 0$

$$\Rightarrow \quad h^2 + k^2 + gh + fk + \frac{c}{2} = 0$$

∴ Locus of (h, k) is $x^2 + y^2 + gx + fy + c = 0$

11. Given that $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$, $i = 1, 2, 3, 4$ are four distinct points on a circle.

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

As the point $\left(mi, \frac{1}{mi}\right)$ lies on it, therefore we have

$$m^2 + \frac{1}{m^2} + 2gm + \frac{2f}{m} + c = 0$$

$\Rightarrow m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

Since m_1, m_2, m_3, m_4 are roots of this equation, therefore product of roots = 1

$$\Rightarrow \quad m_1 m_2 m_3 m_4 = 1$$

12. Let AB be the length of chord intercepted by circle on $y + x = 0$.

Let CM be perpendicular to BA from centre $C(h, k)$

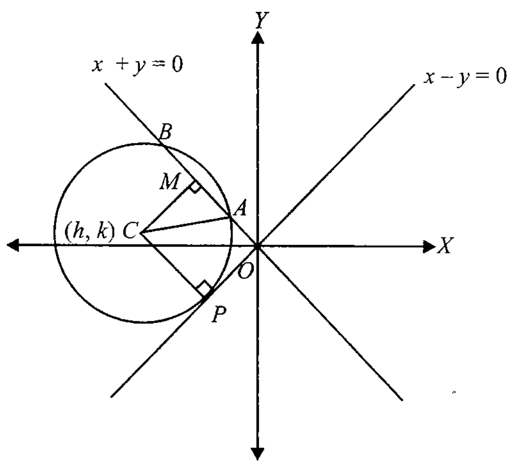


Fig. 2.206

Also $y - x = 0$ and $y + x = 0$ are perpendicular to each other.

$\therefore$ $OPCM$ is a rectangle.

$\therefore$ $CM = OP = 4\sqrt{2}$.

Let r be the radius of circle.

Also $AM = 3\sqrt{2}$

$\therefore$ In ΔCAM, $AC^2 = AM^2 + MC^2$

$\Rightarrow$ $r^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2$

$\Rightarrow$ $r^2 = (5\sqrt{2})^2$

$\Rightarrow$ $r = 5\sqrt{2}$

Also coordinates of P are

$(0 - 4\sqrt{2}\cos 45°, 0 - 4\sqrt{2}\sin 45°)$ or $(-4, -4)$

Slope of PC is -1 and $CP = 5\sqrt{2}$

$\Rightarrow$ coordinates of C are

$(-4 + 5\sqrt{2}\cos 135°, -4 + 5\sqrt{2}\sin 135°)$ or $(-9, 1)$

Hence, equation of circle is $(x + 9)^2 + (y - 1)^2 = 50$

or $x^2 + y^2 + 18x - 2y + 32 = 0$.

13.

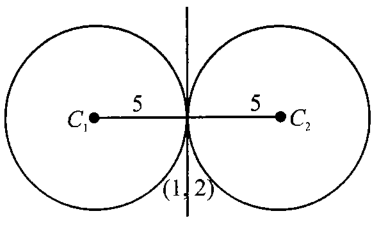


Fig. 2.207

Since the circles touch at (1, 2), the line of centres and the point (1, 2) are collinear and $\perp$ to the common tangent $4x + 3y = 10$. Also centres are at a distance 5 units from (1, 2). Now slope of the line $\perp$ to common tangent is $3/4 = \tan\alpha$

$\therefore$ Equation of line $\perp$ to common tangent through (1, 2) (in symmetrical form) is given by

$$= \frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \pm 5$$

$$\Rightarrow \quad \frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

$\Rightarrow$ Centres are (5, 5) and (– 3, – 1).

Hence, equations of the circles are

$(x - 5)^2 + (y - 5)^2 = 25$ and $(x + 3)^2 + (y + 1)^2 = 25$

i.e. $x^2 + y^2 - 10x - 10y + 25 = 0$ and $x^2 + y^2 + 6x - 2y - 15 = 0$

14.

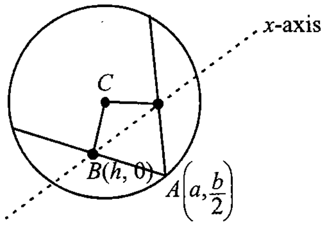


Fig. 2.208

Given circle is $2x(x - a) + y(2y - b) = 0$ $(a, b \neq 0)$

or $x^2 + y^2 - ax - \frac{b}{2}y = 0$ (i)

Its centre is $C\left(\frac{a}{2}, \frac{b}{4}\right)$

Also point $A\left(a, \frac{b}{2}\right)$ lies on the circle.

Since chord is bisected at point $B(h, 0)$ on the x-axis we have (slope BC) $\times$ (slope of AB) $= -1$

$$\Rightarrow \left(\frac{\frac{b}{4} - 0}{\frac{a}{2} - h}\right)\left(\frac{\frac{b}{2} - 0}{a - h}\right) = -1$$

$$\Rightarrow h^2 - 3\frac{a}{2}h + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Since two such chords exist, then above equation must have two distinct real roots, for which its discriminant must be zero.

$$\Rightarrow \frac{9a^2}{4} - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\Rightarrow a^2 > 2b^2$$

15. See passage 3 in Comprehension Type problems.

16. The given circles are

$$x^2 + y^2 - 4x - 2y + 4 = 0 \quad \text{(i)}$$

$$x^2 + y^2 - 12x - 8y + 36 = 0 \quad \text{(ii)}$$

with centres $C_1(2, 1)$ and $C_2(6, 4)$ and radii 1 and 4, respectively.

Also $C_1C_2 = 5$

As $r_1 + r_2 = C_1C_2$

$\Rightarrow$ Two circles touch each other externally at P.

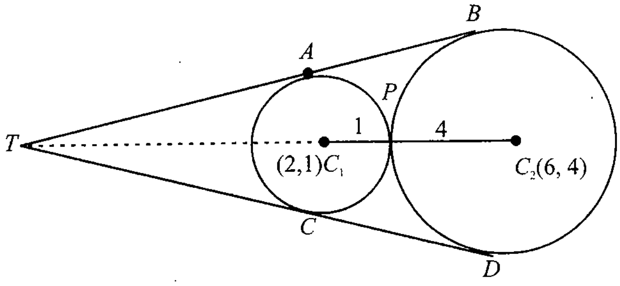


Fig. 2.209

Clearly P divides C_1C_2 in the ratio 1 : 4.

Therefore, coordinates of P are

$$\left(\frac{1\times 6+4\times 2}{1+4}, \frac{1\times 4+4\times 1}{1+4}\right)$$

$$=\left(\frac{14}{5}, \frac{8}{5}\right)$$

Let AB and CD be two common tangents of given circles, meeting each other at T.

Then T divides C_1C_2 externally in the ratio 1 : 4.

Hence, $T\equiv\left(\frac{1\times 6-4\times 2}{1-4}, \frac{1\times 4-4\times 1}{1-4}\right)\equiv\left(\frac{2}{3}, 0\right)$

Let m be the slope of the tangent, then equation of tangent through (2/3, 0) is

$$y-0=m\left(x-\frac{2}{3}\right)$$

$$\Rightarrow \quad y-mx+\frac{2}{3}m=0$$

Now, length of perpendicular from (2, 1) to the above tangent is radius of the circle

$$\therefore \quad \left|\frac{1-2m+\frac{2}{3}m}{\sqrt{m^2+1}}\right|=1$$

$$\Rightarrow \quad (3-4m)^2=9(m^2+1)$$

$$\Rightarrow \quad 9-24m+16m^2=9m^2+9$$

$$\Rightarrow \quad 7m^2-24m=0$$

$$\Rightarrow \quad m=0, \frac{24}{7}$$

Thus, the equations of the tangents are $y=0$ and $7y-24x+16=0$.

17. Let the given point be $(p, \bar{p})\equiv\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$

Therefore, the equation of the circle becomes

$$x^2+y^2-px-\bar{p}x=0$$

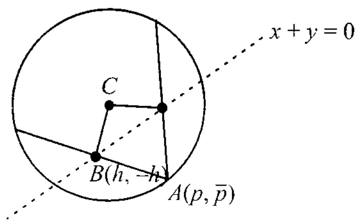


Fig. 2.210

Since the chord is bisected by the line $x+y=0$, its midpoint can be chosen as $(h, -h)$. From the figure, $BC\perp AB$

$$\Rightarrow \quad \left(\frac{\frac{\bar{p}}{2}+h}{\frac{p}{2}-h}\right)\left(\frac{\bar{p}+h}{p-h}\right)=-1$$

$$\Rightarrow \quad \left(\frac{\bar{p}+2h}{p-2h}\right)\left(\frac{\bar{p}+h}{p-h}\right)=-1$$

$$\Rightarrow \quad \bar{p}^2+3\bar{p}h+2h^2=-p^2+3ph-2h^2$$

$$\Rightarrow \quad 4h^2-3(\bar{p}-p)h+\bar{p}^2+p^2=0$$

Since two such chords exist, then above equation must have two distinct real roots, for which its discriminant must be zero.

$$\Rightarrow \quad 9(\bar{p}-p)^2-16(\bar{p}^2+p^2)>0$$

$$\Rightarrow \quad 7(\bar{p}^2+p^2)+18\,\bar{p}p<0$$

$$\Rightarrow \quad (7/4)(2+4a^2)+(18/4)(1-2a^2)<0$$

$$\Rightarrow \quad 32-8a^2<0$$

v $\quad a^2>4$

$$\Rightarrow \quad a\in(-\infty, -2)\cup(2, \infty)$$

18. The given curve is $ax^2+2hxy+by^2=1$ (i)

Let the point P not lying on curve (i) be (x_1, y_1).

Let θ be the inclination of line through P which intersects the given curve at Q and R.

The equation of line through P is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

$$\Rightarrow \quad x=\cos\theta+x_1,$$

$$y=r\sin\theta+y_1$$

For point Q or R, above point must lie on curve (i).

$\Rightarrow a[x_1+r\cos\chi]^2+2h[x_1+r\cos\theta][y_1+r\sin\theta]+b[y_1+r\sin\theta]^2=1$

$\Rightarrow (a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)r^2+2(ax_1\cos\theta+hx_1\sin\theta+hy_1\cos\theta+by_1\sin\theta)r$

$$+\left(ax_1^2+2hx_1y_1+by_1^2-1\right)=0$$

It is a quadratic in r giving two values of r for PQ and PR. Therefore, product of roots $=PQ.PR$

$$\Rightarrow \quad PQ.\,PR=\frac{ax_1^2+2hx_1y_1+by_1^2-1}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}$$

Here, $ax_1^2+2hx_1y_1+by_1^2-1\neq 0$ as (x_1, y_1) does not lie on curve (i).

Also denominator

$=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta$

$=a+2h\sin\theta\cos\theta+(b-a)\sin^2\theta$

$=a+h\sin 2\theta+\frac{(b-a)}{2}(1-2\cos\theta)$

$=\left(\frac{a+b}{2}\right)+h\sin 2\theta+\left(\frac{a-b}{2}\right)\cos 2\theta$

which is independent of θ if $h=0$, and $\frac{a-b}{2}=0$

$\Rightarrow \quad h=0$ and $a=b$

Hence, given equation is a circle.

19. See Example 2.4.

20. Let r be the radius C_1 and $2r$ be of C_2, then

$$OA = OB = r$$

and $$OP = 2r$$

Since PA and PB are tangents to C_1

$\therefore$ $$\angle OAP = \angle OBP = 90^\circ$$

Let OP meet C_1 at G

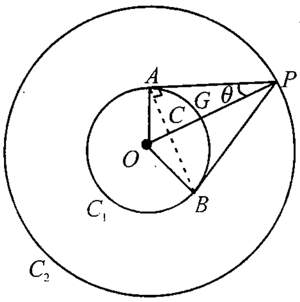


Fig. 2.211

Let $\angle OPA$ = q, then $\sin\theta = \frac{OA}{OP} = \frac{r}{2r} = \frac{1}{2}$

$\Rightarrow$ $$\theta = 30^\circ$$

$\Rightarrow$ $$\angle AOP = 60^\circ$$

In $$\Delta OAC, \cos 60^\circ = \frac{OC}{OA}$$

$\Rightarrow$ $$\frac{1}{2} = \frac{OC}{r} \Rightarrow OC = \frac{r}{2}$$

$\therefore$ $$PC = 2r - \frac{r}{2} = \frac{3r}{2}$$

Also $$CG = r - OC = r - \frac{r}{2} = \frac{r}{2}$$

Clearly $CG = \frac{1}{3}PC \Rightarrow G$ is centroid of ΔABP

21. The given circle is $x^2 + y^2 = 1$ (i)

Centre $O(0, 0)$ radius = 1

Let T_1 and T_2 be the tangents drawn from (–2, 0) to the circle (i)

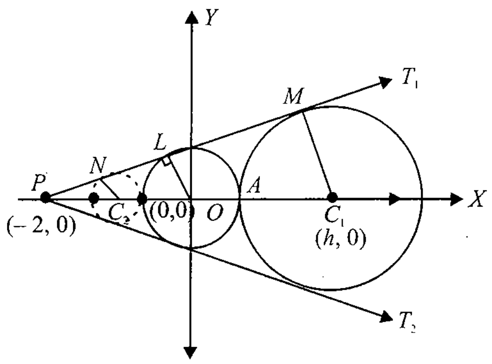


Fig. 2.212

From the figure, ΔPLO and ΔPMC_1 are similar

$\Rightarrow$ $$\frac{OL}{OP} = \frac{C_1M}{C_1P}$$

$\Rightarrow$ $$\frac{1}{2} = \frac{r_1}{h_1 + 2}$$

$\Rightarrow$ $$2r_1 = h_1 + 2 \quad \text{(ii)}$$

Also circles are touching externally,

$\Rightarrow$ $$h_1 = r_1 + 1 \quad \text{(iii)}$$

From (ii) and (iii), $r_1 = 3$ and $h_1 = 4$

Hence, equation of circle is

$$(x - 3)^2 + y^2 = 9 \quad \text{(iv)}$$

Also ΔPLO and ΔPNC_2 are similar

$\Rightarrow$ $$\frac{OL}{OP} = \frac{C_2N}{C_2P}$$

$\Rightarrow$ $$\frac{1}{2} = \frac{r_2}{2 - h_2}$$

$\Rightarrow$ $$2r_2 = 2 - h_2 \quad \text{(v)}$$

Also circles are touching externally,

$\Rightarrow$ $$h_2 = 1 + r_2 \quad \text{(vi)}$$

From (v) and (vi), $r_2 = \frac{1}{3}$ and $h_2 = \frac{4}{3}$

Hence, equation of circle is

$$\left(x - \frac{4}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2 \quad \text{(vii)}$$

Since circles (i) and (iv) are two touching circles, they have three common tangents T_1, T_2 and $x - 1 = 0$

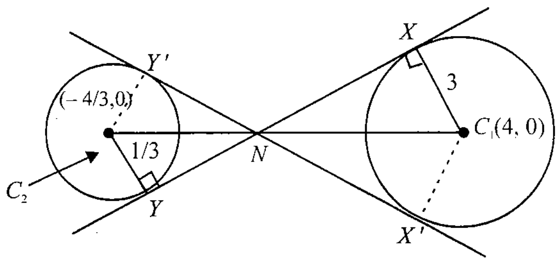


Fig. 2.213

Similarly common tangents of circles (i) and (vii) are T_1, T_2 and $x = -1$

For the circles (iv) and (vii), there will be four common tangents two direct and two indirect.

Two common direct tangents are T_1 and T_2

Let us find two common indirect tangents.

$$\frac{C_1N}{C_2N} = \frac{3}{1/3} = 9$$

$\Rightarrow N$ divides C_1C_2 in the ratio 9 : 1

$$\therefore N\left(\frac{9 \times (-4/3) + 1 \times 4}{10}, 0\right) = \left(\frac{-4}{5}, 0\right)$$

Any line through N is $y = m\left(x + \frac{4}{5}\right)$

or $\quad 5mx - 5y + 4m = 0$

If it is tangent to circle (iii), then $\left|\dfrac{20m + 4m}{\sqrt{25m^2 + 25}}\right| = 3$

$\Rightarrow \quad 24m = 15\sqrt{m^2 + 1}$

$\Rightarrow \quad 64m^2 = 25m^2 + 25$

$\Rightarrow \quad 39m^2 = 25$

$\Rightarrow \quad m = \pm 5/\sqrt{39}$

$\therefore$ Required tangents are $\quad y = \pm \dfrac{5}{\sqrt{39}}\left(x + \dfrac{4}{5}\right)$

22. The equation $2x^2 - 3xy + y^2 = 0$ represents pair of tangents OA and OA'.

Let angle between these two tangents be 2θ.

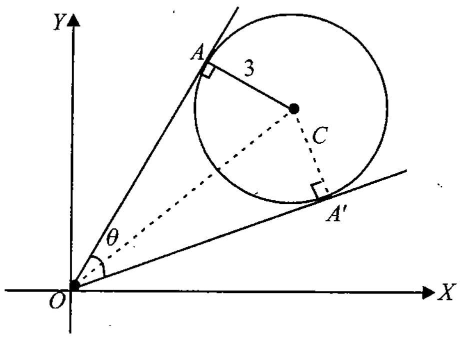


Fig. 2.214

Then, $\quad \tan 2\theta = \dfrac{2\sqrt{\left(\frac{-3}{2}\right)^2 - 2 \times 1}}{2 + 1}$

$\Rightarrow \quad \dfrac{2\tan\theta}{1 - \tan^2\theta} = \dfrac{1}{3}$

$\Rightarrow \quad \tan^2\theta + 6\tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = \dfrac{-6 \pm \sqrt{36 + 4}}{2}$

$= -3 \pm \sqrt{10}$

As θ is acute, $\therefore \quad \tan\theta = \sqrt{10} - 3$

Now we know that line joining the point through which tangents are drawn to the centre bisects the angle between the tangents.

$\therefore \quad \angle AOC = \angle A\angle OC = \theta.$

In $DOAC$, $\quad \tan\theta = \dfrac{3}{OA}$

$\Rightarrow \quad OA = \dfrac{3}{\sqrt{10} - 3} \times \dfrac{\sqrt{10} + 3}{\sqrt{10} + 3}$

$\therefore \quad OA = 3(3 + \sqrt{10})$

23. The given circle is $x^2 + y^2 = r^2$ From point (6, 8) tangents are drawn to this circle.

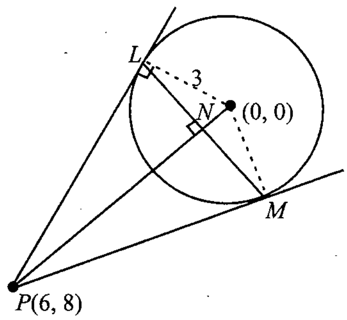


Fig. 2.215

Then, length of tangent

$PL = \sqrt{6^2 + 8^2 - r^2}$

$= \sqrt{100 - r^2}$

Also, equation of chord of contact LM is $6x + 8y - r^2 = 0$

PN = length of $\perp$ from P to LM

$= \dfrac{36 + 64 - r^2}{\sqrt{36 + 64}}$

$= \dfrac{100 - r^2}{10}$

Now in right angled ΔPLN,

$LN^2 = PL^2 - PN^2$

$= (100 - r^2) - \dfrac{(100 - r^2)^2}{100}$

$= \dfrac{(100 - r^2)r^2}{100}$

$\Rightarrow \quad LN = \dfrac{r\sqrt{100 - r^2}}{10}$

$\therefore \quad LM = \dfrac{r\sqrt{100 - r^2}}{5}$

$(\because LM = 2LN)$

$\therefore \quad$ Area of $DPLM = A \times \dfrac{1}{2} \times LM \times PN$

$= \dfrac{1}{2} \times \dfrac{r\sqrt{100 - r^2}}{5} \times \dfrac{100 - r^2}{10}$

$= \dfrac{1}{100}[r(100 - r^2)^{3/2}]$

For max value of A, we should have

$\dfrac{dA}{dr} = 0$

$\Rightarrow \quad \dfrac{1}{100}\left[(100 - r^2)^{3/2} + r\dfrac{3}{2}(100 - r^2)^{1/2}(-2r)\right] = 0$

$\Rightarrow \quad (100 - r^2)^{1/2}[100 - r^2 - 3r^2] = 0$

$\Rightarrow \quad r = 10$ or $r = 5$

but $r = 10$ gives length of tangent PL

$\therefore \quad r \neq 10$

Hence, $\quad r = 5$

24. We are given that line $2x + 3y + 1 = 0$ touches a circle $S = 0$ at $(1, -1)$.

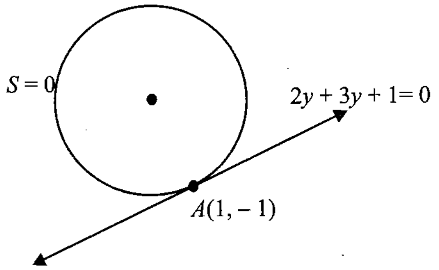


Fig. 2.216

So equation of this circle can be given by

$(x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0, \lambda \in R$

[Here $(x-1)^2 + (y+1)^2 = 0$ represents a point circle at $(1, -1)$]

or $\quad x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0 \quad$ (i)

But given that this circle is orthogonal to the circle, the extremities of whose diameter are (0, 3) and (– 2, –1)

i.e., $x(x+2) + (y-3)(y+1) = 0$

or $\quad x^2 + y^2 + 2x - 2y - 3 = 0 \quad$ (ii)

Applying the condition of orthogonality for Eqs. (i) and (ii), we get

$2(\lambda - 1) \times 1 + 2\left(\frac{3\lambda + 2}{2}\right) \times (-1)$

$= \lambda + 2 + (-3) \; [2g_1g_2 + 2f_1f_2 = c_1 + c_2]$

$\Rightarrow \quad 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

$\Rightarrow \quad 2\lambda = -3$

$\Rightarrow \quad \lambda = -\frac{3}{2}$

Substituting this value of l in Eq. (i), we get the required circle as

$$x^2 + y^2 - 5x - \frac{5}{2}y + \frac{1}{2} = 0$$

or $\quad 2x^2 + 2y^2 - 10x - 5y + 1 = 0$

25.

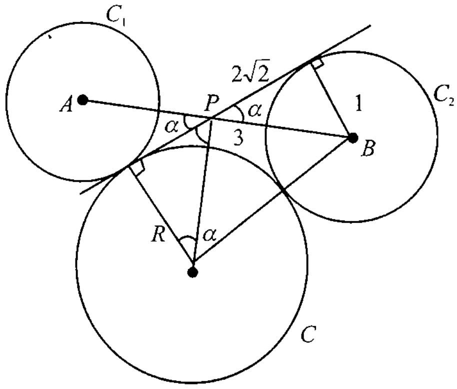


Fig. 2.217

$$\cos\alpha = \frac{2\sqrt{2}}{3}$$

$$\sin\alpha = \frac{1}{3}$$

$$\tan\alpha = \frac{2\sqrt{2}}{R}$$

$$\Rightarrow \quad R = \frac{2\sqrt{2}}{\tan a} = 8 \text{ units}$$

Objective Type

Fill in the blanks

1. As P lies on a circle and A and B two points in the plane such that $\frac{PA}{PB} = k$.

Then, k can be any positive real number except 1 as otherwise P will lie on perpendicular bisector of AB.

2. Solving line $4x - 3y - 10 = 0 \quad$ (i)

and circle $x^2 + y^2 - 2x + 4y - 20 = 0 \quad$ (ii)

Solving Eqs. (i) and (ii), we get

$$\left(\frac{3y+10}{4}\right)^2 + y^2 - 2\left(\frac{3y+10}{4}\right) + 4y - 20 = 0$$

$\Rightarrow \quad 9y^2 + 60y + 100 + 16y^2 - 24y - 80 + 64y - 320 = 0$

$\Rightarrow \quad 25y^2 + 100y - 300 = 0$

$\Rightarrow \quad y^2 + 4y - 12 = 0$

$\Rightarrow \quad y = 2, -6$

$\Rightarrow \quad x = 4, -2$

Hence, points are (4, 2) and (– 2, – 6).

3. Since given lines $6x - 8y + 8 = 0$ and $6x - 8y - 7 = 0$ are parallel and touching the circle, diameter of the circle is distance between parallel lines

$$= \left|\frac{8+7}{\sqrt{36+64}}\right| = \frac{15}{10} = \frac{3}{2}$$

$\therefore \quad$ Radius of circle $= \frac{1}{2}(AB) = \frac{3}{4}$

4. The equation of circle is, $x^2 + y^2 - 4x - 2y - 11 = 0$.

It's centre is (2, 1), radius $= \sqrt{4 + 1 + 11} = 4 = BC$

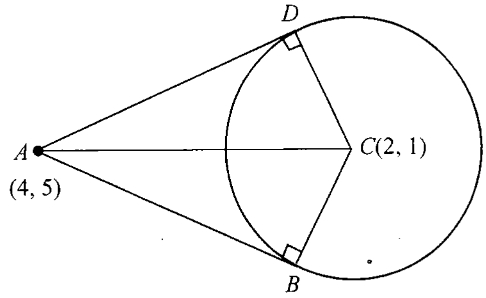


Fig. 2.218

Length of tangent from the point (4, 5) is

$$= \sqrt{16 + 25 - 16 - 10 - 11}$$

$$= \sqrt{4} = 2 = AB$$

$\therefore$ Area of quadrilateral $ABCD$

$$= 2\ (\text{Area of } \Delta ABC)$$

$$= 2 \times \frac{1}{2} \times AB \times BC$$

$$= 2 \times \frac{1}{2}\ 2 \times 4$$

$$= 8 \text{ sq. units}$$

5. Origin $O(0, 0)$ satisfies the circle.

If midpoint of the chord from origin is $P(h, k)$, then P is midpoint of OR, where $R(2h, 2k)$ lies on the circle.

Hence, $(2h - 1)^2 + (2k)^2 = 1$

or $\quad x^2 + y^2 - x = 0$

6. The equation of two circle are

$S_1 \equiv x^2 + y^2 - \frac{2}{3}x + 4y - 3 = 0 \quad$ (i)

and $S_2 \equiv x^2 + y^2 + 6x + 2y - 15 = 0 \quad$ (ii)

Now we know equation of common chord of two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$

$\Rightarrow \quad \frac{20x}{3} - 2y - 12 = 0$

$\Rightarrow \quad 10x - 3y - 18 = 0$

7. The equation of circle is

$x^2 + y^2 + 4x - 6y + 9 = 0 \quad$ (i)

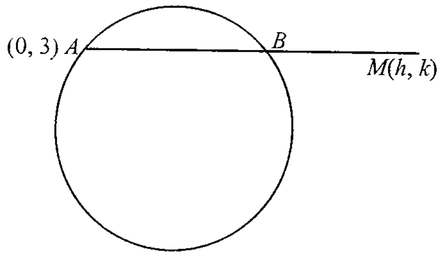


Fig. 2.219

Given that $\quad AM = 2AB$

$\Rightarrow \quad AB = BM$

Let the co-ordiantes of M be (h, k)

Then B is midpoint of AM

$\therefore \quad B\left(\frac{0+h}{2}, \frac{3+k}{2}\right) \equiv \left(\frac{h}{2}, \frac{k+3}{2}\right)$

As B lies on circle (i)

$\therefore \quad \left(\frac{h}{2}\right)^2 + \left(\frac{k+3}{2}\right)^2 + 4\left(\frac{h}{2}\right) - 6\left(\frac{k+3}{2}\right) + 9 = 0$

$\Rightarrow \quad h^2 + k^2 + 6k + 9 + 8h - 12k - 36 + 36 = 0$

$\Rightarrow \quad h^2 + k^2 + 8h - 6k + 9 = 0$

$\therefore \quad$ Locus of (h, k) is $x^2 + y^2 + 8x - 6y + 9 = 0$

8.

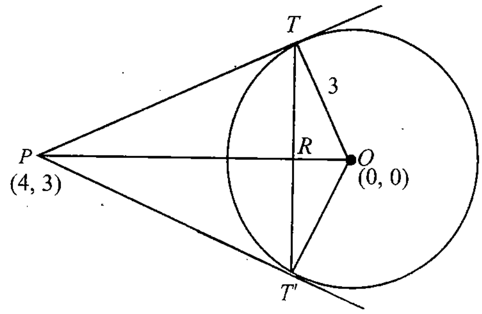


Fig. 2.220

Let R be the point of intersection of OP and TT'.

Equation of chord of contact TT' is

$4x + 3y = 9 \quad$ (i)

Now, OR = length of the perpendicular from O to TT'

$= \left|\frac{4 \times 0 + 3 \times 0 - 9}{\sqrt{4^2 + 3^2}}\right| = \frac{9}{5}$

OT = radius of circle = 3

$\therefore \quad TR = \sqrt{OT^2 - OR^2} = \sqrt{9 - \frac{81}{25}} = \frac{12}{5}$

Also $\quad OP = 5$

$\therefore \quad PR = OP - OR = 5 - \frac{9}{5} = \frac{16}{5}$

Area of the $\quad \Delta PTT' = PR \times TR$

$= \frac{16}{5} \times \frac{12}{5} = \frac{192}{25}$

9. See the solution of similar question 7 in the concept application exercise 2.4.

10. See solution of problem 89 in Objective Type.

11. The given lines are $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ which meet x-axis at $A\left(-\frac{1}{\lambda}, 0\right)$ and $B(-3, 0)$ and y-axis at $C(0, 1)$ and $D\left(0, \frac{3}{2}\right)$, respectively.

Then, we must have

$OA \times OB = OC \times OD$

$\Rightarrow \quad \left(\frac{1}{\lambda}\right)(3) = 1 \times \frac{3}{2}$

$\Rightarrow \quad \lambda = 2$

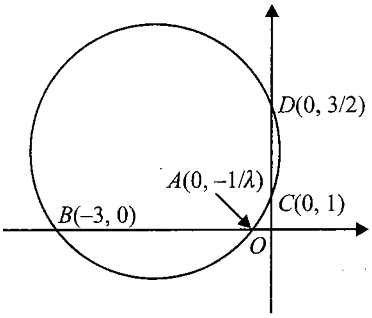


Fig. 2.221

12. See the solution of similar problem (question 1) in the concept application exercise 2.6.

13. Points of intersection of line $y = x$ and circle $x^2 + y^2 - 2x = 0$ can be obtained by solving these two equations or $x^2 + y^2 - 2x = 0 \Rightarrow x = 0, 1$

$\Rightarrow \quad y = 0, 1$

$\therefore A(0, 0), B(1, 1)$

Now equation of circle with AB as diameter is

$$x(x-1) + y(y-1) = 0$$

14. Let ABC be the given equilateral Δ, then C must lie on y-axis.

Let $(0, a)$, Also $AC = AB$

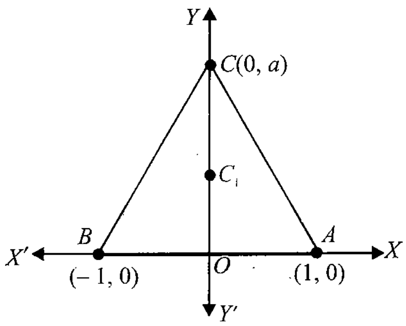


Fig. 2.222

$\Rightarrow \quad \sqrt{1+a^2} = 2$

$\Rightarrow \quad 1 + a^2 = 4$

$\Rightarrow \quad a = \sqrt{3}$

$\therefore C(0, \sqrt{3})$

Then, centroid of ΔABC is $\left(0, \frac{1}{\sqrt{3}}\right)$.

But in an equilateral Δ circumcentre, coincides with centroid

$\therefore$ Circumcentre is $\left(0, \frac{1}{\sqrt{3}}\right)$

Also, radius of circumcircle $= C_1B$

$$= \sqrt{(1-0)^2 + \left(0 - \frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$$

$\therefore$ Equation of circumcircle is

$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

$\Rightarrow \quad x^2 + y^2 - \frac{2y}{\sqrt{3}} + \frac{1}{3} = \frac{4}{3}$

$\Rightarrow \quad x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$

15. Let (h, k) be any point on the given line.

$\therefore 2h + k = 4$ and chord of contact is $hx + ky = 1$

or $\quad hx + (4 - 2h)y = 1$

or $\quad (4y - 1) + h(x - 2y) = 0$

It passes through the point of intersection of $4y - 1 = 0$ and $x - 2y = 0$ or $\left(\frac{1}{2}, \frac{1}{4}\right)$.

True or false

1. The circle passes through the points $A(1, \sqrt{3})$, $B(1, -\sqrt{3})$ and $C(3, -\sqrt{3})$.

Here line AB is parallel to y-axis and BC is parallel to x-axis.

Hence, $\quad \angle ABC = 90°$

$\therefore AC$ is a diameter of circle

$\therefore$ Equation of circle is

$$(x-1)(x-3) + (y - \sqrt{3})(y + \sqrt{3}) = 0$$

$\Rightarrow \quad x^2 + y^2 - 4x = 0 \quad$ (i)

Let us check the position of point (5/2,1) with respect to the circle (i), we get

$$S_1 = \frac{25}{4} + 1 - 10 < 0$$

$\therefore$ Point lies inside the circle.

$\therefore$ No tangent can be drawn to the given circle from point (5/2. 1).

$\therefore$ Given statement is true.

2. The centre of the circle $x^2 + y^2 - 6x + 2y = 0$ is $(3, -1)$ which lies on the $x + 3y = 0$.

$\therefore$ The statement is true.

Multiple choice questions with one correct answer

1. d.

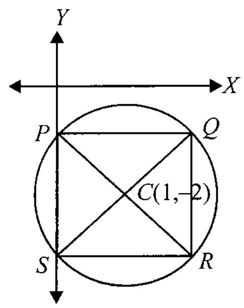


Fig. 2.223

Radius of the circle $CQ = \sqrt{2}$

Since $\angle QSR = 45°$

Coordinates of Q and S are given by $(1 \pm \sqrt{2}\cos 45°, -2 \pm \sqrt{2}\sin 45°)$

or $Q(2, -1)$ and $S(0, -3)$

Coordinates of P and R are given by $(1 \pm \sqrt{2}\cos 135°, -2 \pm \sqrt{2}\sin 135°)$

or $P(0, -1)$ and $S(2, -3)$

2. b. The circle through points of intersection of the two circle $x^2 + y^2 - 6 = 0$ and $x^2 + y^2 - 6x + 8 = 0$ is

$$(x^2 + y^2 - 6) + \lambda(x^2 + y^2 - 6x + 8) = 0$$

As it passes through (1, 1)

$$(1 + 1 - 6) + \lambda(1 + 1 - 6 + 8) = 0$$

$\Rightarrow$ $\lambda = 1$

$\therefore$ The required circle is

$$2x^2 + 2y^2 - 6x + 2 = 0$$

or $x^2 + y^2 - 3x + 1 = 0$

3. b. Use similar method as in the above question.

4. c. Let AB be the chord with its midpoint $M(h, k)$.

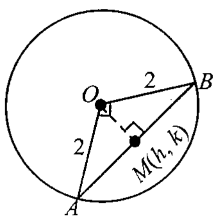


Fig. 2.224

As $\angle AOB = 90°$

$\therefore$ $AB = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\therefore$ $AM = \sqrt{2}$

By property of right angled Δ, $AM = MB = OM$

$\therefore$ $OM = \sqrt{2} \Rightarrow h^2 + k^2 = 2$

$\therefore$ Locus of (h, k) is $x^2 + y^2 = 2$

5. a. Let the equation of the circle through (a, b) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

So, $a^2 + b^2 + 2ag + 2bf + c = 0$ (ii)

Since circle (i) cuts $x^2 + y^2 = k^2$ orthogonally,

$\therefore$ $2g(0) + 2f(0) = c - k^2 \Rightarrow c = k^2$

Putting $c = k^2$ in Eq. (ii), we get

$2ag + 2bf + (a^2 + b^2 + k^2) = 0$

So, the locus of the centre $(-g, -f)$ is

$-2ax - 2by + (a^2 + b^2 + k^2) = 0$

or $2ax + 2by - (a^2 + b^2 + k^2) = 0$

6. a. If d is the distance between the centres of two circles of radii r_1 and r_2, then they intersect in two distinct point if $|r_1 - r_2| < d < r_1 + r_2$

Here, radii of two circles are r and 3 and distance between the centres is 5.

Thus, $|r - 3| < 5 < r + 3$

$\Rightarrow$ $2 < r < 8$ and $r > 2$

$\Rightarrow$ $2 < r < 8$

7. c. The given diameter are $2x - 3y = 5$ (i)

and $3x - 4y = 7$ (ii)

Solving Eqs. (i) and (ii), $x = 1, y = -1$

Thus $(1, -1)$ is the centre.

Now area of the circle, $\pi r^2 = 154 \Rightarrow r^2 = \frac{154}{22} \times 7 = 49$

Hence, the equation of the circle is: $(x - 1)^2 + (y + 1)^2 = 49$

$\Rightarrow$ $x^2 + y^2 - 2x + 2y = 47$

8. d. See the solution of problem 25 in multiple type questions.

9. d. Let the centre of the circle be (h, k).

Since the circle touches the axis of y, therefore radius $= h$.

The radius of the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ is 2 and it has its centre at (3, 3).

Since the two circles touch each other externally, therefore

Distance between the centres = sum of the radii

$\Rightarrow$ $\sqrt{(h-3)^2 + (k-3)^2} = |h + 2|$

$\Rightarrow$ $k^2 - 10h - 6k + 14 = 0$

Hence, the locus of (h, k) is $y^2 - 10x - 6y + 14 = 0$

10. d. Centre of the circle

$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$

is $C(-2, 3)$ and its radius is

$\sqrt{2^2 + (-3)^2 - 9\sin^2\alpha - 13\cos^2\alpha}$

$\sqrt{4 + 9 - 9\sin^2\alpha - 13\cos^2\alpha} = |2\sin\alpha|$

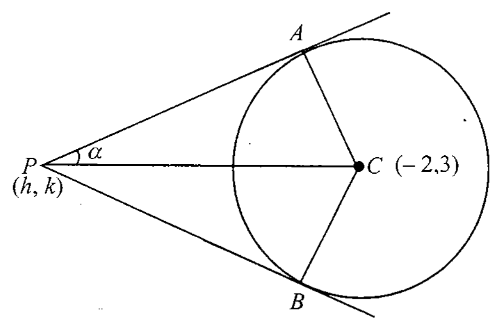


Fig. 2.225

Let $P(h, k)$ be any point on the locus. Then $\angle APC = \alpha$.

From the diagram $\sin \alpha = \dfrac{AC}{PC} = \dfrac{2 \sin \alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$

$\Rightarrow \quad (h+2)^2 + (k-3)^2 = 4$

or $\quad h^2 + k^2 + 4h - 6k + 9 = 0$

Thus, required equation of the locus is

$x^2 + y^2 + 4x - 6y + 9 = 0$

11. d. Let $B(h, 0)$ is the midpoint of the chord drawn from point $A(p, q)$.

Also centre is $C\left(\dfrac{p}{2}, \dfrac{q}{2}\right)$

Then, we have $BC \perp AB$

$$\Rightarrow \left(\frac{\frac{q}{2} - 0}{\frac{p}{2} - h}\right)\left(\frac{q-0}{p-h}\right) = -1$$

$$\Rightarrow \left(\frac{q}{p-2h}\right)\left(\frac{q-0}{p-h}\right) = -1$$

$\Rightarrow 2h^2 - 3ph + p^2 + q^2 = 0$

Since two such chords exist, the above equation must have two distinct real roots,

$\Rightarrow$ Discriminant > 0

$\Rightarrow 9p^2 - 8(p^2 + q^2) > 0$

$\Rightarrow p^2 > 8q^2$

12. c. See solution of example 2.8

13. c. See problem 3 in Multiple Correct Answers Type.

14. b. $x^2 + y^2 = r^2$ is a circle with centre at (0, 0) and radius r units.

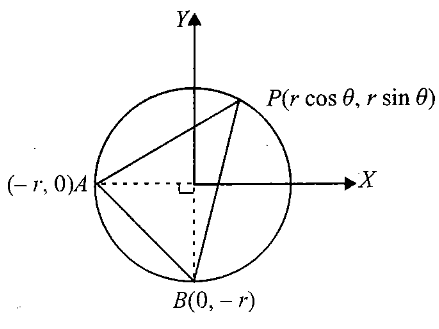


Fig. 2.226

Any arbitrary point P on it is $(r \cos \theta, r \sin \theta)$ choosing A and B as $(-r, 0)$ and $(0, -r)$.

For locus of centroid of ΔABP

$$\left(\frac{r\cos\theta - r}{3}, \frac{r\sin\theta - r}{3}\right) = (x, y)$$

$\Rightarrow \quad r \cos \theta - r = 3x$ and $r \sin \theta - r = 3y$

$\Rightarrow \quad r \cos \theta = 3x + r$ and $r \sin \theta = 3y + r$

Squaring and adding $(3x + r)^2 + (3y + r)^2 = r^2$ which is a circle.

16. a.

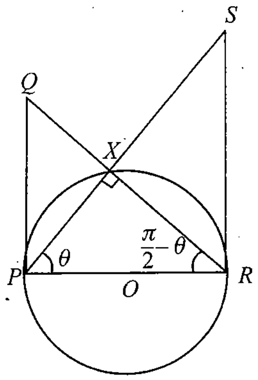


Fig. 2.227

From the above figure, we have $\dfrac{PQ}{PR} = \tan(\pi/2 - \theta) = \cot \theta$

and $\quad \dfrac{RS}{PR} = \tan \theta$

$\Rightarrow \dfrac{PQ}{PR} \cdot \dfrac{RS}{PR} = 1$

$\Rightarrow \quad (PR)^2 = PQ \cdot RS$

$\Rightarrow \quad (2r)^2 = PQ \cdot RS$

$\Rightarrow \quad 2r = \sqrt{PQ \cdot RS}$

16. c. Line $5x - 2y + 6 = 0$ is interested by tangent at P to circle $x^2 + y^2 + 6x + 6y - 2 = 0$ on y-axis at $Q(0, 3)$.

In other words, tangent passes through (0, 3).

$\therefore PQ$ = length of tangent to circle from (0, 3)

$= \sqrt{0 + 9 + 0 + 18 - 2}$

$= \sqrt{25} = 5$

17. a. $x^2 - 8x + 12 = 0 \quad \Rightarrow (x-6)(x-2) = 0$

$y^2 - 14y + 45 = 0 \Rightarrow (y-5)(y-9) = 0$

Thus, sides of squares are

$x = 2, x = 6, y = 5, y = 9$

Then centre of circle inscribed in square will be $\left(\dfrac{2+6}{2}, \dfrac{5+9}{2}\right) = (4, 7)$

18. c. Given circle is $x^2 + y^2 - 2x - 6y + 6 = 0$

Its centre is $H(1, 3)$ and radius = 2

$AH = 2$

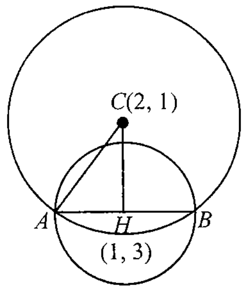


Fig. 2.228

Radius of the required circle

$$= AC = \sqrt{AH^2 + CH^2}$$
$$= \sqrt{2^2 + 5} = 3$$

19. d. Let the centre of circle C be (h, k).

Then as this circle touches axis of x, its radius $= |k|$

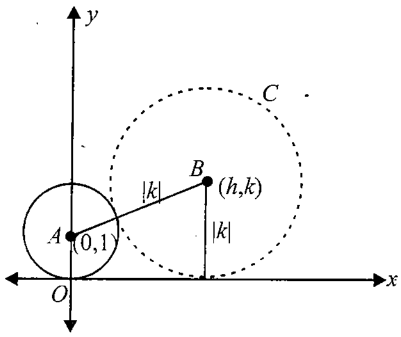


Fig. 2.229

Also it touches the given circle $x^2 + (y - 1)^2 = 1$, centre $(0, 1)$ radius 1, externally

Therefore, the distance between centres = sum of radii

$$\Rightarrow \quad \sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$$

$$\Rightarrow \quad h^2 + k^2 - 2k + 1 = 1 + 2|k| + k^2$$

$$\Rightarrow \quad h^2 = 2k + 2|k|$$

$\therefore$ Locus of (h, k) is, $x^2 = 2y + 2|y|$

Now if $y > 0$, it becomes $x^2 = 4y$

and if $y \le 0$, it becomes $x = 0$

$\therefore$ Combining the two, the required locus is

$\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$

20. b. The centre of the cirlce is $C(3, 2)$.

Since CA and CB are perpendicular to PA and PB, CP is the diameter of the circumcircle of triangle PAB. Its equation is

$(x - 3)(x - 1) + (y - 2)(y - 8) = 0$

or $x^2 + y^2 - 4x - 10y + 19 = 0$.

21. d. Circle touching y–axis at $(0, 2)$ is $(x - 0)^2 + (y - 2)^2 + \lambda x = 0$ passes through $(-1, 0)$

$\therefore \quad 1 + 4 - \lambda = 0 \Rightarrow \lambda = 5$

$\therefore \quad x^2 + y^2 + 5x - 4y + 4 = 0$

Put $y = 0 \Rightarrow x = -1, -4$

$\therefore$ Circle passes through $(-4, 0)$

Multiple choice questions with one or more than one correct answer

1. a., c. Equation of any line through the origin $(0, 0)$ is

$$y = mx \qquad \text{(i)}$$

If line (i) is tangent to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$,

then the length of $\perp$ from centre (r, h) on (i) = radius of circle

$$\Rightarrow \quad \frac{|mr - h|}{\sqrt{m^2 + 1}} = \sqrt{r^2 + h^2 - h^2}$$

$$\Rightarrow \quad (mr - h)^2 = (m^2 + 1)r^2$$

$$\Rightarrow 0 \cdot m^2 + (2hr)m + (r^2 - h^2) = 0$$

$$\Rightarrow \quad m = \infty, \frac{h^2 - r^2}{2hr}$$

Substituting these values in Eq. (i), we get tangents as $x = 0$ and $(h^2 - r^2)x - 2rhy = 0$

2. a., b., c., d.

Putting $y = c^2/x$ in $x^2 + y^2 = a^2$, we obtain $x^2 + c^4/x^2 = a^2$

$$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0 \qquad \text{(i)}$$

As x_1, x_2, x_3 and x_4 are roots of Eq. (i),

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 0 \text{ and } x_1x_2x_3x_4 = c^4$$

Similarly, forming equation in y, we get $y_1 + y_2 + y_3 + y_4 = 0$ and $y_1y_2y_3y_4 = c^4$

Comprehension type

For Problems 1–3

1. a. Let A, B, C and D be the complex number $\sqrt{2}$, $-\sqrt{2}$, $-\sqrt{2}i$ and $\sqrt{2}\,a$ respectively

$$\Rightarrow \frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$$

$$= \frac{|z_1 - \sqrt{2}|^2 + |z_1 + \sqrt{2}|^2 + |z_1 - \sqrt{2}\,i|^2 + |z_1 + \sqrt{2}\,i|^2}{|z_1 - \sqrt{2}|^2 + |z_2 + \sqrt{2}|^2 + |z_2 - \sqrt{2}\,i|^2 + |z_2 + \sqrt{2}\,i|^2}$$

$$= \frac{|z_1|^2 + 2}{|z_2|^2 + 2} = \frac{3}{4}$$

2. c.

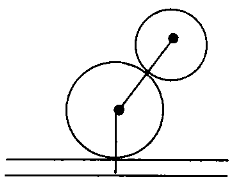


Fig. 2.230

Let C be the centre of the required circle.

Now draw line parallel to L at a distance of r_1 (radius of C_1) from it.

Now $CP_1 = AC \Rightarrow C$ lies on a parabola

3. c.

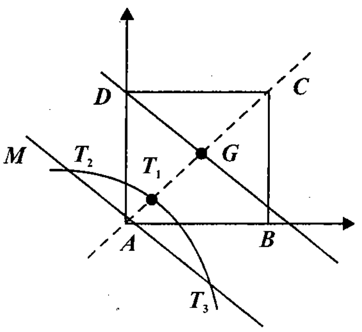


Fig. 2.231

$\because \quad AG = \sqrt{2}$

$\therefore AT_1 = T_1G =$ (as A is the focus, T_1 is the vertex and BD is the directrix of parabola)

Also T_2T_3 is a latus rectum

$\therefore \quad T_2T_3 = 4 \times \frac{1}{\sqrt{2}}$

$\therefore$ Area of $\quad \Delta T_1T_2T_3 = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 1$

For Problems 4–6

4. d.

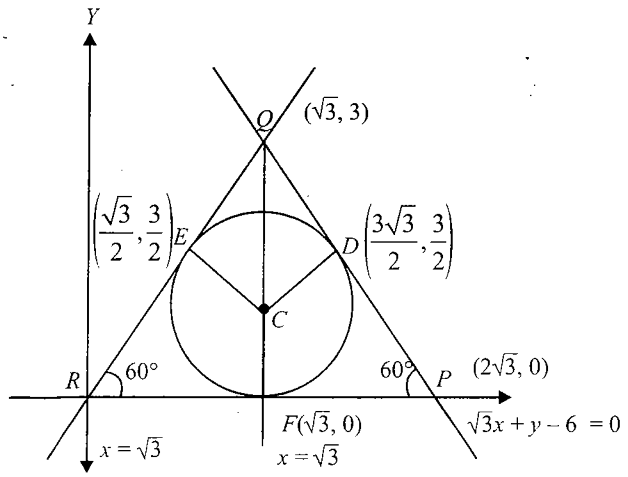


Fig. 2.232

Incentre is centaoid which is $(\sqrt{3}, 1)$. Also inradius = 1

$\Rightarrow$ Equation of the circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

5. a.

6. d. Equation of QR is $y - 3 = \sqrt{3}(x - \sqrt{3})$

$\Rightarrow \quad y = \sqrt{3}x$

Equation of RP is $y = 0$.

Assertion and reasoning

1. a. Equation of director circle of the given circle $x^2 + y^2 = 169$ is $x^2 + y^2 = 2 \times 169 = 338$

We know from every point on director circle, the tangents drawn to given circle are perpendicular to each other

Here (17, 7) lies on director circle.

$\therefore$ The tangent from (17, 7) to given circle are mutually perpendicular.

2. c.

$$\text{Circle} \equiv (x + 3)^2 + (y - 5)^2 = 4$$

Distance between L_1 and L_2

$\Rightarrow \quad \frac{6}{\sqrt{13}} <$ radius

$\Rightarrow$ Statement 2 is false.

But statement 1 is correct.

Matrix-match

1. a → p, q: b → p, q; c → q, r; d → q, r

a.

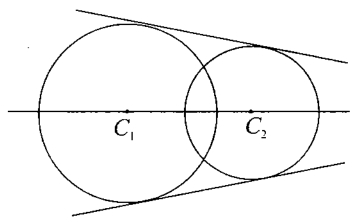


Fig. 2.233

It is clear from Fig. 2.234 that two intersecting circles have a common tangents and a common normal joining the centres.

b.

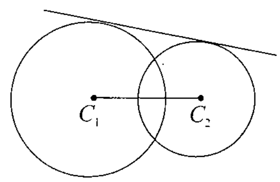


Fig. 2.234

c. Two circles, when one is completely inside the other have a common normal C_1C_2, but no common tangent.

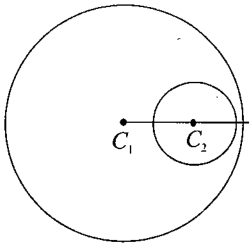


Fig. 2.235

d. Two branches of hyperbola have no common tangent, but have a common normal joining S_1S_2.

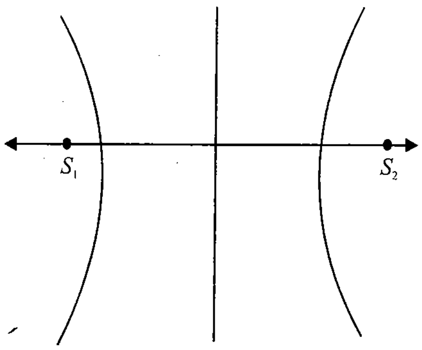


Fig. 2.236

Integer type

1. (2)

$L : 2x - 3y - 1$

$S : x^2 + y^2 - 6$

If $L_1 > 0$ and $S_1 < 0$

then point lies in the smaller part.

$\therefore \left(2, \frac{1}{4}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$ lie inside

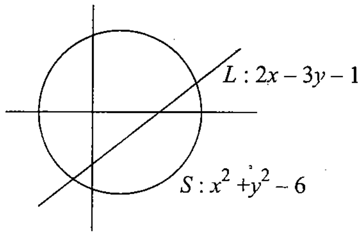


Fig. 2.237

CHAPTER

3

Parabola

- Introduction
- Standard Equation of Parabola
- General Equation of a Parabola

INTRODUCTION

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α. Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as cone extending indefinitely far in both directions. The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

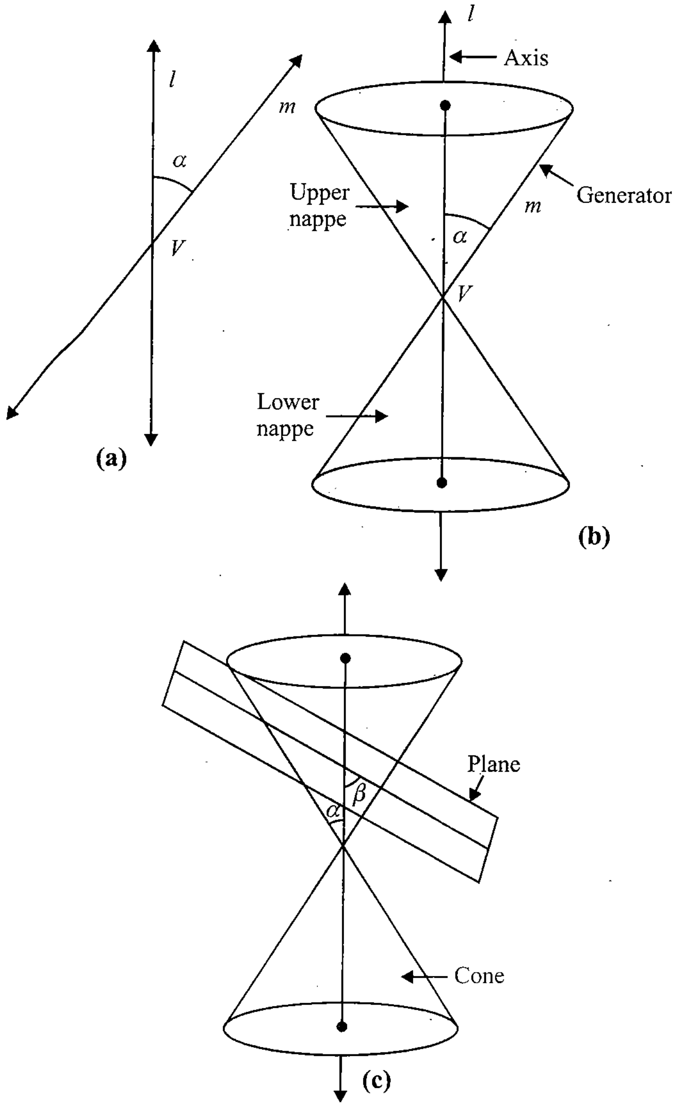


Fig. 3.1

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone. The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

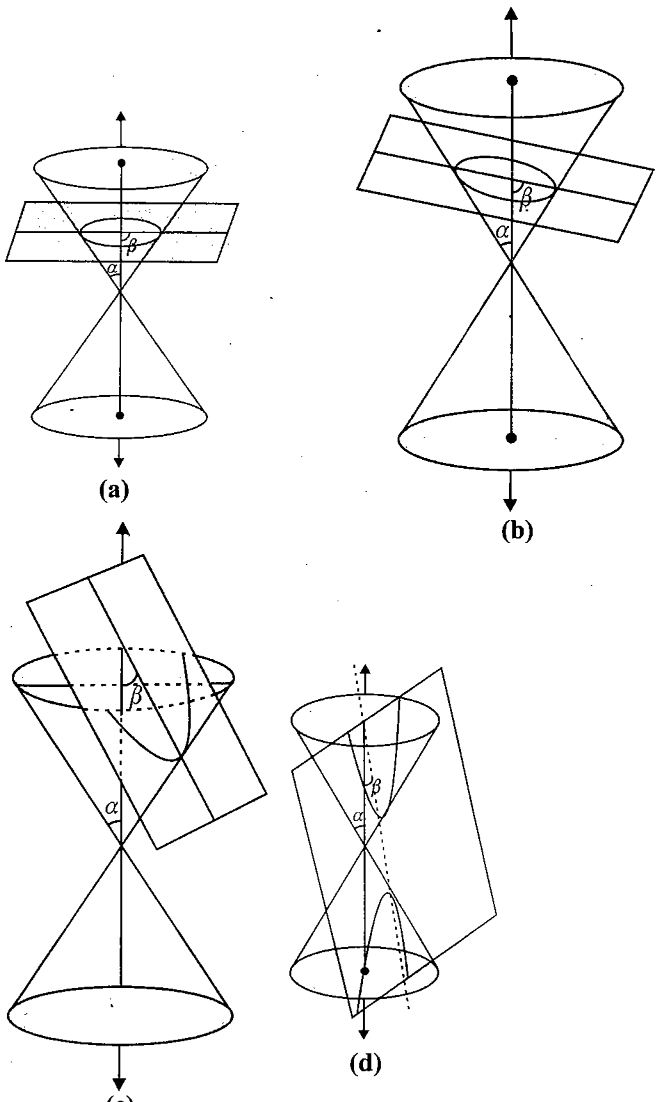


Fig. 3.2

Circle, Ellipse, Parabola and Hyperbola

When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

a. When $\beta = 90°$, the section is a *circle*.

b. When $\alpha < \beta < 90°$, the section is an *ellipse*.

c. When $\beta = \alpha$, the section is a *parabola* (in each of the above three situations, the plane cuts entirely across one nappe of the cone).

d. When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola*.

Conic Section as a Locus of a Point

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line not passing through given fixed point is always constant, is known as a conic section or conic.

The fixed point is called the *focus* of the conic and fixed line is called the *directrix* of the conic.

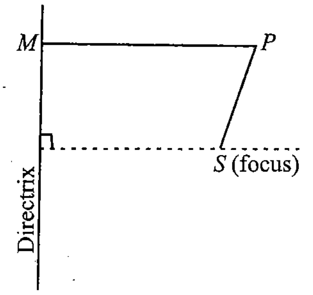


Fig. 3.3

Also this constant ratio is called the eccentricity of the conic and is denoted by e.

If $e = 1$, the conic is called parabola.
If $e < 1$, the conic is called ellipse.
If $e > 1$, the conic is called hyperbola.
If $e = 0$, the conic is called circle.
If $e = \infty$, the conic is called pair of straight lines.
In Fig. 3.3 $\frac{SP}{PM}$ = constant = e or $SP = ePM$

Equation of Conic Section

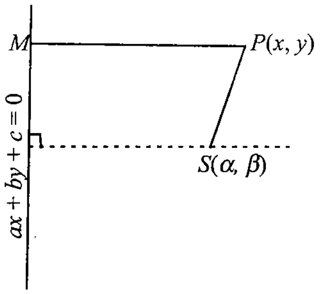


Fig. 3.4

If the focus is (α, β) and the directrix is $ax + by + c = 0$ then the equation of the conic section whose eccentricity = e is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\frac{|ax+by+c|}{\sqrt{(a^2+b^2)}}$$

or $$(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(ax+by+c)^2}{(a^2+b^2)}$$

Important Terms

Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex: The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal chord: Any chord passing through the focus is called focal chord of the conic section.

Double ordinate: A straight line that is drawn perpendicular to the axis and terminates at both ends of the curve is a double ordinate of the conic section.

Latus rectum: The double ordinate passing through the focus is called the latus rectum of the conic section.

Centre: The point that bisects every chord of the conic passing through it is called the centre of the conic section.

Note:
Parabola has no centre, but circle, ellipse, hyperbola have centre.

STANDARD EQUATION OF PARABOLA

Consider the focus of the parabola as $S(a, 0)$ and directrix be $x + a = 0$, and axis as x-axis.

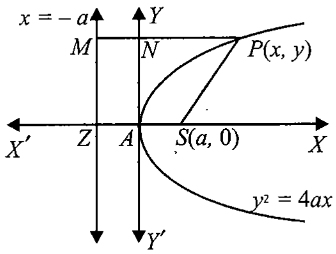


Fig. 3.5

Now according to the definition of the parabola, for any point on the parabola, we must have

$$SP = PM$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + (y-0)^2} = PN + NM = x + a$$

$$\Rightarrow \quad (x-a)^2 + y^2 = (a+x)^2$$

$$\Rightarrow \quad y^2 = (a+x)^2 - (x-a)^2$$

$$\Rightarrow \quad y^2 = 4ax$$

Vertex:	$(0, 0)$
Tangent at vertex:	$x = 0$
Equation of latus rectum:	$x = a$
Extremities of latus rectum:	$(a, 2a), (a, -2a)$
Length of latus rectum:	$4a$
Focal distance (SP):	$SP = PM = x + a$
Parametric form:	$x = at^2$ and $y = 2at$, where t is parameter

Other Standard Forms of Parabola

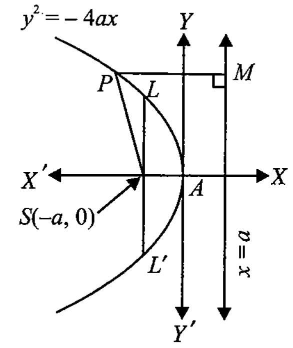


Fig. 3.6

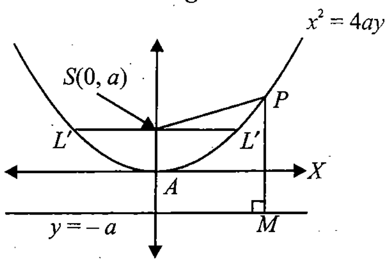


Fig. 3.7

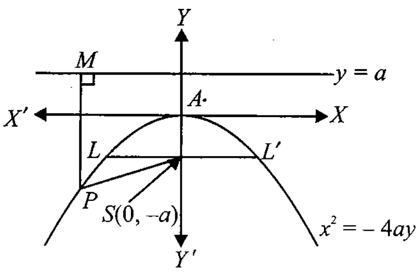


Fig. 3.8

Equation of curve:	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex:	(0, 0)	(0, 0)	(0, 0)
Focus:	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix:	$x - a = 0$	$y + a = 0$	$y - a = 0$
Axis:	$y = 0$	$x = 0$	$x = 0$
Tangent at vertex:	$x = 0$	$y = 0$	$y = 0$
Parametric form:	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$

If focus and vertex of the parabola are $(p, 0)$ and $(q, 0)$, then its equation is

$y^2 = 4(q - p)(x - p)$(when $p < q$)

or $y^2 = -4(p - q)(x - p)$ (where $q < p$)

If focus and vertex of the parabola are $(p, 0)$ and $(q, 0)$, then its equation is

$x^2 = 4(q - p)(y - p)$ (when $q > p$)

or $x^2 = -4(p - q)(x - p)$ (when $q < p$)

Example 3.1 **Find the equation of a parabola**

i. having its vertex at A (1, 0) and focus at S (3, 0)

ii. having its focus at S (2, 0) and one extremity of its latus rectum as (2, 2)

iii. having focus at (0, − 3) and its directrix is $y = 3$

Sol. **i.** Clearly the axis of the parabola is x-axis. Corresponding value of $a = 3 - 1 = 2$. Thus equation of the parabola is $y^2 = 8(x - 1)$.

ii. Clearly the other extremity of latus rectum is $(2, -2)$. Its axis is x-axis. Corresponding value of $a = \frac{2-0}{2} = 1$. Hence, its vertex is (1, 0) or (3, 0). Thus its equation is $y^2 = 4(x - 1)$ or $y^2 = -4(x - 3)$.

iii.

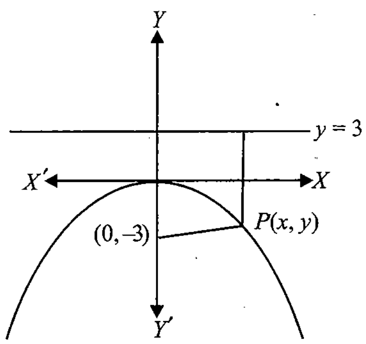


Fig. 3.9

Let $P(x, y)$ be any point on the parabola.

Then by definition $\sqrt{(x-0)^2 + (y+3)^2} = y - 3$

$\Rightarrow \quad x^2 = -12y.$

Example 3.2 **A beam is supported at its ends by two supports which are 12 m apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?**

Sol.

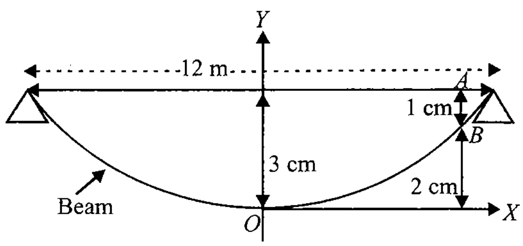


Fig. 3.10

Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig. 3.10.

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $\left(6, \frac{3}{100}\right)$, we have $6^2 = 4a \times \frac{3}{100}$, i.e., $a = 300$ m.

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $\left(x, \frac{2}{100}\right)$.

Therefore, $$x^2 = 4 \times 300 \times \frac{2}{100} = 24.$$

$$\Rightarrow \quad x = 2\sqrt{6} \text{ m}$$

Example 3.3 **If the two ends of the latus rectum are given. How many parabolas can be drawn?**

Sol.

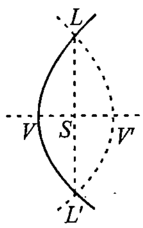


Fig. 3.11

L, L' are the ends of the latus rectum. S bisects LL'. VSV' is the perpendicular bisector of LL', where $VS = \frac{1}{4} LL' = V'S$. Clearly, two parabolas are possible.

Example 3.4 **Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.**

Sol. If the coordinates of a point on the parabola $y^2 = 4ax$ are $P(x, y)$, then its focal distance is $SP = x + a$.

Here, $a = 2$ and $SP = 4$.

$\therefore \quad 4 = x + 2$

$\Rightarrow \quad x = 2$

$\Rightarrow \quad y^2 = 8 \times 2$

$\Rightarrow \quad y = \pm 4$

Thus, the coordinates of the required point are $(2, \pm 4)$.

Example 3.5 **If the vertex of a parabola is the point $(-3, 0)$ and the directrix is the line $x + 5 = 0$, then find its equation.**

Sol. Since the line passing through the focus and perpendicular to the directirx is x-axis, therefore axis of the required parabola is x-axis.

Let the coordinates of the focus be $S(a, 0)$.

Since the vertex is the midpoint of the line joining the focus and the point $(-5, 0)$ where the directrix $x + 5 = 0$ meets the axis.

Therefore,
$$-3 = \frac{a-5}{2}$$
$$\Rightarrow \quad a = -1$$

Thus, the coordinates of the focus are $(-1, 0)$.

Let $P(x, y)$ be a point on the parabola.

Then by definition
$$\sqrt{(x+1)^2 + y^2} = (x+5)^2$$
$$\Rightarrow \quad y^2 = 8(x+3)$$

Example 3.6 **The chord AB of the parabola $y^2 = 4ax$ cuts the axis of the parabola at C. If $A = (at_1^2, 2at_1)$, $B = (at_2^2, 2at_2)$ and $AC:AB = 1:3$, then prove that $t_2 + 2t_1 = 0$.**

Sol.

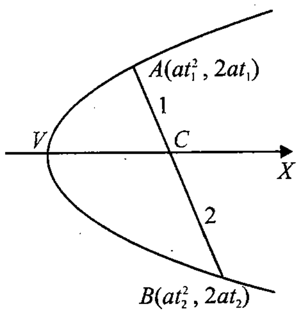


Fig. 3.12

$$\frac{AC}{AB} = \frac{1}{3} \Rightarrow \frac{AC}{BC} = \frac{1}{2}$$

Here, $$C = \left(\frac{2at_1^2 + at_2^2}{3}, \frac{4at_1 + 2at_2}{3}\right)$$

It lies on $$y = 0$$

$$\therefore \quad \frac{4at_1 + 2at_2}{3} = 0 \Rightarrow t_2 + 2t_1 = 0$$

Example 3.7 **M is the foot of the perpendicular from a point P on a parabola $y^2 = 4ax$ to its directrix and SPM is an equilateral triangle, where S is the focus. Then find SP.**

Sol.

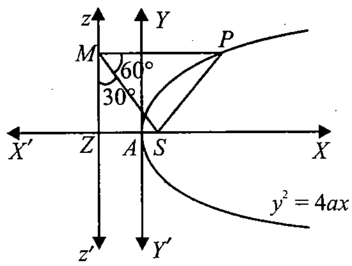


Fig. 3.13

From the definition of the parabola, we have
$$SP = PM$$

SPM is an equilateral triangle.

Therefore,
$$SP = PM = SM$$
$$\Rightarrow \quad \angle PMS = 60°$$
$$\Rightarrow \quad \angle SMZ = 30°$$

In ΔSMZ, we have
$$\sin 30° = \frac{SZ}{SM}$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{2a}{SM}$$

$$\Rightarrow \qquad SM = 4a$$

Hence, $\qquad SP = SM = 4a$

Example 3.8 **An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.**

Sol.

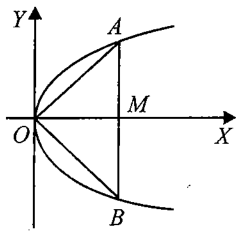


Fig. 3.14

If triangle OAB is equilateral then $OA = OB = AB = p$.

Thus AB will be a double ordinate of the parabola.

Thus $\qquad \angle AOM = \angle MOB = \frac{\pi}{6}$

$$\Rightarrow \qquad OM = p\cos\frac{\pi}{6} \text{ and } AM = p\sin\frac{\pi}{6}$$

Then A has coordinates $\left(\frac{\sqrt{3}p}{2}, \frac{p}{2}\right)$

A lies on the parabola, then $\frac{p^2}{4} = 4a\frac{\sqrt{3}p}{2}$

$$\Rightarrow \qquad p = 8\sqrt{3}a$$

Example 3.9 **A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola.**

Sol.

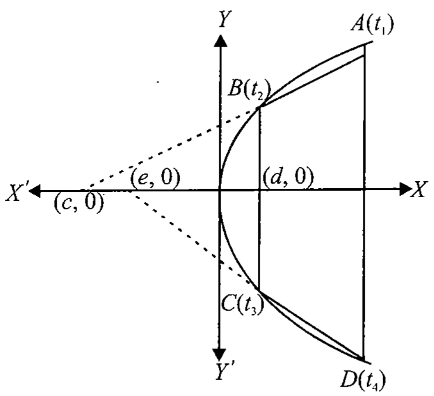


Fig. 3.15

Equation of chord AB is

$$2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

It passes through the point $(c, 0)$,

then given $\qquad t_1t_2 = -\frac{c}{a} \qquad$ (i)

Similarly for chord BC, $t_2t_3 = -\frac{d}{a} \qquad$ (ii)

And for chord CD, $\qquad t_3t_4 = -\frac{e}{a} \qquad$ (iii)

Multiplying (i) and (iii), we get

$$t_1t_2t_3t_4 = +\frac{ec}{a^2}$$

$$\Rightarrow \qquad t_1t_4\left(-\frac{d}{a}\right) = \frac{ec}{a^2}$$

$$\Rightarrow \qquad t_1t_4 = -\frac{ec}{ad}$$

Hence, chord AD passes through the fixed point.

Example 3.10 **Find the locus of middle points of chords of a parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.**

Sol. Here, $h = \frac{at_1^2 + at_2^2}{2}$, $k = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$

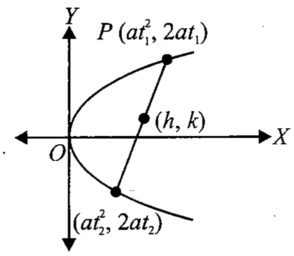


Fig. 3.16

Also, $\frac{2at_1 - 0}{at_1^2 - 0} \times \frac{2at_2 - 0}{at_2^2 - 0} = -1$

$$\Rightarrow \qquad t_1t_2 = -4$$

Now, $\qquad \frac{2h}{a} = t_1^2 + t_2^2 = (t_1 + t_2)^2 - 2t_1t_2$

$$= \left(\frac{k}{a}\right)^2 - 2(-4)$$

Therefore, the locus of (h, k) is

$$\frac{2x}{a} = \frac{y^2}{a^2} + 8$$

which is a parabola.

Example 3.11 **Let there be two parabolas $y^2 = 4ax$ and $y^2 = -4bx$ (where $a \neq b$ and $a, b > 0$). Then find the locus of**

the middle points of the intercepts between the parabolas made on the lines parallel to the common axis.

Sol. Let the line parallel to common axis be $y = h$.

Then coordinates of A and B are $\left(\frac{h^2}{4a}, h\right)$ and $\left(-\frac{h^2}{4b}, h\right)$, respectively.

If $P(\alpha, \beta)$ is a midpoint of AB, then $\alpha = \frac{1}{2}\left(\frac{h^2}{4a} - \frac{h^2}{4b}\right)$ and $\beta = h$

$$\therefore \quad 2\alpha = \frac{\beta^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Hence, locus of P is

$$2x = \frac{y^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Example 3.12 **A right-angled triangle ABC is inscribed in parabola $y^2 = 4x$, where A is vertex of parabola and $\angle BAC = \pi/2$. If $AB = \sqrt{5}$, then find the area of ΔABC.**

Sol.

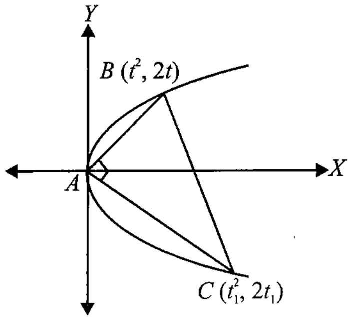


Fig. 3.17

From the figure, $AB = \sqrt{5}$

$\Rightarrow \quad t^4 + 4t^2 = 5$

$\Rightarrow \quad (t^2 - 1)(t^2 + 4) = 0$

$\Rightarrow \quad t = \pm 1$

$\Rightarrow \quad B \equiv (1, 2)$

Also $m_{AB} \times m_{AC} = -1$

$\therefore \frac{2}{t} \cdot \frac{2}{t_1} = -1$

$\therefore t_1 = -4$

$\Rightarrow$ C has coordinates $(16, -8)$

Now $AC = \sqrt{256 + 64} = \sqrt{320}$

Then, the area of $\angle ABC$ is $\frac{1}{2}\sqrt{5}\sqrt{320} = \frac{1}{2}\sqrt{1600} = 20.$

Example 3.13 **AP is perpendicular to PB, where A is vertex of parabola $y^2 = 4x$ and P on the parabola. B is on the axis of parabola. Then find the locus of centroid of ΔPAB.**

Sol.

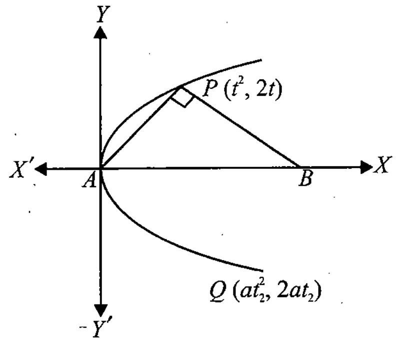


Fig. 3.18

Let P be $(t^2, 2t)$

Slope of $AP = \frac{2}{t}$

$\Rightarrow$ Slope of $BP = -\frac{t}{2}$

$\Rightarrow$ Equation of line BP is $y - 2t = -\frac{t}{2}(x - t^2)$

$\Rightarrow$ Point B is $(t^2 + 4, 0)$

Now let centroid of ΔPAB is (h, k)

$$\Rightarrow \quad h = \frac{t^2 + t^2 + 4}{3} \text{ and } k = \frac{2t}{3}$$

$$\Rightarrow \quad 3h - 4 = 2\left(\frac{3k}{2}\right)^2.$$

Hence, the required locus is

$$3x - 4 = \frac{9y^2}{2}$$

which is a parabola.

Position of a Point With Respect to a Parabola $y^2 = 4ax$

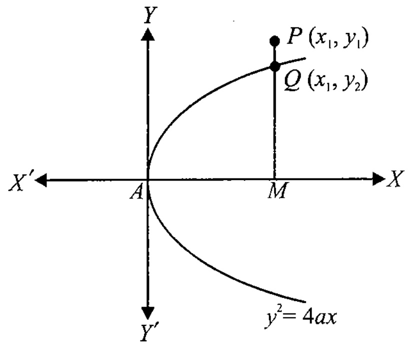


Fig. 3.19

Let $P(x_1, y_1)$ be a point in the plane (in 1st or 4th quadrant) From P draw $PM \perp AX$, meeting the parabola $y^2 = 4ax$ at Q. Then the coordinates of Q be (x_1, y_2). Since Q lies on the parabola, we have $y_2^2 = 4ax_1$

Now, point (x_1, y_1) will be outside, on or inside the parabola $y^2 = 4ax$ according to

$$PM >, = \text{ or } < QM$$

$$\Rightarrow \quad PM^2 >, = \text{ or } < QM^2$$

$$\Rightarrow \quad y_1^2 >, = \text{ or } < y_2^2$$

$$\Rightarrow \quad y_1^2 >, = \text{ or } < 4ax_1$$

For P lying in 3rd or 4th quadrant, $y_1^2 - 4ax_1 > 0$ ($\because x_1 < 0$)

Example 3.14 **The equation of a parabola is $y^2 = 4x$. P (1, 3) and Q (1, 1) are two points in the xy plane. Then, for the parabola**

a. P and Q are exterior points
b. P is an interior point while Q is an exterior point
c. P and Q are interior points
d. P is an exterior point while Q is an interior point

Sol. d. Here, $S \equiv y^2 - 4x = 0$

$$S(1, 3) \equiv 3^2 - 4\cdot 1 > 0$$

$\Rightarrow$ $P(1, 3)$ is an exterior point.

$$S(1, 1) = 1^2 - 4\cdot 1 < 0$$

$\Rightarrow$ $Q(1, 1)$ is an interior point.

Example 3.15 **The point (a, $2a$) is an interior point of the region bounded by the parabola $y^2 = 16x$ and the double ordinate through the focus. Then find the values of a.**

Sol.

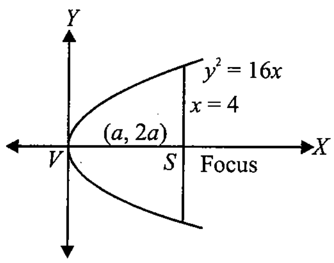


Fig. 3.20

$(a, 2a)$ is an interior point of $y^2 - 16x = 0$ if

$$(2a)^2 - 16a < 0, \text{ i.e., } a^2 - 4a < 0$$

$V(0, 0)$ and $(a, 2a)$ are on the same side of $x - 4 = 0$.

So, $a - 4 < 0$, i.e., $a < 4$.

Now,

$$a^2 - 4a < 0$$

$$\Rightarrow \quad 0 < a < 4$$

Equation of Parabola When Vertex is (*h*, *k*) and Axis is Parallel to *x*-Axis

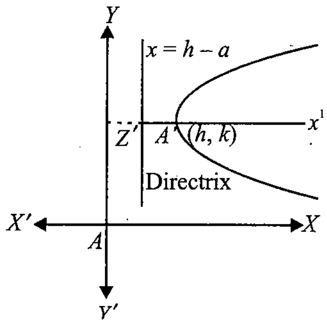


Fig. 3.21

The parabola

$$y^2 = 4ax \qquad \text{(i)}$$

can be written as

$$(y - 0)^2 = 4a(x - 0)$$

The vertex of this parabola is A (0, 0).

Now when origin is shifted to A' (h, k) without changing the direction of axes, its equation becomes

$$(y - k)^2 = 4a(x - h) \qquad \text{(ii)}$$

This is called general form of the parabola Eq. (i) and axis $A'X' \parallel AX$ with its vertex at A' (h, k). Its focus is at $(a + h, k)$ and length of latus rectum $= 4a$.

The equation of the directrix is

$$x = h - a$$

or $$x + a - h = 0$$

Equation of Parabola When Vertex is (*h*, *k*) and Axis is Parallel to *y*-Axis

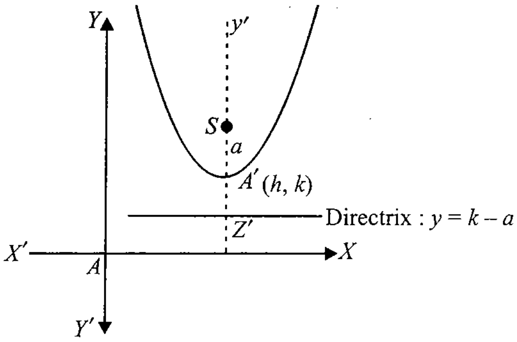


Fig. 3.22

The equation of parabola with vertex (h, k) is

$$(x - h)^2 = 4a(y - k)$$

Its focus is at $(h, a + k)$ and length of latus rectum $= 4a$.

The equation of the directrix is

$$y = k - a$$

or $$y + a - k = 0$$

Parabolic Curve

The equations $y = Ax^2 + Bx + C$ and $x = Ay^2 + By + C$ $(A \neq 0)$ represent parabola and are called parabolic curve.

Now,

$$y = Ax^2 + Bx + C$$
$$= A\left\{x^2 + \frac{B}{A}x + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{(B^2 - 4AC)}{4A^2}\right\}$$

or
$$\left(x + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(y + \frac{B^2 - 4AC}{4A}\right)$$

Comparing it with $(x - h)^2 = 4a(y - k)$ it represents a parabola with vertex at $(h, k) \equiv \left(-\frac{B}{2A}, -\frac{B^2 - 4AC}{4A}\right)$, axis parallel to y-axis, latus rectum $= \frac{1}{|A|}$ and the curve opening upwards and downwards depending upon the sign of A (for $A > 0$ curve opens upward, for $A < 0$ curve opens downward).

Similarly, $x = Ay^2 + By + C$ can be simplified to

$$\left(y + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(x + \frac{B^2 - 4AC}{4A}\right)$$

Comparing it with $(y - k)^2 = 4a(x - h)$ it represents a parabola with vertex at $(h, k) \equiv \left(-\frac{B^2 - 4AC}{4A}, -\frac{B}{2A}\right)$ axis parallel to x-axis and latus rectum $= \frac{1}{|A|}$ and the curve opening left and right according to $A < 0$ and $A > 0$ respectively.

Note:
Parametric form of the parabola $(y - k)^2 = 4a(x - h)$ is $x = h + at^2$ and $y = k + 2at$.

Example 3.16 **$y^2 + 2y - x + 5 = 0$ represents a parabola. Find its vertex, equation of axis, equation of latus rectum, coordinates of the focus, equation of the directrix, extremities of the latus rectum and the length of the latus rectum.**

Sol.

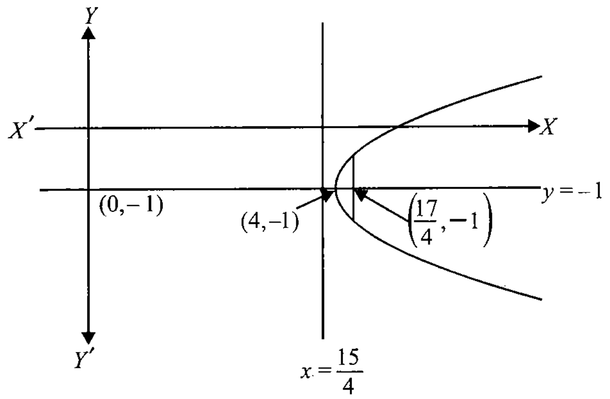


Fig. 3.23

$$y^2 + 2y - x + 5 = 0$$
$$\Rightarrow \quad y^2 + 2y + 1 = x - 4$$
$$\Rightarrow \quad (y + 1)^2 = (x - 4)$$

Comparing this equation with $(y - k)^2 = 4a(x - h)$ we have $a = \frac{1}{4}$.

Vertex: $(4, -1)$.

Equation of axis: $y + 1 = 0$

Equation of latus rectum: $x - 4 = \frac{1}{4}$ or $x = \frac{17}{4}$

Focus: Intersection point of axis and latus rectum is $\left(\frac{17}{4}, -1\right)$.

Directrix: $x - 4 = -\frac{1}{4}$ or $4x - 15 = 0$

Extremities of its latus rectum: These points lie at distance $2a$ from the focus on the latus rectum line which are $\left(\frac{17}{4}, -1 + \frac{1}{2}\right)$ and $\left(\frac{17}{4}, -1 - \frac{1}{2}\right)$, or $\left(\frac{17}{4}, \frac{-1}{2}\right)$ and $\left(\frac{17}{4}, \frac{-3}{2}\right)$

Length of its latus rectum: 1 unit.

Example 3.17 **Find the equation of parabola which has axis parallel to y-axis and which passes through points (0, 2), (– 1, 0) and (1, 6).**

Sol. General equation of such parabola is

$$y = Ax^2 + Bx + C$$

It passes through points (0, 2), (– 1, 0) and (1, 6). Then we have,

$$C = 2 \quad \text{(i)}$$
$$A - B + C = 0 \quad \text{(ii)}$$
$$A + B + C = 6 \quad \text{(iii)}$$

Solving (i), (ii) and (iii) we get $C = 2$, $A = 1$ and $B = 3$.

Hence equation of parabola is

$$y = x^2 + 3x + 2$$

Example 3.18 **Prove that the locus of centre of circle which touches given circle externally and given line is parabola.**

Sol.

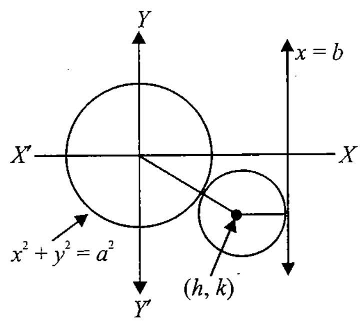


Fig. 3.24

Let given circle be $x^2 + y^2 = a^2$ and given line be $x = b$.

From the diagram radius of variable circle is $b - h$.

If it touches $x^2 + y^2 = a^2$,

then

$$a + (b - h) = \sqrt{h^2 + k^2}$$

$$\Rightarrow (a + b)^2 - 2(a + b)h + h^2 = h^2 + k^2$$

$$\Rightarrow \quad y^2 = (a + b)^2 - 2(a + b)x$$

which is equation of parabola.

Example 3.19 **If on a given base BC a triangle be described such that the sum of the tangents of the base angles is m, then prove that locus of opposite vertex A is parabola.**

Sol.

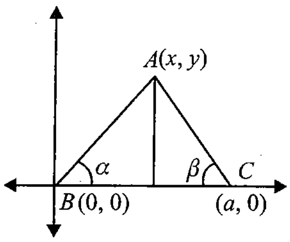


Fig. 3.25

Let the given points B and C are (0, 0) and $(a, 0)$.

According to the given condition

$\tan\alpha + \tan\beta = m$ (where m is constant)

$$\Rightarrow \quad \frac{y}{x} + \frac{y}{a - x} = m$$

$$\Rightarrow \quad y\frac{a}{x(a - x)} = m$$

$$\Rightarrow \quad ay = mx(a - x)$$

which is equation of parabola.

Example 3.20 **The parametric equation of a parabola is $x = t^2 + 1, y = 2t + 1$. Then find the equation of directrix.**

Sol. Eliminating t, we have

$$x = \left(\frac{y - 1}{2}\right)^2 + 1$$

or

$$(y - 1)^2 = 4(x - 1)$$

Putting $y - 1 = Y,\ x - 1 = X$, the equation becomes

$$Y^2 = 4X$$

So, the equation of the directrix is

$$X + 1 = 0 \Rightarrow x = 0$$

Example 3.21 **Find the points on the parabola $y^2 - 2y - 4x = 0$ whose focal length is 6.**

Sol. $y^2 - 2y - 4x = 0$

or

$$(y - 1)^2 = 4x + 1$$

or

$$(y - 1)^2 = 4\left(x + \frac{1}{4}\right)$$

Vertex of parabola is $\left(-\frac{1}{4}, 1\right)$. Corresponding parabola with vertex at origin is $y^2 = 4x$.

For point on this parabola having focal length 6, we have

$$6 = a + x = 1 + x$$

or

$$\therefore \quad x = 5$$

Hence points on this parabola are $(5, \pm 2\sqrt{5})$.

Hence corresponding points on the parabola

$$(y - 1)^2 = 4\left(x + \frac{1}{4}\right) \text{ are } \left(5 - \frac{1}{4}, 1 \pm 2\sqrt{5}\right)$$

$$\equiv \left(\frac{19}{4}, 1 \pm 2\sqrt{5}\right)$$

Example 3.22 **If the length of chord of circle $x^2 + y^2 = 4$ and $y^2 = 4(x - h)$ is maximum, then find the value of h.**

Sol.

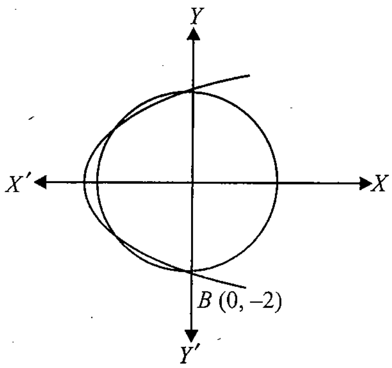


Fig. 3.26

Obviously, maximum lengths of chord occur when the parabola passes through (0, 2) and (0, –2).

Hence, from $y^2 = 4(x - h)$ we have,

$$4 = 4(0 - h)$$

$$\Rightarrow \quad h = -1$$

Concept Application Exercise 3.1

1. If the focus and vertex of a parabola are the points (0, 2) and (0, 4), respectively, then find its equation.
2. Find the coordinates of any point on the parabola whose focus is (0, 1) and the directrix is $x + 2 = 0$.
3. Find the length of the common chord of the parabola $y^2 = 4(x + 3)$ and the circle $x^2 + y^2 + 4x = 0$.

4. Find the vertex of the parabola $x^2 = 2(2x + y)$.
5. Find the equation of the directrix of the parabola $x^2 - 4x - 3y + 10 = 0$.
6. The vertex of a parabola is (2, 2) and the coordinates of its two extremities of latus rectum are (–2, 0) and (6, 0). Then find the equation of the parabola.
7. Find the length of latus rectum of parabola whose focus is at (2, 3) and directrix is the line $x - 4y + 3 = 0$.
8. Find the angle made by a double ordinate of length $8a$ at the vertex of the parabola $y^2 = 4ax$.
9. LOL' and MOM' are two chords of parabola $y^2 = 4ax$ with vertex A passing through a point O on its axis. Prove that the radical axis of the circles described on LL' and MM' as diameters passes though the vertex of the parabola.
10. If (a, b) is the midpoint of a chord passing through the vertex of the parabola $y^2 = 4(x + 1)$, then prove that $2(a + 1) = b^2$.
11. Find the range of values of λ for which the point $(\lambda, -1)$ is exterior to both the parabolas $y^2 = |x|$.
12. Prove that the locus of a point, which moves so that its distance from a fixed line is equal to the length of the tangent drawn from it to a given circle, is a parabola.
13. Prove that the locus of the centre of a circle, which intercepts a chord of given length $2a$ on the axis of x and passes through a given point on the axis of y distant b from the origin, is parabola.

GENERAL EQUATION OF A PARABOLA

Let $S(a, b)$ be the focus and $lx + my + n = 0$ is the equation of the directrix.

Let $P(x, y)$ be any point on the parabola.

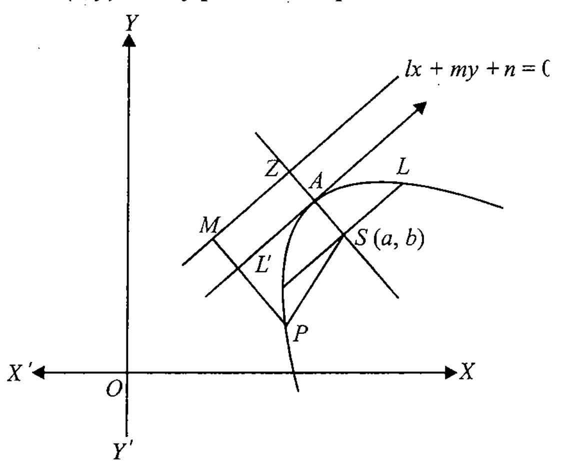


Fig. 3.27

By definition,

$$SP = PM$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + (y-b)^2} = \frac{|lx + my + n|}{\sqrt{(l^2 + m^2)}}$$

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = \frac{(lx + my + n)^2}{(l^2 + m^2)}$$

$$\Rightarrow \quad m^2x^2 + l^2y^2 - 2lmxy + x \text{ term} + y \text{ term} + \text{constant} = 0$$

This is of the form $(mx - ly)^2 + 2gx + 2fy + c = 0$

This equation is the general equation of parabola.

Note:

Second degree terms in the general equation of a parabola forms of a perfect square.

Recognition of Conics

The equation of conics represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

can be recognized easily by the condition given in the tabular form. For this, first we have to find $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$. When $\Delta \neq 0$, Eq. (i) represents the non-degenerate conic whose nature is given in the following table:

Condition	Nature of conics
$\Delta \neq 0, h = 0, a = b$	A circle
$\Delta \neq 0, ab - h^2 = 0$ ($\because$ 2nd degree terms form a perfect square)	A parabola
$\Delta \neq 0, ab - h^2 > 0$	An ellipse or empty set
$\Delta \neq 0, ab - h^2 < 0$	A hyperbola
$\Delta \neq 0, ab - h^2 < 0$ and $a + b = 0$	A rectangular hyperbola

Example 3.23 **Find the value of λ if equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents parabola.**

Sol. Comparing this equation with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ we have $a = 9, b = 4, c = 3, h = \lambda, g = 2, f = -1$.

If the equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents the parabola then its second degree terms must form the perfect square.

$\Rightarrow \lambda^2 = 36$ (using $h^2 - ab = 0$)

$\Rightarrow \lambda = \pm 6$

Also for these values of λ, $\Delta \neq 0$.

Example 3.24 **Find the value of λ if the equation $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2$ represents a parabola. Also, find its focus, the equation of its directrix, the equation of its axis, the coordinates of its vertex, the equation of its latus rectum, length of the latus rectum and the extremities of the latus rectum.**

Sol. $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2 = 2\lambda\left(\frac{x+y+3}{\sqrt{2}}\right)^2$

$\therefore \quad \sqrt{(x-1)^2+(y-2)^2} = \sqrt{2\lambda}\,\frac{|x+y+3|}{\sqrt{2}}$

This represents parabola if $\sqrt{2\lambda} = 1$

$\Rightarrow \quad \lambda = \frac{1}{2}$

Its focus is (1, 2). Its directirx is

$$x + y + 3 = 0.$$

Axis of parabola is a line through focus (1, 2) and perpendicular to the directrix $x + y + 3 = 0$

Hence, its axis is the line $y = x + 1$.

Axis $y = x + 1$ and the directrix $x + y + 3 = 0$ meet at (–2, –1), thus its vertex is $\left(\frac{1-2}{2}, \frac{2-1}{2}\right)$, i.e., $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Its latus rectum will be in the form of $x + y + b = 0$,

which passes through focus (1, 2). Hence its equation is $x + y - 3 = 0$

Distance between focus and directrix is $\frac{|1+2+3|}{\sqrt{2}} = \frac{6}{\sqrt{2}}$.

Thus length of its latus rectum is $\frac{12}{\sqrt{2}}$, i.e., $6\sqrt{2}$ units.

If (x_1, y_1) is the extremity of its latus rectum, then

$$\frac{x_1 - 1}{-\frac{1}{\sqrt{2}}} = \frac{y_1 - 2}{\frac{1}{\sqrt{2}}} = \pm\frac{6}{\sqrt{2}}$$

$\Rightarrow (x_1, y_1)$ is (–2, 5) or (4, –1)

Example 3.25 **Find the equation to the parabola whose focus is S (–1, 1) and directrix is $4x + 3y - 24 = 0$. Also find its axis, the vertex, the length and the equation of the latus rectum.**

Sol. Let $P(x, y)$ be any point on the parabola. Since the distance of P from the focus is equal to its distance from the directrix, i.e., $PS = PQ$ or $PS^2 = PQ^2$

or $\quad (x+1)^2 + (y-1)^2 = \left[\frac{(4x+3y-24)}{5}\right]^2$

i.e., $9x^2 + 16y^2 - 24xy + 242x + 94y - 526 = 0$ (i)

This is the required equation of the parabola.

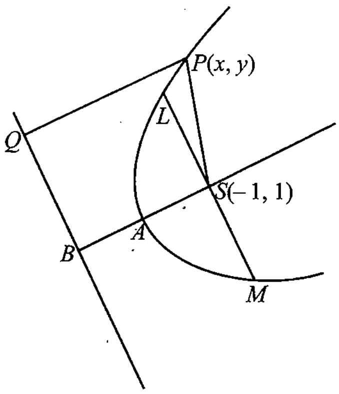


Fig. 3.28

The axis is a line through S (–1, 1) and $\perp$ to the directrix $4x + 3y - 24 = 0$. Thus the equation of the axis is

$$3(x+1) - 4(y-1) = 0 \text{ or } 3x - 4y + 7 = 0 \quad \text{(ii)}$$

The axis and the directrix intersect at B. Solving them, we get B (3, 4).

The vertex A is the midpoint of S (–1, 1) and B (3, 4)

Thus vertex A is $\left(1, \frac{5}{2}\right)$ (iii)

Also $\quad AS = \sqrt{\left[2^2 + \left(\frac{3}{2}\right)^2\right]} = \frac{5}{2}$

Hence length of the latus rectum $= 4AS = 10$ (iv)

Now, latus rectum is a straight line through the focus S and parallel to the directrix.

Hence its equation is $4x + 3y + 1 = 0$ (v)

Properties of Focal Chord

Any point on the parabola $y^2 = 4ax$ can be taken as $(at^2, 2at)$, where t is parameter and $t \in R$. Any line passing through the focus of the parabola is called focal chord of the parabola.

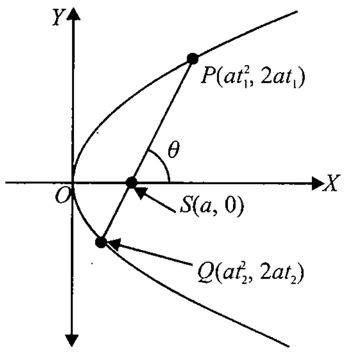


Fig. 3.29

1. If the chord joining $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$ is the focal chord then $t_1t_2 = -1$.

Proof:

$$P \equiv (at_1^2, 2at_1) \text{ and } Q \equiv (at_2^2, 2at_2)$$

Since PQ passes through the focus $S(a, 0)$

$\therefore$ Q, S, P are collinear

$\therefore$ Slope of PS = Slope of QS

$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$

$$\Rightarrow \quad \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

$$\Rightarrow \quad t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$$

$$t_2 - t_1 \neq 0 \therefore t_1t_2 + 1 = 0$$

$$\therefore \quad t_1t_2 = -1 \text{ or } t_2 = -\frac{1}{t_1}, \quad \text{(i)}$$

which is the required relation.

Note:

If one extremity of a focal chord is $(at_1^2, 2at_1)$ *then the other extremity* $(at_2^2, 2at_2)$ *becomes* $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.

2. If point P is $(at^2, 2at)$, then length of focal chord PQ is $a\left(t + \frac{1}{t}\right)^2$.

Proof:

$$PQ = SP + SQ = a + at^2 + a + \frac{a}{t^2}$$

$$= a\left(t + \frac{1}{t}\right)^2$$

3. The length of the focal chord which makes an angle θ with positive direction of x-axis is $4a \operatorname{cosec}^2 \theta$.

Proof:

$$PQ = a\left(t + \frac{1}{t}\right)^2$$

Now slope of $\quad PQ = \frac{2}{t - \frac{1}{t}} = \tan \theta$

$$\Rightarrow \quad 2 \cot \theta = t - \frac{1}{t}$$

$$\Rightarrow \quad PQ = a\left(t + \frac{1}{t}\right)^2 = a\left[\left(t - \frac{1}{t}\right)^2 + 4\right]$$

$$= a[4 \cot^2 \theta + 4]$$

$$= 4a \operatorname{cosec}^2 \theta$$

From this we can conclude that the minimum length of focal chord is $4a$, which is the length of latus rectum.

4. Semi-latus rectum is harmonic mean of SP and SQ, where P and Q are extremities of focal chord.

Proof:

PQ is focal chord.

$$SP = a + at^2 \text{ and } SQ = a + \frac{a}{t^2}$$

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a + at^2} + \frac{1}{a + a/t^2} = \frac{1}{a}$$

$$\Rightarrow \quad 2a = \frac{2SP \times SQ}{SP + SQ}$$

5. Circle described on the focal length as diameter touches the tangent at vertex.

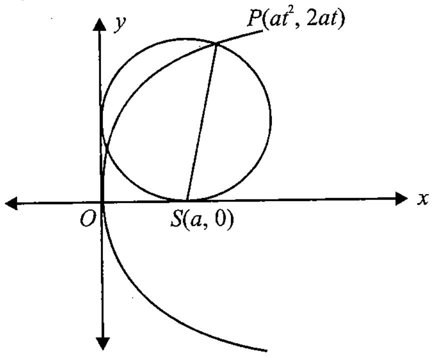


Fig. 3.30

Proof:

Equation of the circle described on SP as diameter is

$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$

Solving it with y-axis, $x = 0$, we have

$(0 - at^2)(0 - a) + (y - 2at)(y - 0) = 0$

or $\quad y^2 - 2aty + a^2t^2 = 0$ which has equal roots.

Hence, y-axis touches the circle.

Also point of contact is $(0, at)$.

6. Circle described on the focal chord as diameter touches directrix.

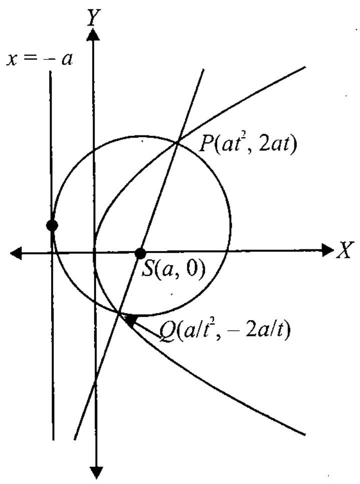


Fig. 3.31

Proof:

Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Solving it with $x = -a$, we have

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

or $y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right) = 0$, which is the perfect square.

Hence, $x = -a$ touches the circle.

Example 3.26 **If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c. Then prove that $b^2c = 4a^3$.**

Sol.

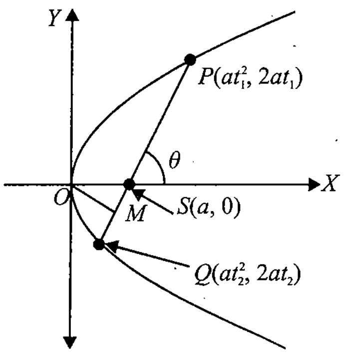


Fig. 3.32

In the figure, $OM = b$ = distance of focal chord from vertex.

Now let focal chord makes an angle θ with positive x- axis.

Then its length, $\quad PQ = 4a \operatorname{cosec}^2 \theta$

Now in right-angled triangle OMS, $\sin\theta = \frac{OM}{OS} = \frac{b}{a}$

$\Rightarrow \quad PQ = 4a\left(\frac{a}{b}\right)^2 = \frac{4a^3}{b^2}$, i.e., $b^2c = 4a^3$

Example 3.27 **If (2, –8) is at an end of a focal chord of the parabola $y^2 = 32x$, then find the other end of the chord.**

Sol. For $y^2 = 32x$, $a = 8$.

Then any point on the parabola is $(8t^2, 16t)$.

Comparing this with point P (2, –8), we get $t = -\frac{1}{2}$.

Now if PQ is the focal chord and point Q is $(8t_1^2, 16t_1)$, then $t_1 = -\frac{1}{t} = 2$.

Hence, point Q has coordinates (32, 32).

Example 3.28 **Circles are drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line, find its equation.**

Sol. $y^2 - 4x - y - 4 = 0$

$\Rightarrow \quad y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$

$\Rightarrow \quad \left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$ circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus circle will touch the line $x + \frac{17}{16} = -1$, i.e., $16x + 33 = 0$.

Example 3.29 **If AB is a focal chord of $x^2 - 2x + y - 2 = 0$ whose focus is 'S'. If $AS = l_1$ then find BS.**

Sol. $x^2 - 2x + y - 2 = 0$

$\Rightarrow \quad x^2 - 2x + 1 = 3 - y$

$\Rightarrow \quad (x - 1)^2 = -(y - 3)$.

Length of its latus rectum is 1 unit.

Since AS, $\frac{1}{2}$, BS are in H.P., thus

$$\frac{1}{2} = \frac{2AS \times BS}{AS + BS}$$

$$\Rightarrow \quad BS = \frac{l_1}{(4l_1 - 1)}$$

Concept Application Exercise 3.2

1. Circle drawn having its diameter equal to focal distance of any point lying on the parabola $x^2 - 4x + 6y + 10 = 0$ will touch a fixed line, find its equation.
2. A circle is drawn to pass through the extremities of the latus rectum of the parabola $y^2 = 8x$. It is given that this circle also touches the directrix of the parabola. Find the radius of this circle.
3. If a focal chord of $y^2 = 4ax$ makes an angle $\alpha \in \left[0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then find the minimum length of this focal chord.
4. If the line passing through the focus S of the parabola $y = ax^2 + bx + c$ meets the parabola at P and Q and if $SP = 4$ and $SQ = 6$ then find the value of a.
5. The coordinates of the ends of a focal chord of a parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then find the value of $x_1x_2 + y_1y_2$.

Intersection of a Line and a Parabola

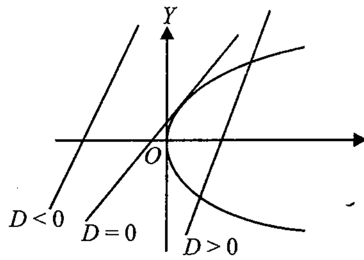


Fig. 3.33

Let the parabola be $y^2 = 4ax$ (i)

and the given line be $y = mx + c$ (ii)

Eliminating y from (i) and (ii), then $(mx + c)^2 = 4ax$

or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ (iii)

This equation, being quadratic in x, gives two values of x and shows that every straight line will cut the parabola in two points may be real, coincident or imaginary according as discriminant of Eq. (iii) is greater, equal or less than 0.

That is, $4(mc - 2a)^2 - 4m^2c^2 >, =, < 0$

or $4a^2 - 4amc >, =, < 0$ in equation (ii),

or $a >, =, < mc$ (iv)

Equation of Tangent

Equation of Tangent at Point $P(x_1, y_1)$ to Parabola $y^2 = 4ax$

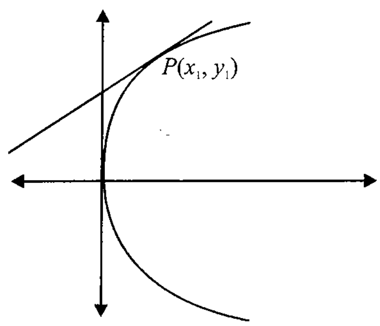


Fig. 3.34

Differentiating $y^2 = 4ax$ with respect to x, we have

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

then equation of tangent at point P is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

or $$yy_1 - 2ax = y_1^2 - 2ax_1$$

$$\Rightarrow \quad yy_1 - 2ax - 2ax_1 = y_1^2 - 4ax_1$$

(adding $-2ax_1$ both sides)

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = 0 \quad \text{(i)}$$

(since point (x_1, y_1) lies on the parabola)

Hence, equation of tangent at point $P(x_1, y_1)$ is given by

$$T = 0$$

where T is an expression which we get by replacing y^2 by yy_1, and $2x$ by $x + x_1$.

Equation of Tangent at Point P(t) or P(at^2, 2at)

In Eq. (i) replace y_1 by $2at$ and x_1 by at^2

we have $2aty = 2a(x + at^2)$ or $ty = x + at^2$ (ii)

Equation of Tangent if Slope of Tangent is m

In Eq. (ii), slope of tangent $m = \frac{1}{t}$

In Eq. (ii) replacing t by $\frac{1}{m}$ we have $y = mx + \frac{a}{m}$ which is equation of tangent in terms of slope.

This is tangent at point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

If line $y = mx + c$ touches parabola $y^2 = 4ax$ we must have $c = \frac{a}{m}$ (comparing equation with $y = mx + \frac{a}{m}$)

Note:

- *Equation of tangent to the parabola $(y - k)^2 = 4a(x - h)$ having slope m is $y - k = m(x - h) + \frac{a}{m}$.*

  Equation of tangent at point p(t) on different parabolas:

-

Equations of parabola	Parametric co-ordinates t	Tangent at $P(t)$
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

-

Equation of parabola	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangency for line $y = mx + c$
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$

$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

Pair of Tangents from Point (x_1, y_1)

Let T (h, k) be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$.

Equation of PT is

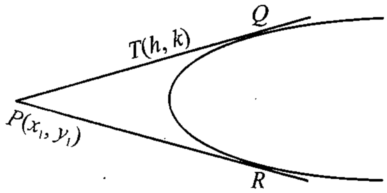


Fig. 3.35

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

or $$y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is tangent to the parabola $y^2 = 4ax$

$\therefore$ $$c = \frac{a}{m}$$

or $$cm = a$$

or $$\left(\frac{hy_1 - kx_1}{h - x_1}\right)\left(\frac{k - y_1}{h - x_1}\right) = a$$

or $$(k - y_1)(hy_1 - kx_1) = a(h - x_1)^2$$

$\therefore$ Locus of (h, k), equation of pair of tangents is

$$(y - y_1)(xy_1 - x_1 y) = a(x - x_1)^2$$

or $$(y^2 - 4ax)(y_1^2 - 4ax_1) = \{(yy_1 - 2a(x + x_1)\}^2$$

or $$SS_1 = T^2,$$

where $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

Example 3.30 **Find equation of the tangent to the parabola $y^2 = 8x$ having slope 2 and also find the point of contact.**

Sol. Equation of the tangent to $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$.

Hence, for the given parabola, equation of the tangent is $y = 2x + \frac{2}{2}$ or $y = 2x + 1$ and point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv \left(\frac{2}{2^2}, \frac{2(2)}{2}\right) \equiv \left(\frac{1}{2}, 2\right).$$

Example 3.31 **Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2, 5).**

Sol. Equation of the given parabola is $y^2 = 12x$ (i)

Comparing with $y^2 = 4ax$, we get $a = 3$.

$\therefore$ Let the equation of the tangent from (2, 5)

i.e., $$y = mx + \frac{3}{m} \quad \text{(ii)}$$

Eq. (ii) passes through the point (2, 5), then

$$5 = 2m + \frac{3}{m}$$

$\Rightarrow$ $$2m^2 - 5m + 3 = 0$$

$\Rightarrow$ $$m = 1, \frac{3}{2}$$

Therefore, From Eq. (ii), the equations of the required tangents are

$$y = x + 3 \text{ and } 2y = 3x + 4.$$

Example 3.32 **If the line $y = 3x + c$ touches the parabola $y^2 = 12x$ at point P, then find the equation of the tangent at point Q where PQ is a focal chord.**

Sol. Line $y = 3x + c$ touches $y^2 = 12x$ then we must have $c = \frac{a}{m}$ or $c = \frac{3}{3} = 1$ and the point of contact is

$$P\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv P\left(\frac{3}{3^2}, \frac{2(3)}{3}\right) \equiv P\left(\frac{1}{3}, 2\right)$$

Comparing this point to $(at^2, 2at)$, we have $2at = 2$

$\Rightarrow$ $$t = \frac{1}{3}$$

Hence, point P has parameter $\frac{1}{3}$, then point Q has parameter -3.

Now tangent at point Q is $(-3)y = x + 3(-3)^2$

or $$x + 3y - 27 = 0.$$

Example 3.33 **Find the equation of tangent to parabola $y = x^2 - 2x + 3$ at point (2, 3).**

Sol. Since the equation of parabola is not in the standard form, we use calculus method to find the equation of tangent.

$y = x^2 - 2x + 3$

Differentiating, w.r.t. x, we have $\frac{dy}{dx} = 2x - 2$

We want to find tangent at point (2, 3)

Then $\left(\frac{dy}{dx}\right)_{(2,3)} = 2(2) - 2 = 2$

Hence using point slope form equation of tangent is

$y-3=2(x-2)$ or $y=2x-1$

Example 3.34 **Find the equation of tangent to parabola $x=y^2+3y+2$ having slope 1.**

Sol. $x=y^2+3y+2$

Differentiating both sides w.r.t. x, we have

$$1=2y\frac{dy}{dx}+3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y+3}$$

Now slope of tangent is 1

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y+3}=1$$

$\Rightarrow$ $y=-1$, which is the ordinate of the point on the curve where slope of tangent is 1.

Putting $y=-1$, in equation of parabola we get $x=0$

Hence using point-slope form we have $y-(-1)=1(x-0)$ or $x-y-1=0$

Example 3.35 **Find the equation of tangents drawn to parabola $y=x^2-3x+2$ from the point $(1,-1)$.**

Sol. Tangents are drawn to the parabola from the point $(1,-1)$.

Now equation of line from $(1,-1)$ having slope m is $y-(-1)=m(x-1)$

or $mx-y-m-1=0$ or $y=mx-m-1$

Since this line touches the parabola, when we solve line and parabola and the resulting quadratic will have equal roots

Solving we have $mx-m-1=x^2-3x+2$

$x^2-(3+m)x+3+m=0$

This equation has equal roots

$$\Rightarrow \quad (m+3)^2-4(m+3)=0$$

$$\Rightarrow \quad m=-3 \text{ or } m=1$$

Hence equation of tangents are $y+1=-3(x-1)$ and $y+1=x-1$

or $3x+y-2=0$ and $x-y-2=0$

Example 3.36 **Find the shortest distance between the line $y=x-2$ and the parabola $y=x^2+3x+2$.**

Sol. Let $P(x_1,y_1)$ be a point closest to the line $y=x-2$

then $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of given line

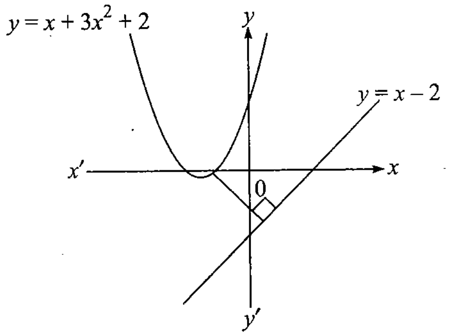


Fig. 3.36

$2x_1+3=1\Rightarrow x_1=-1 \Rightarrow y_1=0$

Hence point $(-1, 0)$ is the closest and its perpendicular distance from the line $y=x-2$ will be the shortest distance $\Rightarrow$ Shortest distance $=\frac{3}{\sqrt{2}}$

Example 3.37 **Find the equation of common tangent of $y^2=4ax$ and $x^2=4ay$.**

Sol.

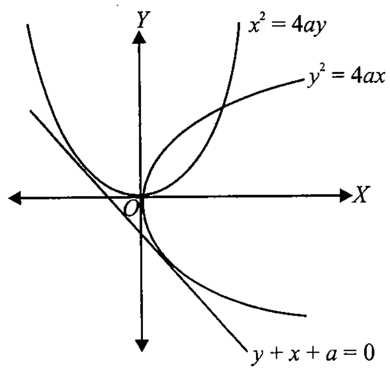


Fig. 3.37

Equation of the tangent to $y^2=4ax$ having slope m is $y=mx+\frac{a}{m}$.

It will touch $x^2=4ay$, if $x^2=4a\left(mx+\frac{a}{m}\right)$ has equal roots.

Thus, $\quad 16a^2m^2=-16\frac{a^2}{m}\ (\because D=0)$

$\Rightarrow \quad m=-1.$

Thus, common tangent is $y+x+a=0$.

Example 3.38 **A tangent to the parabola $y^2=8x$ makes an angle of 45° with the straight line $y=3x+5$. Then find one of the points of contact.**

Sol. Equation of the tangent to the parabola

$y^2=4ax$ at $(at^2, 2at)$ is

$ty=x+at^2$

Here $a=2$, so the equation of the tangent at $(2t^2, 4t)$ to the parabola

$$y^2=8x \text{ is } ty=x+2t^2 \qquad \text{(i)}$$

Slope of Eq. (i) is $\frac{1}{t}$ and that of given line is 3.

$$\Rightarrow \quad \frac{\frac{1}{t}-3}{1+\frac{1}{t}\times 3} = \pm \tan 45° = \pm 1$$

$$\Rightarrow \quad t = -\frac{1}{2} \text{ or } 2$$

For $\quad t = -\frac{1}{2}$, tangent is

$$\left(-\frac{1}{2}\right)y = x + 2\left(\frac{1}{4}\right),$$

i.e., $2x + y + 1 = 0$ at point of contact $\left(\frac{1}{2}, -2\right)$.

For $t = 2$, tangent is $2y = x + 8$, at point of contact (8, 8).

Example 3.39 **Show that $x \cos \alpha + y \sin \alpha = p$ touches the parabola $y^2 = 4ax$ if $p \cos \alpha + a \sin^2 \alpha = 0$ and that the point of contact is $(a \tan^2 \alpha, -2a \tan \alpha)$.**

Sol. The given line is $x \cos \alpha + y \sin \alpha = p$

or $\quad y = -x \cot \alpha + p \text{ coses } \alpha$

$\therefore \quad m = -\cot \alpha$ and $c = p \operatorname{cosec} \alpha$

since the given line touches the parabola

$\therefore \quad c = \frac{a}{m}$ or $cm = a$

$\Rightarrow \quad (p \operatorname{cosec} \alpha)(-\cot \alpha) = a$

and the point of contact is

$$\left(\frac{a}{\cot^2 \alpha}, -\frac{2a}{\cot \alpha}\right) \equiv (a \tan^2 \alpha, -2a \tan \alpha)$$

Example 3.40 **The tangents to a parabola $y^2 = 4ax$ at the vertex V and any point P meet at Q. If S be the focus, then prove that SP, SQ, SV are in G.P.**

Sol. Let the parabola be $y^2 = 4ax$.

Q is the intersection of the lines $x = 0$ and tangent at point $P(at^2, 2at)$, $ty = x + at^2$.

Solving these, we get $Q = (0, at)$. Also $S = (a, 0)$.

Now focal length $\quad SP = a + at^2$

$$SQ^2 = a^2 + a^2t^2 = a^2(t^2 + 1)$$

and $\quad SV = a$

$\therefore \quad SQ^2 = SP \times SV$

$\Rightarrow SP, SQ, SV$ are in G.P.

Example 3.41 **Parabola $y^2 = 4x$ and the circle having its centre at (6, 5) intersect at right angle. Then find the possible points of intersection of these curves.**

Sol. Let the possible point be $(t^2, 2t)$. Equation of the tangent at this point is $yt = x + t^2$.

It must pass through centre of the circle (6, 5).

$\Rightarrow \quad t^2 - 5t + 6 = 0$

$\Rightarrow \quad t = 2, 3$

$\Rightarrow$ Possible points are (4, 4), (9, 6).

Example 3.42 **If a tangent to the parabola $y^2 = 4ax$ meets the x-axis at T and intersect tangent at vertex A at P, and the rectangle $TAPQ$ be completed. Then find the locus of point Q.**

Sol.

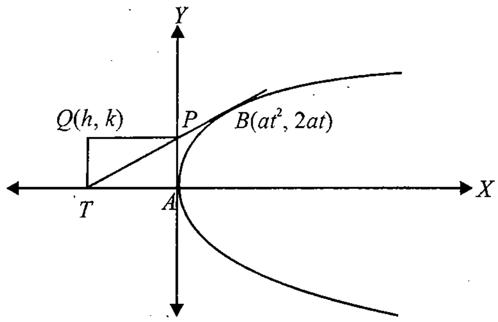


Fig. 3.38

The tangent at any point B $(at^2, 2at)$ to the parabola is

$$ty = x + at^2. \quad \text{(i)}$$

Since tangent at the vertex A is y-axis, so T and P are $(-at^2, 0)$ and $(0, at)$, respectively. Clearly A is (0, 0).

If Q be (h, k), then $h = AT = -at^2$ and $k = AP = at$

Eliminating t, we get $k^2 + ah = 0$.

Hence, the locus of Q is $y^2 + ax = 0$,

which is a parabola.

Example 3.43 **Two tangents are drawn from the point (–2, –1) to the parabola $y^2 = 4x$. If α is the angle between these tangents, then find the value of $\tan \alpha$.**

Sol. Here $a = 1$. Any tangent having slope m is $y = mx + \frac{1}{m}$.

It passes through (– 2, –1)

$\Rightarrow \quad 2m^2 - m - 1 = 0$

$\Rightarrow \quad m = 1, -\frac{1}{2}$

$$\Rightarrow \quad \tan \alpha = \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$$

Example 3.44 **If two tangents drawn from the point (α, β) to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then prove that $\alpha = \frac{2}{9}\beta^2$.**

Sol. Any tangent to the parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

It passes through (α, β), therefore,

$$\beta = m\alpha + \frac{1}{m}$$

or $\quad \alpha m^2 - \beta m + 1 = 0$

According to the question, it has roots m_1, $2m_1$

Now, $\quad m_1 + 2m_1 = \frac{\beta}{\alpha}$ and $m_1 \cdot 2m_1 = \frac{1}{\alpha}$

$$\Rightarrow \quad 2\left(\frac{\beta}{3\alpha}\right)^2 = \frac{1}{\alpha} \Rightarrow \alpha = \frac{2}{9}\beta^2$$

Example 3.45 **A pair of tangents are drawn to parabola $y^2 = 4ax$ which are equally inclined to a straight line $y = mx + c$, whose inclination to the axis is α. Prove that the locus of their point of intersection is the straight line $y = (x - a) \tan 2\alpha$.**

Sol.

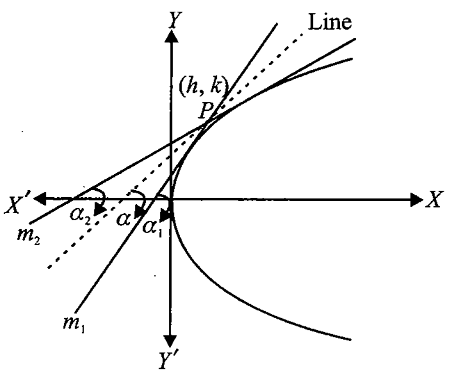


Fig. 3.39

We have $\quad \theta = \alpha_1 - \alpha = \alpha - \alpha_2$

$\therefore \quad 2\alpha = \alpha_1 + \alpha_2$

$$\Rightarrow \quad \tan 2\alpha = \tan(\alpha_1 + \alpha_2) = \frac{m_1 + m_2}{1 - m_1 m_2}$$

but (h, k) lies on $\quad y = mx + \frac{a}{m}$

$\Rightarrow \quad m^2 h - km + a = 0$

Hence, $\quad m_1 + m_2 = \frac{k}{h}$ and $m_1 m_2 = \frac{a}{h}$;

$$\tan 2\alpha = \frac{\frac{k}{h}}{1 - \frac{a}{h}} = \frac{k}{h - a}$$

$$\Rightarrow \quad y = (x - a) \tan 2\alpha$$

Example 3.46 **Tangents are drawn from the point $(-1, 2)$ on the parabola $y^2 = 4x$. Find the length that these tangents will intercept on the line $x = 2$.**

Sol.

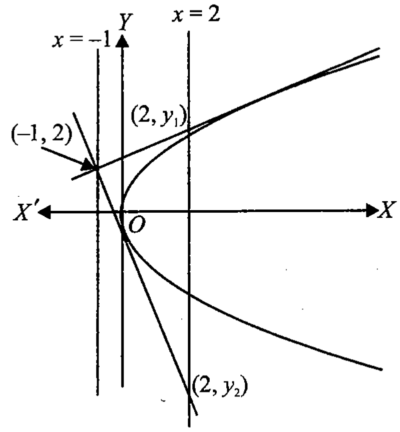


Fig. 3.40

Equation of the pair of tangents from point (x_1, y_1) is $SS_1 = T^2$

or $\quad (y^2 - 4x)(y_1^2 - 4x_1) = (yy_1 - 2(x + x_1))^2$

Then equation of the pair of tangents from $(-1, 2)$ is

$$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2$$
$$= 4(y - x + 1)^2$$

or $\quad 2(y^2 - 4x) = (y - x + 1)^2$

Solving with the line $\quad x = 2$, we get

$$2(y^2 - 8) = (y - 1)^2$$

or $\quad y^2 + 2y - 17 = 0$

where $\quad y_1 + y_2 = -2$ and $y_1 y_2 = -17$

Now $\quad |y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1 y_2$

$$= 4 - 4(-17) = 72$$

$\therefore \quad |y_1 - y_2| = \sqrt{72} = 6\sqrt{2}$

Properties of Tangents

Point of Intersection of Tangents at Any Two Points on the Parabola

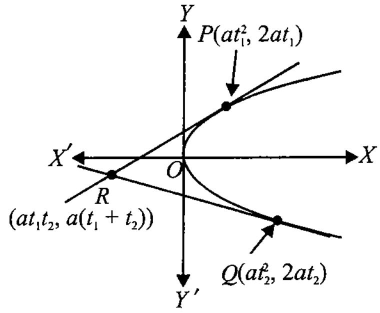


Fig. 3.41

Let the equation of the parabola be $y^2 = 4ax$

The two points on the parabola are $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$

Equation of the tangents at P and Q are

$$t_1 y = x + at_1^2 \quad \text{(i)}$$

and

$$t_2 y = x + at_2^2 \quad \text{(ii)}$$

Solving these equations, we get

$$x = at_1t_2,\ y = a(t_1 + t_2).$$

Thus, the co-ordinates of the point of intersection of tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are $(at_1t_2, a(t_1 + t_2))$.

Note:

- *The geometric mean of the x-coordinates of P and Q (i.e.,* $\sqrt{at_1^2 \times at_2^2} = at_1t_2$*) is x-coordinates of the point of intersection of tangents at P and Q on the parabola.*
- *The arithmetic mean of the y-coordinates of P and Q* $\left(\text{i.e., } \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ *is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.*

Locus of Foot of Perpendicular From Focus Upon Any Tangent is Tangent at Vertex

Equation of the tangent to parabola $y^2 = 4ax$ at point $P(t)$ is $ty = x + at^2$

It meets y-axis at $Q(0, at)$

Now $$m_{SQ} = \frac{at - 0}{0 - a} = -t$$

Slope of the tangent PQ is $\frac{1}{t}$. Hence, SQ is perpendicular to PQ.

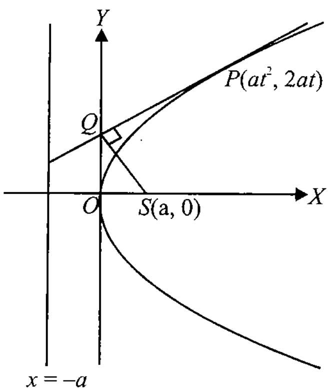


Fig. 3.42

Length of Tangent Between the Point of Contact P(t) and the Point Where it Meets the Directrix Q Subtends Right Angle at Focus

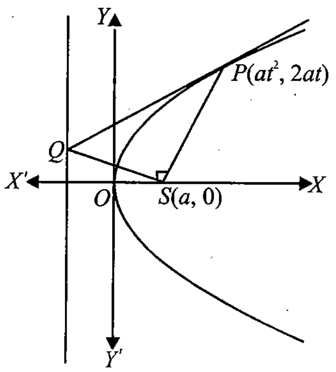


Fig. 3.43

Equation of the tangent to parabola $y^2 = 4ax$ at point $P(t)$ is $ty = x + at^2$

It meets $x = -a$ at $\left(-a, \dfrac{at^2 - a}{t}\right)$

Now the slope of $SP = m_{SP} = \dfrac{2at - 0}{at^2 - a} = \dfrac{2t}{t^2 - 1}$

Slope of $SQ = m_{SQ} = \dfrac{\frac{at^2 - a}{t} - 0}{-a - a} = \dfrac{1 - t^2}{2t}$

Hence, SP is perpendicular to SQ.

Tangents at Extremities of Focal Chord are Perpendicular and Intersect on Directrix

Slope of the tangent at point $P(t)$ is $\frac{1}{t}$

If PQ is focal chord then point Q has parameter $-\frac{1}{t}$

Then slope of the tangent at point Q is $-t$

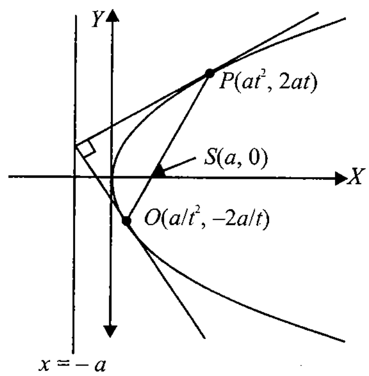


Fig. 3.44

Hence, the tangents are perpendicular.

Moreover, the point of intersection of the tangents at point $P(t_1)$ and $Q(t_2)$ is

$$(at_1t_2, a(t_1+t_2)) \equiv \left(a\frac{1}{t}(-t), a\left(t-\frac{1}{t}\right)\right) \equiv \left(-a, a\left(t-\frac{1}{t}\right)\right)$$

Thus, tangents intersect on the directix.

Example 3.47 **Find the points of contact Q and R of a tangent from the point $P(2, 3)$ on the parabola $y^2 = 4x$.**

Sol. $\left.\begin{matrix} t_1t_2 = 2 \\ t_1+t_2 = 3 \end{matrix}\right] \Rightarrow t_1 = 1 \text{ and } t_2 = 2$

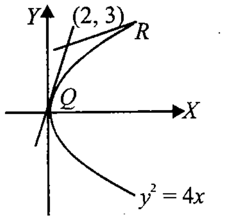


Fig. 3.45

Hence, points $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$. i.e., (1, 2) and (4, 4).

Example 3.48 **Two straight lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents of $y^2 = 4ax$. Prove $m_1m_2 = -1$.**

Sol. Clearly both the lines pass through $(-a, b)$, which is a point lying on the directrix of the parabola.

Thus, $\quad m_1m_2 = -1$.

Because tangents drawn from any point on the directrix are always mutually perpendicular.

Example 3.49 **Mutually perpendicular tangents TA and TB are drawn to $y^2 = 4ax$, then find the minimum length of AB.**

Sol. Chord of contact of mutually perpendicular tangents is always a focal chord. Thus, minimum length of AB is $4a$.

Example 3.50 **Tangents PA and PB are drawn from point P on the directrix of the parabola $(x - 2)^2 + (y - 3)^2 = \frac{(5x - 12y + 3)^2}{169}$. Find the least radius of the circumcircle of the triangle PAB.**

Sol. Tangents from any point on the directrix are perpendicular and the corresponding chord of contact is the focal chord which is the diameter of the circumcircle of the triangle PAB. The least value of the diameter is the latus rectum.

For the given parabola, focus is (2, 3) and directrix is $5x - 12y + 3 = 0$

Hence, latus rectum $= \frac{|10 - 36 + 3|}{13} = \frac{23}{13}$.

Example 3.51 **Tangents are drawn to the parabola $(x - 3)^2 + (x + 4)^2 = \frac{(3x - 4y - 6)^2}{25}$ at the extremities of the chord $2x - 3y - 18 = 0$. Find the angle between the tangents.**

Sol. The given chord $2x - 3y - 18 = 0$ satisfies the point $(3, -4)$ which is focus of the given parabola. Hence, it is focal chord and tangents at extremities are perpendicular.

Locus of Point of Intersection of Tangent under Different Conditions

Tangents to the parabola $y^2 = 4ax$ at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$, respectively. These tangents intersect at point $R(at_1t_2, a(t_1 + t_2))$. If we want to find the locus of point R under some conditions then let point R has coordinates (h, k). We have $h = at_1t_2$ and $k = a(t_1 + t_2)$.

Example 3.52 **Find the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$.**

a. which are inclined at an angle θ to each other

b. which intercept constant length c on the tangent at the vertex

c. such that area of ΔABR is constant c, where A and B are the point of intersection of tangents with y-axis and R is a point of intersection of tangents.

Sol.

a. Given that $\tan\theta = \left|\dfrac{\frac{1}{t_1} - \frac{1}{t_1}}{1 + \frac{1}{t_1}\frac{1}{t_2}}\right| = \left|\dfrac{t_1 - t_2}{1 + t_1t_2}\right|$

$\Rightarrow \tan^2\theta\,(1 + t_1t_2)^2 = (t_1 - t_2)^2$

$\Rightarrow \tan^2\theta\,(1 + t_1t_2)^2 = (t_1 + t_2)^2 + 4t_1t_2$

$\Rightarrow \tan^2\theta\left(1 + \frac{h}{a}\right)^2 = \left(\frac{k}{a}\right)^2 + 4\frac{h}{a}$

$\Rightarrow \tan^2\theta\,(x + a)^2 = y^2 + 4ax$

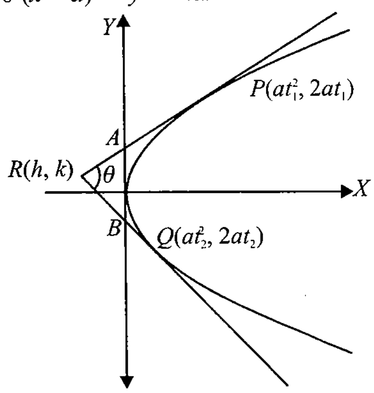


Fig. 3.46

b. Points A and B have coordinates $(0, at_1)$ and $(0, at_2)$

Given $AB = c$ or $|at_1 - at_2| = c$

$\Rightarrow a^2[(t_1 + t_2)^2 - 4t_1t_2] = c^2$

$\Rightarrow a^2\left[\left(\frac{k}{a}\right)^2 - 4\frac{h}{a}\right] = c^2$

$\Rightarrow y^2 - 4ax = c^2$ which is a parabola.

c. Points A and B have coordinates $(0, at_1)$ and $(0, at_2)$

Given area of triangle $ABR = c$

$\Rightarrow \frac{1}{2} AB \times RM = c$

$\Rightarrow \frac{1}{2}|at_1 - at_2|(at_1t_2) = c^2$

$\Rightarrow a^4[(t_1 + t_2)^2 - 4t_1t_2](t_1t_2)^2 = 4c^2$

$\Rightarrow a^4\left[\left(\frac{k}{a}\right)^2 - 4\frac{h}{a}\right]\left(\frac{h}{a}\right)^2 = 4c^2$

$\Rightarrow (y^2 - 4ax)x^2 = 4c^2$

Concept Application Exercise 3.3

1. Find the point on the curve $y^2 = ax$ the tangent at which makes an angle of 45° with x-axis.
2. Find the equation of the straight lines touching both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.
3. Find the angle at which the parabolas $y^2 = 4x$ and $x^2 = 32y$ intersect.
4. How many distinct real tangents that can be drawn from $(0, -2)$ to the parabola $y^2 = 4x$?
5. The tangents to the parabola $y^2 = 4x$ at the points (1, 2) and (4, 4) meet on which of the following lines?

   a. $x = 3$ **b.** $x + y = 4$

   c. $y = 3$ **d.** $y = 4$
6. If the tangents at the points P and Q on the parabola $y^2 = 4ax$ meet at T and S is its focus, then prove that SP, ST and SQ are in G.P.
7. If the line $x + y = a$ touches the parabola $y = x - x^2$, then find the value of a.
8. From an external point P, pair of tangent are drawn to the parabola $y^2 = 4x$. If θ_1 and θ_2 are the inclinations of these tangents with x-axis such that $\theta_1 + \theta_2 = \frac{\pi}{4}$, then find the locus of P.
9. Find the angle between the tangents drawn from (1, 3) to the parabola $y^2 = 4x$.
10. Find the slopes of the tangents to the parabola $y^2 = 8x$ which are normal to the circle $x^2 + y^2 + 6x + 8y - 24 = 0$.
11. Find the angle between the tangents drawn to $y^2 = 4x$, where it is intersected by the line $y = x - 1$.
12. Find the locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$.
13. From a point P on directrix, tangents PA and PB are drawn to the parabola $y^2 = 16x$. Find the minimum radius of the circle circumscribing ΔPAB.
14. Find the angle between the tangents drawn from the origin to the parabolas $y^2 = 4a(x - a)$.
15. Find the locus of point from which the two tangents drawn to a parabola $y^2 = 4ax$ are such that slope of one is thrice of the other.

Equation of Normal

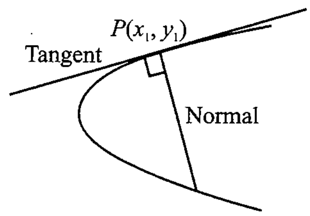


Fig. 3.47

Differentiating $y^2 = 4ax$ with respect to x, we have $\frac{dy}{dx} = \frac{2a}{y}$

The slope of the tangent at $(x_1, y_1) = \frac{2a}{y_1}$

Since the normal at (x_1, y_1) is perpendicular to the tangent at (x_1, y_1)

$\therefore$ Slope of normal at $(x_1, y_1) = -\frac{y_1}{2a}$

Hence, the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{(i)}$$

Parametric Form

Replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes $y - 2at = -t(x - at^2)$ or $y = -tx + 2at + at^3$

The equations of normals of all standard parabolas are as follows:

Equations of parabola	Parametric coordinates t	Normals at t
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Slope Form

The equation of normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3 \quad \text{(i)}$$

Since m is the slope of the normal, then $m = -t$

Then equation of normal is

$$y = mx - 2am - am^3. \quad \text{(ii)}$$

Thus $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal.

The coordinates of the foot of normal are $(am^2, -2am)$.

Comparing (ii) with $y = mx + c$

$\therefore$ $c = -2am - am^3$

which is the condition when $y = mx + c$ is the normal of $y^2 = 4ax$.

Equation of normal for all parabolas in terms of m.

Equations of parabolas	Point of contact in terms of slope (m)	Equations of normals in terms of slope (m)	Condition for line $y = mx + c$ is normal
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Example 3.53 **Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that one of the normals is given by $y = x - 3$ and find the equations of the others.**

Sol. Equation of the normal to $y^2 = 4x$ having slope m is $y = mx - 2m - m^3$.

If it passes through the point (15, 12), then

$$12 = 15m - 2m - m^3$$

$\Rightarrow$ $m^3 - 13m + 12 = 0$

$\Rightarrow (m - 1)(m - 3)(m + 4) = 0$

$\Rightarrow$ $m = 1, 3, -4$

Taking $m = 1$, the equation of normal is $y = x - 3$, which is one of the normals.

Taking $m = 3$, and -4, the equations of other two normals are $y = 3x - 33$ and $y + 4x = 72$.

Example 3.54 **Find the equations of normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum.**

Sol. Differentiating $y^2 = 4ax$ w.r.t. x, we have $\frac{dy}{dx} = \frac{2a}{y}$

Hence, slope of normal at point $P(a, 2a)$ is $-\frac{2a}{2a} = -1$

Slope of normal at point $Q(a, -2a)$ is $-\frac{-2a}{2a} = 1$

Hence, equation of normal at point P and Q are $x + y - 3a = 0$ and $x - y - 3a = 0$.

Example 3.55 **If $y = x + 2$ is normal to parabola $y^2 = 4ax$, then find the value of a.**

Sol. Normal to parabola $y^2 = 4ax$ having slope m is $y = mx - 2am - am^3$

Given normal is $y = x + 2 \Rightarrow m = 1$ and $-2am - am^3 = 2$

$\Rightarrow$ $-2a(1) - a(1)^3 = 1$

$\Rightarrow$ $a = -1/3$

Example 3.56 **Find the equation of normal to the parabola $y = x^2 - x - 1$ which has equal intercept on axis.**

Also find the point where this normal meets the curve again.

Sol. Normal has equal intercept on axis, then its slope is -1.

Now differentiating $y = x^2 - x - 1$ w.r.t. x both sides

we have $\frac{dy}{dx} = 2x - 1$, which is the slope of the tangent to the parabola at any point on the parabola.

Now Slope of normal to curve at any point is

$$m = -\frac{dx}{dy} = \frac{1}{1-2x}$$

Then we want slope of normal as $-1 \Rightarrow \frac{1}{1-2x} = -1$

$\Rightarrow x = 1$

$\Rightarrow y = -1$ (from $y = x^2 - x - 1$)

Hence from point – slope form equation of normal is

$y - (-1) = -1(x - 1)$ or $x + y = 0$

Solving this equation of normal with the equation of parabola.

$-x = x^2 - x - 1$ or $x^2 = 1$ or $x = \pm 1$

Hence normal meets parabola again at point whose abscissa is -1, for which ordinate is 1

Thus normal meets parabola again at $(1, -1)$.

Example 3.57 **Find the minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.**

Sol. Centre and radius of the given circle is $P(6, 0)$ and $\sqrt{5}$, respectively.

Now the shortest distance always occurs along common normal.

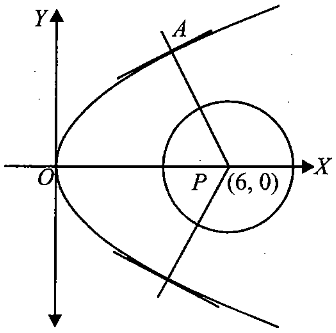


Fig. 3.48

Differentiating $y^2 = 4x$ with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{y}$$

Then the slope of normal at point $A\left(y_1^2/4, y_1\right)$ is $\frac{y_1}{2}$.

Also from definition, the slope of AP is given by

$$\frac{y_1 - 0}{\frac{y_1^2}{4} - 6} = -\frac{y_1}{2}$$

$\Rightarrow$ $\qquad y_1 = 0$ or $y_1 = \pm 4$

Hence, the points are $O(0, 0)$, $A(4, 4)$, $C(4, -4)$.

The shortest distance is $AP - \sqrt{5} = \sqrt{20} - \sqrt{5} = \sqrt{5}$.

Example 3.58 **Prove that the length of the intercept on the normal at the point $P(at^2, 2at)$ of a parabola $y^2 = 4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$.**

Sol.

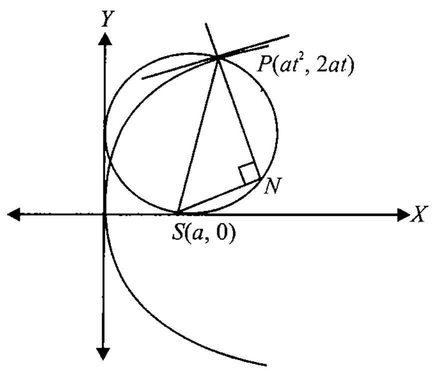


Fig. 3.49

Equation of normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3 \qquad \text{(i)}$$

Let this normal meets the circle on SP as diameter in N, then $\angle SNP = \frac{\pi}{2}$ (angle in a semicircle)

$\therefore$ $\qquad PN^2 = SP^2 - SN^2$

SN is perpendicular to the normal

Now $\qquad SP = a + at^2$

and $\qquad SN = \frac{|at - 2at - at^3|}{\sqrt{1+t^2}}$

$$= at\sqrt{1+t^2}$$

$\therefore$ $\qquad PN^2 = a^2(1+t^2)^2 - a^2t^2(1+t^2)$

$$= a^2(1+t^2)[1+t^2-t^2]$$

$$= a^2(1+t^2)$$

$\therefore$ $\qquad PN = a\sqrt{1+t^2}$

Properties of Normal

1. Normal other than axis of parabola never passes through the focus.

   Proof:

   Let normal at $P(am^2, -2am)$, $y = mx - 2am - am^3$ passes through the focus $(a, 0)$.

   Then $\qquad 0 = am - 2am - am^3$

   $\Rightarrow$ $\qquad m^2 + 1 = 0$, which is not possible.

   Hence, proved.

2. Point of intersection of normal at point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$:

   Solving normal at point P, $y = -t_1x + 2at_1 + at_1^3$ and normal at point Q, $y = -t_2x + 2at_2 + at_2^3$,

   we have point of intersection, which is $[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1+t_2)]$.

3. Normal at the point $P(t_1)$ meets the curve again at point $Q(t_2)$ such that

$$t_2 = -t_1 - \frac{2}{t_1}$$

   Proof:

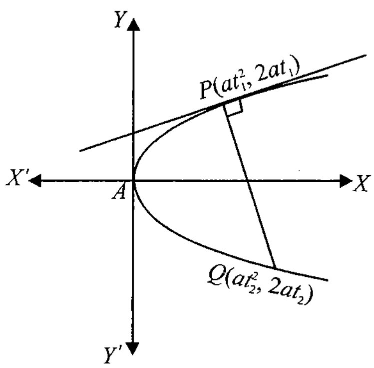


Fig. 3.50

For the parabola be $y^2 = 4ax$, the equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \quad \text{(i)}$$

Since it meets the parabola again at $Q\ (at_2^2, 2at_2)$, Eq. (i) passes through $Q(at_2^2, 2at_2)$.

$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$\Rightarrow 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow a\ (t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because \quad a(t_2 - t_1) \neq 0 \quad (\because t_1 \text{ and } t_2 \text{ are different})$

$\therefore \quad 2 + t_1\ (t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \frac{2}{t_1}$

Example 3.59 **In the parabola $y^2 = 4ax$, the tangent at P whose abscissa is equal to the latus rectum meets its axis at T and normal at P cuts the curve again at Q. Show that PT: $PQ = 4 : 5$.**

Sol. Let P be $(at^2, 2at)$. Since $at^2 = 4a$, $t = \pm 2$. Consider $t = 2$, and $P(4a, 4a)$. Tangent at P is $2y = x + 4a$ which meets x-axis at $T(-4a, 0)$.

If coordinates of Q are $(at_1^2, 2at_1)$ then $t_1 = -t - \frac{2}{t} = -3$

so Q is $(9a, -6a)$

$\therefore \quad (PQ)^2 = 125a^2$ and $(PT)^2 = 80a^2$

or $\quad PT{:}PQ = 4{:}5$

Example 3.60 **If the normal to the parabola $y^2 = 4ax$ at point t_1 cuts the parabola again at point t_2, then prove that $t_2^2 \geq 8$.**

Sol. A normal at point t_1 cuts the parabola again at t_2, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0$

Since, t_1 is real, discriminant ≥ 0

$\Rightarrow \quad t_2^2 - 8 \geq 0$

$\Rightarrow \quad t_2^2 \geq 8$

Example 3.61 **Find the length of normal chord which subtends an angle of 90° at the vertex of the parabola $y^2 = 4x$.**

Sol.

$$t_1 = -t - \frac{2}{t} \text{ also, } tt_1 = -4$$

$(\because OP \perp OQ)$

$\Rightarrow \quad t_1 = -\frac{4}{t}$

$\Rightarrow \quad -\frac{4}{t} = -t - \frac{2}{t}$

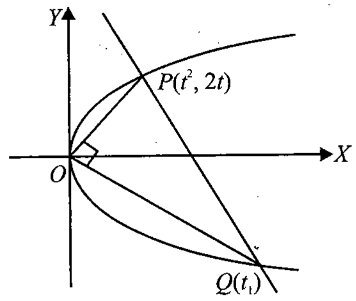


Fig. 3.51

$\Rightarrow \quad \frac{2}{t} = t$

$\Rightarrow \quad t = \sqrt{2}$

$\Rightarrow \quad t_1 = -2\sqrt{2}$

$\Rightarrow \quad Q \equiv (8, -4\sqrt{2}), P \equiv (1, 2)$

$\Rightarrow \quad PQ = \sqrt{7^2 + (2 + 4\sqrt{2})^2} = \sqrt{85 + 16\sqrt{2}}$

Example 3.62 **Find the locus of the point of intersection of the normals at the end of the focal chord of the parabola $y^2 = 4ax$.**

Sol. Point of intersection of normal at point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$R \equiv [2a + a\ (t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)] = (h, k)$$

Also PQ is focal chord, then $t_1t_2 = -1$

Then $\quad k = a(t_1 + t_2) \quad$ (i)

and

$$h = 2a + a[(t_1 + t_2)^2 - t_1t_2]$$

$$= 2a + a[(t_1 + t_2)^2 + 1] \quad \text{(ii)}$$

Eliminating $t_1 + t_2$ from Eqs. (i) and (ii), we have

$$h = 2a + a\left(\frac{k^2}{a^2} + 1\right)$$

$\Rightarrow \quad y^2 = a(x - 3a)$

Example 3.63 **Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another.**

Sol. The equation of the normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

It passes through the point (h, k) if

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \quad$ (i)

Let the roots of the above equation be m_1, m_2 and m_3.

Let the perpendicular normals correspond to the values of m_1 and m_2 so that $m_1m_2 = -1$.

From Eq. (i), $m_1m_2m_3 = -\frac{k}{a}$

Since $m_1m_2 = -1$, $m_3 = \frac{k}{a}$

Since m_3 is a root of Eq. (i), we have

$$a\left(\frac{k}{a}\right)^3 + \frac{k}{a}(2a-h) + k = 0$$

$$\Rightarrow \quad k^2 + a(2a-h) + a^2 = 0$$

$$\Rightarrow \quad k^2 = a(h-3a)$$

Hence, the locus of (h, k) is

$$y^2 = a(x-3a)$$

Example 3.64 Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is a normal to the curve.

Sol. Consider a chord PQ joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on parabola $y^2 = 4ax$.

If slope of PQ is m, then we have

$$m = 2a(t_2 - t_1)/a(t_2^2 - t_1^2)$$

$$= 2/(t_1 + t_2) \quad \text{(i)}$$

Normals at P and Q are

$$y + t_1x = 2at_1 + at_1^3$$

and $$y + t_2x = 2at_2 + at_2^3$$

Let the normal meet at $A(x_1, y_1)$, then $x_1 = 2a + a(t_1^2 + t_2^2 + t_1t_2) = 2a + a[(t_1 + t_2)^2 - t_1t_2]$

and $$y_1 = -at_1t_2(t_1 + t_2) \quad \text{(ii)}$$

Using Eq. (i) and (ii), we get $x_1 - 2a = a\left[4/m^2 + y_1m/2a\right]$

The locus of $A(x_1, y_1)$ is $\frac{1}{2}my = x - 2a - \frac{4a}{m^2}$

i.e., $$y - \left(\frac{2}{m}\right)x = -\frac{4a}{m} - \frac{8a}{m^3} \quad \text{(ii)}$$

Putting $-\frac{2}{m} = t$, locus of Eq. (ii) can be expressed as $y + tx = 2at + at^3$, which is normal to the parabola.

Co-normal Points

Let $P(h, k)$ be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k) then

$$k = mh - 2am - am^3$$

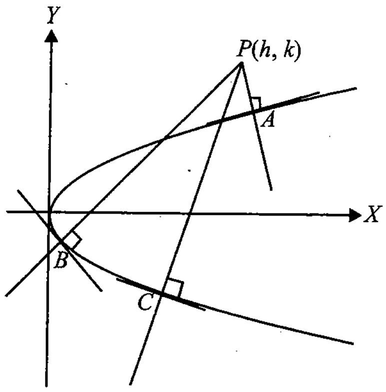


Fig. 3.52

$$\Rightarrow \quad am^3 + m(2a-h) + k = 0 \quad \text{(i)}$$

This is a cubic equation in m. So, it has three roots, say m_1, m_2 and m_3.

$$\therefore \quad m_1 + m_2 + m_3 = 0,$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a} \quad \text{(ii)}$$

Hence, for any given point $P(h, k)$, Eq. (i) has maximum three real roots. Corresponding to each of these three roots, we have one normal passing through $P(h, k)$.

Hence, in total, we have maximum three normals PA, PB and PC drawn through P to the parabola. Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are called co-normal points.

Note:

1. *The algebraic sum of ordinates of the feet of three normals drawn to a parabola from a given point is 0.*

   Proof:

   Let the ordinates of A, B, C be y_1, y_2, y_3 respectively. Then $y_1 = -2am_1$, $y_2 = -2am_2$ and $y_3 = -2am_3$.

   Therefore, algebraic sum of these ordinates is

   $$y_1 + y_2 + y_3$$
   $$= -2am_1 - 2am_2 - 2am_3$$
   $$= -2a(m_1 + m_2 + m_3)$$
   $$= -2a \times 0 \quad \{\textit{from Eq. (i)}\}$$
   $$\Rightarrow y_1 + y_2 + y_3 = 0$$

2. *Centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.*

   Proof:

   If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

   $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$

   Since $y_1 + y_2 + y_3 = 0$.

Therefore, the centroid lies on the x-axis OX, which is the axis of the parabola also.

3. *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.*

Proof:

When normals are real, then all the three roots of Eq. (i) are real and in that case

$m_1^2 + m_2^2 + m_3^2 > 0$ *(for any value of m_1, m_2, m_3)*

$\Rightarrow (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow (0)^2 - \dfrac{2(2a-h)}{a} > 0$

$\Rightarrow h - 2a > 0$

$\Rightarrow h > 2a$

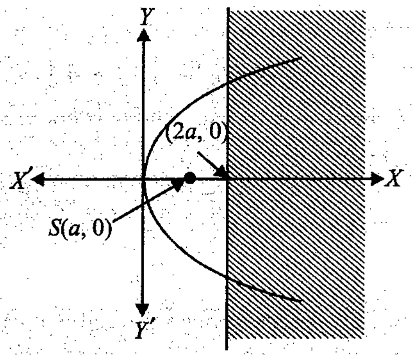


Fig. 3.53

As shown in Fig. 3.53, point (h, k) must lie in the shaded region, from which we can draw three normals to the parabola $y^2 = 4ax$.

Example 3.65 **Find the number of distinct normals that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.**

Sol.

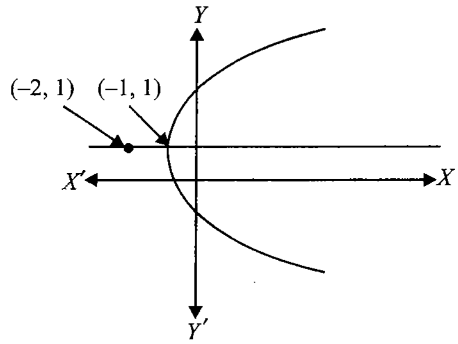


Fig. 3.54

Here, $y^2 - 2y + 1 = 4(x + 1)$

or

$(y - 1)^2 = 4(x + 1)$

So, the axis is $y - 1 = 0$.

Also (–2, 1) lies on the axis, and it is exterior to the parabola because $1^2 - 4(-2) - 2(1) - 3 > 0$.

Hence, only one normal is possible.

Example 3.66 **If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ be (1, 2) and (1, –2), then find the third foot.**

Sol. The sum of the ordinates of the feet $= y_1 + y_2 + y_3 = 0$.

$\therefore \quad 2 + (-2) + y_3 = 0$

$\therefore \quad y_3 = 0$

$\therefore$ third foot is (0, 0)

Example 3.67 **If three distinct normals can be drawn to the parabola $y^2 - 2y = 4x - 9$ from the point $(2a, b)$, then find the range of the value of a.**

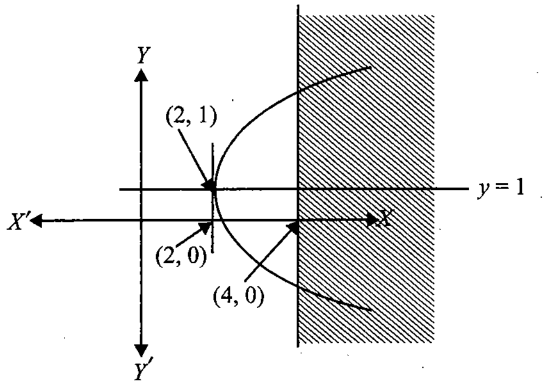


Fig. 3.55

Sol. Given parabola is $(y - 1)^2 = 4(x - 2)$. Hence, vertex is (2, 1).

Focus is (2, 2). Hence, all the points $(2a, b)$ must lie in the shaded region

$\Rightarrow \quad 2a > 4$

$\Rightarrow \quad a > 2$

Example 3.68 **If (h, k) is a point on the axis of the parabola $2(x-1)^2 + 2(y-1)^2 = (x + y + 2)^2$ from where three distinct normals may be drawn, then prove that $h > 2$.**

Sol.

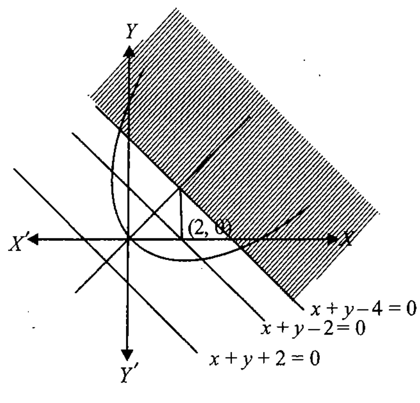


Fig. 3.56

We have, $2(x-1)^2 + 2(y-1)^2 = (x+y+2)^2$

$$\Rightarrow \sqrt{(x-1)^2+(y-1)^2} = \left|\frac{x+y+2}{\sqrt{1+1}}\right|$$

Focus is (1, 1) and directrix is $x + y + 2 = 0$

Equation of latus rectum is $x + y - 2 = 0$

Then the required points lie above the line $x + y - 4 = 0$ which intersect the line $x = y$ (axis of parabola) at (2, 2).

Hence, $h > 2$

Example 3.69 **If the normals from any point to the parabola $y^2 = 4x$ cut the line $x = 2$ in points whose ordinates are in A.P., then prove that slopes of tangents at the co-normal points are in G.P.**

Sol. Equation of the normal to the parabola $y^2 = 4x$ is given by

$$y = -tx + 2t + t^3 \quad \text{(i)}$$

Since it intersects $x = 2$, we get $y = t^3$

Let the three ordinates be t_1^3, t_2^3, t_3^3 are in A.P.

$$\Rightarrow 2t_2^3 = t_1^3 + t_3^3$$
$$= (t_1+t_3)^3 - 3t_1t_3(t_1+t_3) \quad \text{(ii)}$$

Now $t_1 + t_2 + t_3 = 0$

$\Rightarrow$ $t_1 + t_3 = -t_2$

Hence, Eq. (ii) reduces to

$$2t_2^3 = (-t_2)^3 - 3t_1t_3(-t_2)$$
$$= -t_2^3 + 3t_1t_2t_3$$

$$\Rightarrow 3t_2^3 = 3t_1t_2t_3 \Rightarrow t_2^2 = t_1t_3$$

$\Rightarrow t_1, t_2, t_3$ are in G.P.

Hence, slopes of tangents $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ are in G.P.

Circle through the Co-normal Points

To find the equation of the circle passing through the three (co-normal) points on the parabola, normal at which pass through a given point (h, k)

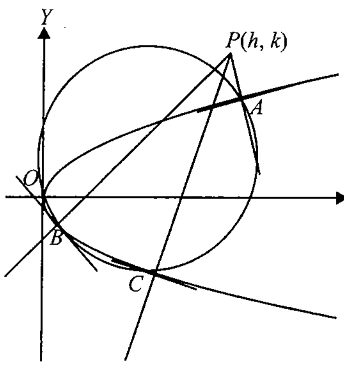


Fig. 3.57

Let $A(am_1^2, -2am_1)$ $B(am_2^2, -2am_2)$ and $C(am_3^2, -2am_3)$ be the three points on the parabola $y^2 = 4ax$. Since the point of intersection of normals at these points is (h, k)

$$\therefore \quad am^3 + (2a-h)m + k = 0 \quad \text{(i)}$$

$$\Rightarrow \quad m_1 + m_2 + m_3 = 0 \quad \text{(ii)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a} \quad \text{(iii)}$$

and

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{(iv)}$$

Let the equation of the circle through A, B and C be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(v)}$$

If the point $(am^2, -2am)$ lies on it, then

$$(am^2)^2 + (-2am)^2 + 2g(am^2) + 2f(-2am) + c = 0$$

or $a^2m^4 + (4a^2 + 2ag)m^2 - 4afm + c = 0$ (vi)

This equation has four roots m_1, m_2, m_3 and m_4 such that the circle passes through the points $A\left(am_1^2, -2am_1\right)$, $B\left(am_2^2, -2am_2\right)$, $C\left(am_3^2, -2am_3\right)$ and $D\left(am_4^2, -2am_4\right)$.

$$\therefore \quad m_1 + m_2 + m_3 + m_4 = 0 \quad \text{(vii)}$$

$$0 + m_4 = 0 \quad \text{\{From Eq. (i)\}}$$

$$\therefore \quad m_4 = 0$$

$$\Rightarrow \quad \left(am_4^2, -2am_4\right) = (0, 0)$$

Thus, the circle passes through the vertex of the parabola $y^2 = 4ax$.

$$\therefore \quad c = 0$$

From Eq. (vi), $a^2m^4 + (4a^2 + 2ag)m^2 - 4afm = 0$

$\Rightarrow am^3 + (4a + 2g)m - 4f = 0$ (viii)

Now, Eqs. (i) and (viii) are identical

$$\therefore \quad 1 = \frac{4a+2g}{2a-h} = -\frac{4f}{k}$$

$$\therefore \quad 2g = -(2a+h),\ 2f = -k/2$$

$\therefore$ The equation of the required circle is

$$x^2 + y^2 - (2a+h)x - \frac{k}{2}y = 0$$

Example 3.70 **A circle and a parabola $y^2 = 4ax$ intersect at four points. Show that the algebraic sum of the ordinates of the four points is zero. Also show that the line joining one pair of these four points are equally inclined to the axis.**

Sol.

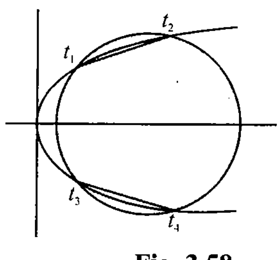


Fig. 3.58

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Solving it with $x = at^2, y = 2at$

$$a^4t^4 + 4a^2t^2 + 2gat^2 + 4aft + c = 0$$

$$a^4t^4 + 2a(2a + g)t^2 + 4aft + c = 0$$

Hence, $t_1 + t_2 + t_3 + t_4 = 0$ (i)

or $2a(t_1 + t_2 + t_3 + t_4) = 0$. Hence, proved

Slope of line joining

$$t_1, t_2 = \frac{2}{t_1 + t_2} = -\frac{2}{t_3 + t_4}$$

$$= m_1 \quad \text{[using Eq. (i)]}$$

Slope of line joining $t_3, t_4 = \frac{2}{t_3 + t_4} = m_2$

Hence, $m_1 + m_2 = 0$

Reflection Property of Parabola

The tangent at any point P to a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

Let the tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the axis of the parabola, i.e., x-axis or $y = 0$ at T.

The equation of tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is $ty = x + at^2$.

For co-ordinate of T solve it with $y = 0$

$\therefore \quad T(-at^2, 0)$

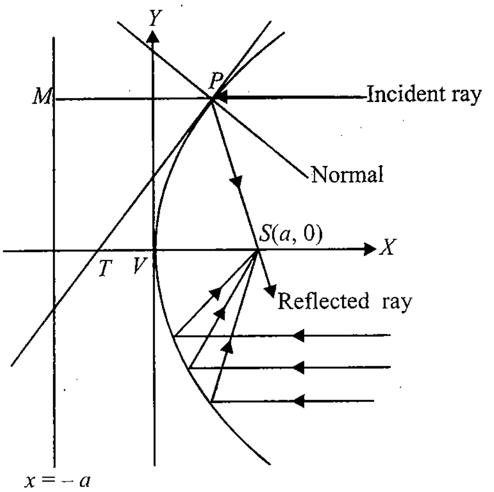


Fig. 3.59

$\therefore \quad ST = SV + VT = a + at^2$

$= a(1 + t^2)$

Also, $SP = PM = a + at^2 = a(1 + t^2)$

$\therefore \quad SP = ST$

i.e. $\angle STP = \angle SPT$

But $\angle STP = \angle MPT$ (alternate angles)

$\therefore \quad \angle SPT = \angle MPT$

Thus, if any light ray is sent along a line parallel to the axis of the parabola then the reflected ray passes through the focus, as the normal bisects the angle between the incident ray and reflected ray.

Example 3.71 **A ray of light moving parallel to the x-axis gets reflected form a parabolic mirror whose equation is $(y - 2)^2 = 4(x + 1)$. Find the point on the axis of parabola through which the ray must pass after reflection.**

Sol. The equation of the axis of the parabola is $y - 2 = 0$ which is parallel to the x-axis. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Hence, it passes through focus (0, 2).

Example 3.72 **If incident from point (–1, 2) parallel to the axis of the parabola $y^2 = 4x$ strikes the parabola, then find the equation of reflected ray.**

Sol. Incident ray as shown in the figure strikes the parabola at $P(1, 2)$.

Reflected ray passes through the focus

Hence, equation of reflected ray is $x = 1$.

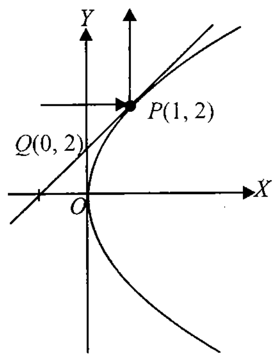


Fig. 3.60

Concept Application Exercise 3.4

1. Prove that the chord $y - x\sqrt{2} + 4a\sqrt{2} = 0$ is a normal chord of the parabola $y^2 = 4ax$. Also find the point on the parabola when the given chord is normal to the parabola.
2. If $y = 2x + 3$ is a tangent to the parabola $y^2 = 24x$, then find its distance from the parallel normal.
3. Find the point where the line $x + y = 6$ is a normal to the parabola $y^2 = 8x$.

4. Find the locus of the midpoints of the portion of the normal to the parabola $y^2 = 4ax$ intercepted between the curve and the axis.
5. Find the angle at which normal at point $P(at^2, 2at)$ to the parabola meets the parabola again at point Q.
6. If tangents are drawn to $y^2 = 4ax$ from any point P on the parabola $y^2 = a(x + b)$, then show that the normals drawn at their point for contact meet on a fixed line.
7. If normal to parabola $y^2 - 4ax = 0$ at α point intersect the parabola again such that sum of ordinates of these two points is 3, then show that the semi-latus rectum is equal to -1.5α.
8. If the parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have a common normal other than x-axis (a, b, c being distinct positive real numbers), then prove that

$$\frac{b}{a-c} > 2.$$

9. If a normal chord subtends a right angle at the vertex of the parabola $y^2 = 4ax$, then find its inclination to the axis.

Chord of Contact

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(h, k)$, then QR is called chord of contact of the parabola $y^2 = 4ax$.

Let $$Q \equiv (x_1, y_1) \text{ and } R \equiv (x_2, y_2)$$

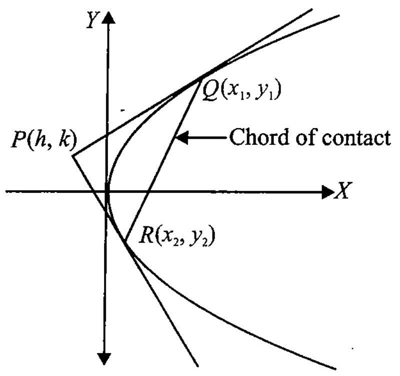


Fig. 3.61

Equation of the tangent PQ is

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

and equation of the tangent PR is

$$yy_2 = 2a(x + x_2) \quad \text{(ii)}$$

Since Eqs. (i) and (ii) pass through (h, k)

$$\therefore \quad ky_1 = 2a(h + x_1) \quad \text{(iii)}$$

and $$ky_2 = 2a(h + x_2) \quad \text{(iv)}$$

Hence, it is clear that $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on $yk = 2a(x + h)$ which is chord of contact of QR.

Example 3.73 **Tangents are drawn to porabola $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets this parabola. Find the point of intersection of these tangents.**

Sol. Let the tangent intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of 'P'. That means $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,

$$\frac{k}{m} = \frac{-2a}{l} = \frac{-2ah}{n}$$

$$\Rightarrow \quad h = \frac{n}{l}, k = -\frac{2am}{l}$$

Example 3.74 **If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, then find the locus of P.**

Sol. Chord of contact of parabola $y^2 = 4ax$ w.r.t. point $P(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

This line touches the parabola $x^2 = 4by$

Solving Eq. (i) with parabola, we have

$$x^2 = 4b\,[(2a/y_1)(x + x_1)]$$

or $$y_1x^2 - 8abx - 8abx_1 = 0$$

According to the question, this equation must have equal roots.

$$\Rightarrow \quad D = 0$$

$$\Rightarrow \quad 64a^2b^2 + 32abx_1y_1 = 0$$

$$\Rightarrow \quad x_1y_1 = -2ab \text{ or } xy = -2ab$$

which is a rectangular hyperbola.

Example 3.75 **Tangents are drawn from any point on the line $x + 4a = 0$ to the parabola $y^2 = 4ax$. Then find the angle subtended by the chord of contact at the vertex.**

Sol. Let $P(at_1^2, 2at_1)$ and $Q(at_1^2, 2at_2)$ be the points of contact for tangents drawn from any point on the line $x + 4a = 0$. Their point of intersection will be on this line.

$$\therefore \quad at_1t_2 + 4a = 0$$

or $$t_1t_2 = -4$$

This is also the condition for chord PQ to subtend a right angle at the vertex.

Equation of Chord Whose Midpoint is (x_1, y_1)

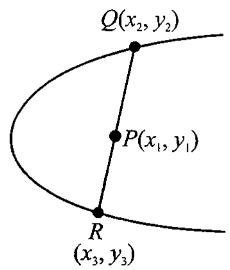


Fig. 3.62

Equation of the parabola is

$$y^2 = 4ax \quad \text{(i)}$$

Let QR be the chord of the parabola whose midpoint is $P(x_1, y_1)$.

Since Q and R lie on parabola (i)

$\therefore \quad y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$\therefore \quad y_3^2 - y_2^2 = 4a(x_3 - x_2)$

or $\quad \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{4a}{y_3 + y_2} = \dfrac{4a}{2y_1}$

($\because P(x_1, y_1)$ is midpoint of QR)

$\therefore \quad \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{2a}{y_1}$ = slope of QR

$\Rightarrow$ Equation of QR is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$

$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

(Subtracting $2ax_1$ from both sides)

$\Rightarrow \quad T = S_1$

where $\quad T = yy_1 - 2a(x + x_1)$

and $\quad S_1 = y_1^2 - 4ax_1$.

Example 3.76 **Find the locus of midpoint of chords of the parabola $y^2 = 4ax$ that pass through the point $(3a, a)$.**

Sol. Let the midpoint of chord be $P(h, k)$, then it's equation is

$$T = S_1$$

i.e., $\quad yk - 2a(x + h) = k^2 - 4ah$

It must pass through $(3a, a)$, hence

$$ak - 2a(3a + h) = k^2 - 4ah$$

Thus, locus of 'P' is

$$y^2 - 2ax - ay + 6a^2 = 0.$$

Example 3.77 **If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then find the midpoint of chord, QR.**

Sol. The equation of the tangent to $y^2 = 8x$ at $P(2, 4)$ is

$$4y = 4(x + 2) \text{ or } x - y + 2 = 0 \quad \text{(i)}$$

Let (x_1, y_1) be the midpoint of chord QR. Then, equation of QR is

$$yy_1 - 4(x + x_1) - 5 = y_1^2 - 8x_1 - 5$$

$$4x - yy_1 - 4x_1 + y_1^2 = 0 \quad \text{(ii)}$$

Clearly, Eqs. (i) and (ii) represent the same line. So,

$$\frac{4}{1} = -\frac{y_1}{-1} = \frac{-4x_1 + y_1^2}{2}$$

$\Rightarrow \quad y_1 = 4$

and $\quad 8 = -4x_1 + y_1^2$

$\Rightarrow \quad y_1 = 4$ and $x_1 = 2$

Example 3.78 **Find the locus of midpoint of normal chord of parabola $y^2 = 4ax$.**

Sol. Normal at point $P(t)$ meets the parabola again at point $Q(t_1 = -t - 2/t)$.

Let its midpoint be $R(h, k)$, then we have

$$h = \frac{at^2 + at_1^2}{2}$$

and

$$k = \frac{2at + 2at_1}{2} = a\left(t - t - \frac{2}{t}\right) = -\frac{2a}{t}$$

$\Rightarrow \quad t = -\dfrac{2a}{k}$

$\Rightarrow \quad h = \dfrac{at^2 + at_1^2}{2} = \dfrac{at^2 + a\left(-t - \frac{2}{t}\right)^2}{2}$

$\Rightarrow \quad \dfrac{2h}{a} = \left(-\dfrac{2a}{k}\right)^2 + a\left(\dfrac{2a}{k} + \dfrac{k}{a}\right)^2$

$\Rightarrow \quad \dfrac{2x}{a} = \left(\dfrac{2a}{y}\right)^2 + a\left(\dfrac{2a}{y} + \dfrac{y}{a}\right)^2$

which is required locus.

Concept Application Exercise 3.5

1. TP and TQ are tangents to the parabola, $y^2 = 4ax$ at P and Q. If the chord PQ passes through the fixed point $(-a, b)$, then find the locus of T.
2. If the distance of the point $(\alpha, 2)$ from its chord of contact w.r.t. parabola $y^2 = 4x$ is 4, then find the value of α.
3. Find the locus of the middle points of the focal chord of the parabola $y^2 = 4ax$.
4. From a variable point on the tangent at the vertex of a parabola $y^2 = 4ax$, a perpendicular is drawn to its chord of contact. Show that these variable perpendicular lines pass through a fixed point on the axis of the parabola.

EXERCISES

Subjective Type

Solutions on page 3.47

1. Prove that line joining the orthocentre to the centroid of a triangle formed by the focal chord of a parabola and tangents drawn at its extremities is parallel to the axis of the parabola.
2. A line AB makes intercepts of length a and b on the coordinate axes. Find the equation of the parabola passing through A, B and the origin, if AB is the shortest focal chord of the parabola.
3. From a point on the circle $x^2 + y^2 = a^2$, two tangents are drawn to the circle $x^2 + y^2 = b^2$ $(a > b)$. If chord of contact touches a variable circle passing through origin, show that locus of the centre of the variable circle is always a parabola.
4. Show that the common tangents to the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 2x = 0$ form an equilateral triangle.
5. The vertices A, B and C of a variable right triangle lie on a parabola $y^2 = 4x$. If the vertex B containing the right angle always remains at the point (1, 2), then find the locus of the centroid of the triangle ABC.
6. If a leaf of a book be folded so that one corner moves along an opposite side, then prove that the line of crease will always touch parabola.
7. A parabola of latus rectum l touches a fixed equal parabola. The axes of two parabolas are parallel. Then find the locus of the vertex of the moving parabola.
8. A variable parabola touches the x and the y-axis at (1, 0) and (0, 1). Then find the locus of the focus of the parabola.
9. Let N by the foot of perpendicular to the x-axis from point p on the parabola $y^2 = 4ax$. A straight line is drawn parallel to the axis which bisects PN and cuts the curve at Q; if NO meets the tangent at the vertex at a point T, then prove that $AT = \frac{2}{3} PN$.
10. Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$. Then find the locus of their point of intersection.
11. Find the area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through (1, 0) and having length $\frac{25}{4}$ unit each.
12. Find the range of parameter a for which a unique circle will pass through the points of intersection of the hyperbola $x^2 - y^2 = a^2$ and the parabola $y = x^2$. Also find the equation of the circle.
13. Find the radius of the largest circle, which passes through the focus of the parabola $y^2 = 4(x + y)$ and also contained in it.
14. A tangent is drawn to the parabola $y^2 = 4ax$ at P such that it cuts the y-axis at Q. A line perpendicular to this tangent is drawn through Q which cuts the axis of the parabola at R. If the rectangle $PQRS$ is completed, then find the locus of S.
15. Tangents are drawn to the parabola at three distinct points. Prove that these tangent lines always make a triangle and that the locus of the orthocenter of the triangle is the directrix of the parabola.
16. A series of chords are drawn so that their projections on the straight line, which is inclined at an angle α to the axis, are of constant length c. Prove that the locus of their middle point is the curve
$(y^2 - 4ax)(y \cos\alpha + 2a \sin\alpha)^2 + a^2c^2 = 0$.
17. A parabola is drawn touching the axis of x at the origin and having its vertex at a given distance k form this axis. Prove that the axis of the parabola is a tangent to the parabola $x^2 = -8k(y - 2k)$.
18. Prove that for a suitable point P on the axis of the parabola, a chord AB through the point P can be drawn such that $\left[\left(\frac{1}{AP^2}\right) + \left(\frac{1}{BP^2}\right)\right]$ is the same for all positions of the chord.

Objective Type

Solutions on page 3.52

Each question has four choices a, b, c and d, out of which *only one* answer is correct. Find the correct answer.

1. The equation of the parabola whose vertex and focus lie on the axis of x at distances a and a_1 from the origin respectively is
   a. $y^2 = 4(a_1 - a)x$
   b. $y^2 = 4(a_1 - a)(x - a)$
   c. $y^2 = 4(a_1 - a)(x - a_1)$
   d. none of these
2. The vertex of a parabola is the point (a, b) and latus rectum is of length l. If the axis of the parabola is along the positive direction of y-axis, then its equation is
   a. $(x + a)^2 = \frac{l}{2}(2y - 2b)$
   b. $(x - a)^2 = \frac{l}{2}(2y - 2b)$
   c. $(x + a)^2 = \frac{l}{4}(2y - 2b)$
   d. $(x - a)^2 = \frac{l}{8}(2y - 2b)$
3. Which one of the following equations represented parametrically equation to a parabolic curve?

a. $x = 3\cos t; y = 4\sin t$

b. $x^2 - 2 = 2\cos t; y = 4\cos^2\frac{t}{2}$

c. $\sqrt{x} = \tan t;\ \sqrt{y} = \sec t$

d. $x = \sqrt{1-\sin t};\ y = \sin\frac{t}{2} + \cos\frac{t}{2}$

4. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is

a. $x^2 + y^2 - 2xy - 4x - 4y - 4 = 0$

b. $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$

c. $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$

d. $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

5. The curve represented by the equation $\sqrt{px} + \sqrt{qy} = 1$, where $p, q \in R, p, q > 0$ is

a. a circle b. a parabola

c. an ellipse d. a hyperbola

6. If parabola $y^2 = \lambda x$ and $25[(x-3)^2 + (y+2)^2] = (3x - 4y - 2)^2$ are equal, then value of λ is

a. 9 b. 3 c. 7 d. 6

7. The length of the latus rectum of the parabola whose focus is $\left(\frac{u^2}{2g}\sin 2\alpha, -\frac{u^2}{2g}\cos 2\alpha\right)$ and directrix is $y = \frac{u^2}{2g}$ is

a. $\frac{u^2}{g}\cos^2\alpha$ b. $\frac{u^2}{g}\cos 2\alpha$

c. $\frac{2u^2}{g}\cos 2\alpha$ d. $\frac{2u^2}{g}\cos^2\alpha$

8. If the segment intercepted by the parabola $y = 4ax$ with the line $lx + my + n = 0$ subtends a right angle at the vertex, then

a. $4al + n = 0$ b. $4al + 4am + n = 0$

c. $4am + n = 0$ d. $al + n = 0$

9. The graph of the curve $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$ falls wholly in the

a. first quadrant b. second quadrant

c. third quadrant d. none of these

10. A point $P(x, y)$ moves in xy plane such that $x = a\cos^2\theta$ and $y = 2a\sin\theta$, where θ is a parameter. The locus of the point P is

a. circle

b. ellipse

c. unbounded parabola

d. part of the parabola

11. Locus of the point $\sqrt{3h}, \sqrt{3k+2}$ if it lies on the line $x - y - 1 = 0$ is a

a. straight line b. circle

c. parabola d. none of these

12. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. The height of the jet above the horizontal OX at a distance of 0.75 m from the point O is

a. 5 m b. 6 m c. 3 m d. 7 m

13. Vertex of the parabola whose parametric equation is $x = t^2 - t + 1, y = t^2 + t + 1; t \in R$, is

a. (1, 1) b. (2, 2)

c. $\left(\frac{1}{2}, \frac{1}{2}\right)$ d. (3, 3)

14. The ratio in which the line segment joining the points (4, –6) and (3, 1) is divided by the parabola $y^2 = 4x$ is

a. $\frac{-20 \pm \sqrt{155}}{11} : 1$

b. $\frac{-2 \pm 2\sqrt{155}}{11} : 2$

c. $-20 \pm 2\sqrt{155} : 11$

d. $-20 \pm \sqrt{155} : 11$

15. If (a, b) is the midpoint of a chord passing through the vertex of the parabola $y^2 = 4x$, then

a. $a = 2b$ b. $2a = b$ c. $a^2 = 2b$ d. $2a = b^2$

16. A set of parallel chords of the parabola $y^2 = 4ax$ have their midpoints on

a. any straight line through the vertex

b. any straight line through the focus

c. a straight line parallel to the axis

d. another parabola

17. A line L passing through the focus of the parabola $y^2 = 4(x-1)$ intersects the parabola in two distinct points. If 'm' be the slope of the line L then

a. $-1 < m < 1$ b. $m < -1$ or $m > 1$

c. $m \in R$ d. none of these

18. If PSQ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$. Then the length of SQ is

a. 6 b. 4

c. 3 d. none of these

19. The circle $x^2 + y^2 + 2\lambda x = 0, \lambda \in R$, touches the parabola $y^2 = 4x$ externally. Then

a. $\lambda > 0$ b. $\lambda < 0$

c. $\lambda > 1$ d. none of these

20. If y_1, y_2 and y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$, then its area is

a. $\frac{1}{2a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

b. $\frac{1}{4a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

c. $\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

d. none of these

21. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

a. $d^2 + (2b + 3c)^2 = 0$

b. $d^2 + (3b + 2c)^2 = 0$

c. $d^2 + (2b - 3c)^2 = 0$

d. none of these

22. Let P be the point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of the midpoint of PQ is

a. $y^2 + 4x + 2 = 0$ b. $y^2 - 4x + 2 = 0$

c. $x^2 - 4y + 2 = 0$ d. $x^2 + 4y + 2 = 0$

23. The locus of the vertex of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is

a. $xy = \frac{105}{64}$ b. $xy = \frac{3}{4}$

c. $xy = \frac{35}{16}$ d. $xy = \frac{64}{105}$

24. A circle touches the x-axis and also touches the circle with centre (0, 3) and radius 2. The locus of the centre of the circle is

a. a circle b. an ellipse

c. a parabola d. a hyperbola

25. Parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$, where c_1 and c_2 are variable, are such that they touch each other. Locus of their point of contact is

a. $xy = 2a^2$ b. $xy = 4a^2$

c. $xy = a^2$ d. none of these

26. The locus of a point on the variable parabola $y^2 = 4ax$, whose distance from focus is always equal to k, is equal to: (a is parameter)

a. $4x^2 + y^2 - 4kx = 0$

b. $x^2 + y^2 - 4kx = 0$

c. $2x^2 + 4y^2 - 8kx = 0$

d. $4x^2 - y^2 + 4kx = 0$

27. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at P and Q, then $AP \cdot AQ$ is equal to [where $A \equiv (\sqrt{3}, 0)$]

a. $\frac{2(\sqrt{3}+2)}{3}$ b. $\frac{4\sqrt{3}}{2}$

c. $\frac{4(2-\sqrt{2})}{3}$ d. $\frac{4(\sqrt{3}+2)}{3}$

28. A line is drawn form A (– 2, 0) to intersect the curve $y^2 = 4x$ in P and Q in the first quadrant such that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$, then slope of the line is always

a. $> \sqrt{3}$ b. $< \frac{1}{\sqrt{3}}$ c. $> \sqrt{2}$ d. $> \frac{1}{\sqrt{3}}$

29. Let $y = f(x)$ be a parabola, having its axis parallel to y-axis, which is touched by the line $y = x$ at $x = 1$, then

a. $2f(0) = 1 - f'(0)$ b. $f(0) + f'(0) + f''(0) = 1$

c. $f'(1) = 1$ d. $f'(0) = f'(1)$

30. An equilateral triangle SAB is inscribed in the parabola $y^2 = 4ax$ having it's focus at 'S'. If chord AB lies towards the left of S, then side length of this triangle is

a. $2a(2-\sqrt{3})$ b. $4a(2-\sqrt{3})$

c. $a(2-\sqrt{3})$ d. $8a(2-\sqrt{3})$

31. Let S be the focus of $y^2 = 4x$ and a point P is moving on the curve such that its abscissa is increasing at the rate of 4 units/sec, then the rate of increase of projection of SP on $x + y = 1$ when P is at (4, 4) is

a. $\sqrt{2}$ b. -1

c. $-\sqrt{2}$ d. $-\frac{3}{\sqrt{2}}$

32. Two parabolas have the same focus. If their directrices are the x-axis and the y-axis, respectively, then the slope of their common chord is

a. ± 1 b. $\frac{4}{3}$

c. $\frac{3}{4}$ d. none of these

33. C is the centre of the circle with centre (0, 1) and radius unity. P is the parabola $y = ax^2$. The set of values of 'a' for which they meet at a point other than the origin, is

a. $a > 0$ b. $a \in \left(0, \frac{1}{2}\right)$

c. $\left(\frac{1}{4}, \frac{1}{2}\right)$ d. $\left(\frac{1}{2}, \infty\right)$

34. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is

a. $2\sqrt{3}$ b. $4\sqrt{3}$ c. $3\sqrt{2}$ d. $2\sqrt{5}$

35. The circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q. Then the length PQ is equal to

a. 2 b. $2\sqrt{2}$

c. 4 d. none of these

36. The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modules of the difference of the ordinates of the points Q and R is

a. $\frac{A}{2a}$ **b.** $\frac{A}{a}$ **c.** $\frac{2A}{a}$ **d.** $\frac{4A}{a}$

37. If A_1B_1 and A_2B_2 are two focal chords of the parabola $y^2 = 4ax$, then the chords A_1A_2 and B_1B_2 intersect on

a. directrix **b.** axis
c. tangent at vertex **d.** none of these

38. If a line $y = 3x + 1$ cuts the parabola $x^2 - 4x - 4y + 20 = 0$ at A and B, then the tangent of the angle subtended by line segment AB at origin is

a. $\frac{8\sqrt{3}}{205}$ **b.** $\frac{8\sqrt{3}}{209}$
c. $\frac{8\sqrt{3}}{215}$ **d.** none of these

39. $P(x, y)$ is a variable point on the parabola $y^2 = 4ax$ and $Q(x + c, y + c)$ is another variable point, where 'c' is a constant. The locus of the midpoint of PQ is

a. parabola **b.** ellipse
c. hyperbola **d.** circle

40. If a and c are the lengths of segments of any focal chord of the parabola $y^2 = 2bx$ $(b > 0)$, then the roots of the equation $ax^2 + bx + c = 0$ are

a. real and distinct **b.** real and equal
c. imaginary **d.** none of these

41. AB is a chord of the parabola $y^2 = 4ax$ with vertex A. BC is drawn is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is

a. a **b.** $2a$ **c.** $4a$ **d.** $8a$

42. Set of values of α for which the point $(\alpha, 1)$ lies inside the curves $c_1: x^2 + y^2 - 4 = 0$ and $c_2: y^2 = 4x$ is

a. $|\alpha| < \sqrt{3}$ **b.** $|\alpha| < 2$
c. $\frac{1}{4} < \alpha < \sqrt{3}$ **d.** none of these

43. If P be a point on the parabola $y^2 = 3(2x - 3)$ and M is the foot perpendicular drawn from P on the directrix of the parabola, then length of each side of an equilateral triangle SMP, where S is focus of the parabola is

a. 2 **b.** 4 **c.** 6 **d.** 8

44. If $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$, then

a. $c = \frac{a}{m}$ **b.** $c = am + \frac{a}{m}$
c. $c = a + \frac{a}{m}$ **d.** none of these

45. The angle between the tangents to the parabola $y^2 = 4ax$ at the points where it intersects with the line $x - y - a = 0$ is

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{6}$ **d.** $\frac{\pi}{2}$

46. The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$, both drawn at the same end of the latus rectum, and the axis of the parabola is

a. $2\sqrt{2}a^2$ **b.** $2a^2$
c. $4a^2$ **d.** none of these

47. Double ordinate AB of the parabola $y^2 = 4ax$ subtends an angle $\pi/2$ at the focus of the parabola, then tangents drawn to parabola at A and B will intersect at

a. $(-4a, 0)$ **b.** $(-2a, 0)$
c. $(-3a, 0)$ **d.** none of these

48. $y = x + 2$ is any tangent to the parabola $y^2 = 8x$. The point P on this tangent is such that the other tangent from it which is perpendicular to it is

a. $(2, 4)$ **b.** $(-2, 0)$
c. $(-1, 1)$ **d.** $(2, 0)$

49. The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x-axis in T and G, respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through P, T, G is

a. $\tan^{-1}(t^2)$ **b.** $\cot^{-1}(t^2)$ **c.** $\tan^{-1}(t)$ **d.** $\cot^{-1}(t)$

50. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ is

a. $\frac{\pi}{2}$ **b.** $\frac{\pi}{3}$ **c.** $\frac{\pi}{6}$ **d.** $\frac{\pi}{4}$

51. Radius of the circle that passes through origin and touches the parabola $y^2 = 4ax$ at the point $(a, 2a)$ is

a. $\frac{5}{\sqrt{2}}a$ **b.** $2\sqrt{2}\,a$ **c.** $\sqrt{\frac{5}{2}}\,a$ **d.** $\frac{3}{\sqrt{2}}a$

52. If the line $x + y = 1$ touches the parabola $y^2 - y + x = 0$, then the coordinates of the point of contact are

a. $(1, 1)$ **b.** $\left(\frac{1}{2}, \frac{1}{2}\right)$ **c.** $(0, 1)$ **d.** $(1, 0)$

53. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other touches $y^2 = 4b(x + b)$. Their point of intersection lies on the line

a. $x - a + b = 0$ **b.** $x + a - b = 0$
c. $x + a + b = 0$ **d.** $x - a - b = 0$

54. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point $(1, 2)$ is

a. $(x-1)^2 = 4(y+1)$

b. $(x+1)^2 = 4(y+1)$

c. $(x+1)^2 = 4(y-1)$

d. $(x-1)^2 = 4(y-1)$

55. Consider the parabola $y^2 = 4x$. $A \equiv (4, -4)$ and $B \equiv (9, 6)$ be two fixed points on the parabola. Let 'C' be a moving point on the parabola between A and B such that the area of the triangle ABC is maximum, then coordinate of 'C' is

a. $\left(\frac{1}{4}, 1\right)$ b. $(4, 4)$

c. $(3, 2\sqrt{3})$ d. $(3, -2\sqrt{3})$

56. $\min\left[(x_1 - x_2)^2 + \left(5 + \sqrt{1 - x_1^2} - \sqrt{4x_2}\right)^2\right] \forall x_1, x_2 \in R$ is

a. $4\sqrt{5} + 1$ b. $4\sqrt{5} - 1$

c. $\sqrt{5} + 1$ d. $\sqrt{5} - 1$

57. Two mutually perpendicular tangent of the parabola $y^2 = 4ax$ meet the axis in P_1 and P_2. If S is the focus of the parabola, then $\frac{1}{(SP_1)} + \frac{1}{(SP_2)}$ is equal to

a. $\frac{4}{a}$ b. $\frac{2}{a}$ c. $\frac{1}{a}$ d. $\frac{1}{4a}$

58. A tangent is drawn to the parabola $y^2 = 4ax$ at the point 'P' whose abscissa lies in the interval (1, 4). The maximum possible area of the triangle formed by the tangent at 'P' ordinates of the point 'P' and the x-axis is equal to

a. 8 b. 16 c. 24 d. 32

59. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. The length of a tangent from the origin to the circle is

a. $\sqrt{\frac{bc}{a}}$ b. ac^2 c. $\frac{b}{a}$ d. $\sqrt{\frac{c}{a}}$

60. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is

a. $x^2 + 2y^2 - ax = 0$

b. $2x^2 + y^2 - 2ax = 0$

c. $2x^2 + 2y^2 - ay = 0$

d. $2x^2 + y^2 - 2ay = 0$

61. Through the vertex O of the parabola $y^2 = 4ax$, two chords OP and OQ are drawn and the circles on OP and OQ as diameters intersect in R. If θ_1, θ_2 and ϕ are the angles made with axis by the tangents at P and Q on the parabola and by OR, then the value of, $\cot\theta_1 + \cot\theta_2$

a. $-2\tan\phi$ b. $-2\tan(\pi - \phi)$

c. 0 d. $2\cot\phi$

62. A line of slope $\lambda (0 < \lambda < 1)$ touches parabola $y + 3x^2 = 0$ at P. If S is the focus and M is the foot of the perpendicular of directrix from P, then $\tan\angle MPS$ equals

a. 2λ b. $\frac{2\lambda}{-1 + \lambda^2}$

c. $\frac{1 - \lambda^2}{1 + \lambda^2}$ d. none of these

63. If $y = 2x - 3$ is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then '$a$' is equal to

a. $\frac{22}{3}$ b. -1 c. $\frac{14}{3}$ d. $\frac{-14}{3}$

64. AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to parabola at A and B meet y-axis at A_1 and B_1, respectively. If the area of trapezium AA_1B_1B is equal to $12a^2$, then angle subtended by A_1B_1 at the focus of the parabola is equal to

a. $2\tan^{-1}(3)$ b. $\tan^{-1}(3)$

c. $2\tan^{-1}(2)$ d. $\tan^{-1}(2)$

65. If $y + 3 = m_1(x + 2)$ and $y + 3 = m_2(x + 2)$ are two tangents to the parabola $y^2 = 8x$, then

a. $m_1 + m_2 = 0$ b. $m_1 m_2 = -1$

c. $m_1 m_2 = 1$ d. none

66. The tangent at any point P on the parabola $y^2 = 4ax$ intersects the y-axis at Q. The tangent to the circum circle of triangle PQS (S is the focus) at Q is

a. a line parallel to x-axis

b. y-axis

c. a line parallel to y-axis

d. none of these

67. If $y = m_1x + c$ and $y = m_2x + c$ are two tangents to the parabola $y^2 + 4a(x + a) = 0$, then

a. $m_1 + m_2 = 0$ b. $1 + m_1 + m_2 = 0$

c. $m_1 m_2 - 1 = 0$ d. $1 + m_1 m_2 = 0$

68. If the parabola $y = ax^2 - 6x + b$ passes through (0, 2) and has its tangent at $x = \frac{3}{2}$ parallel to the x-axis then

a. $a = 2, b = -2$ b. $a = 2, b = 2$

c. $a = -2, b = 2$ d. $a = -2, b = -2$

69. If the angle between the tangents from the point $(\lambda, 1)$ to the parabola $y^2 = 16x$ be $\frac{\pi}{2}$ then λ is

a. 4 b. -4 c. -1 d. 2

70. Minimum area of circle which touches the parabolas $y = x^2 + 1$ and $y^2 = x - 1$ is

a. $\frac{9\pi}{16}$ sq. unit **b.** $\frac{9\pi}{32}$ sq. unit

c. $\frac{9\pi}{8}$ sq. unit **d.** $\frac{9\pi}{4}$ sq. unit

71. If the locus of middle of point of contact of tangent drawn to the parabola $y^2 = 8x$ and foot of perpendicular drawn from its focus to the tangents is a conic then length of latus rectum of this conic is

a. $\frac{9}{4}$ **b.** 9 **c.** 18 **d.** $\frac{9}{2}$

72. If d is the distance between parallel tangents with positive slope to $y^2 = 4x$ and $x^2 + y^2 - 2x + 4y - 11 = 0$, then

a. $10 < d < 2$ **b.** $4 < d < 6$

c. $d < 4$ **d.** none of these

73. If bisector of the angle APB, where PA and PB are the tangents to the parabola $y^2 = 4ax$, is equally inclined to the coordinate axes, then the point P lies on

a. tangent at vertex of the parabola

b. directrix of the parabola

c. circle with centre at the origin and radius a

d. the line of latus rectum

74. The locus of the centre of a circle which cuts orthogonally the parabola $y^2 = 4x$ at (1, 2) will pass though points

a. (3, 4) **b.** (4, 3) **c.** (5, 3) **d.** (2, 4)

75. From a point $A(t)$ on the parabola $y^2 = 4ax$, a focal chord and a tangent is drawn. Two circles are drawn in which one circle is drawn taking focal chord AB as diameter and other is drawn by taking intercept of tangent between point A and point P on the directrix, as diameter. Then the common chord of the circles is

a. line joining focus and P

b. line joining focus and A

c. tangent to the parabola at point A

d. none of these

76. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then

a. $p^2 = 2$ **b.** $q^2 = 2$ **c.** $p = 2q$ **d.** $q = 2p$

77. The set of points on the axis of the parabola $y^2 = 4x + 8$ from which the three normals to the parabola are all real and different is

a. $\{(k, 0) \mid k \le -2\}$ **b.** $\{(k, 0) \mid k > -2\}$

c. $\{(0, k) \mid k > -2\}$ **d.** none of these

78. Tangents and normal drawn to parabola $y^2 = 4ax$ at point $P(at^2, 2at)$, $t \ne 0$, meet the x-axis at points T and N, respectively. If 'S' is the focus of the parabola, then

a. $SP = ST \ne SN$ **b.** $SP \ne ST = SN$

c. $SP = ST = SN$ **d.** $SP \ne ST \ne SN$

79. Locus of the midpoint of any normal chords of $y^2 = 4ax$ is

a. $x = a\left(\frac{4a^2}{y^2} - 2 + \frac{y^2}{2a^2}\right)$

b. $x = a\left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2}\right)$

c. $x = a\left(\frac{4a^2}{y^2} - 2 - \frac{y^2}{2a^2}\right)$

d. $x = a\left(\frac{4a^2}{y^2} + 2 - \frac{y^2}{2a^2}\right)$

80. Normals AO, AA_1, AA_2 are drawn to parabola $y^2 = 8x$ from the point $A(h, 0)$. If triangle OA_1A_2 is equilateral, then possible values of 'h' is

a. 26 **b.** 24

c. 28 **d.** none of these

81. If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is

a. 12 **b.** -12 **c.** 24 **d.** -24

82. At what point on the parabola $y^2 = 4x$ the normal makes equal angle with axes?

a. (4, 4) **b.** (9, 6) **c.** (4, − 4) **d.** (1, ± 2)

83. If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S be the focus, then $SP.SQ.SR$ is equal to

a. a^2SA **b.** SA^3

c. aSA^2 **d.** none of these

84. If the normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum meet the parabola at Q, Q', then QQ' is

a. $10a$ **b.** $4a$ **c.** $20a$ **d.** $12a$

85. From a point $(\sin\theta, \cos\theta)$ if three normals can be drawn to the parabola $y^2 = 4ax$, then the value of 'a' is

a. $\left(\frac{1}{2}, 1\right)$ **b.** $\left[-\frac{1}{2}, 0\right)$

c. $\left[\frac{1}{2}, 1\right]$ **d.** $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

86. If the normals at points 't_1' and 't_2' meet on the parabola, then

a. $t_1t_2 = -1$ **b.** $t_2 = -t_1 - \frac{2}{t_1}$

c. $t_1t_2 = 2$ **d.** none of these

87. Length of the normal chord of the parabola $y^2 = 4x$ which makes an angle of $\frac{\pi}{4}$ with the axis of x is

a. 8 **b.** $8\sqrt{2}$ **c.** 4 **d.** $4\sqrt{2}$

88. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angle α and β, respectively, with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

a. 0 **b.** -2 **c.** $-\frac{1}{2}$ **d.** -1

89. If two normals to a parabola $y^2 = 4ax$ intersect at right angles, then the chord joining their feet passes through a fixed point whose co-ordinates are

a. $(-2a, 0)$ **b.** $(a, 0)$ **c.** $(2a, 0)$ **d.** none

90. PQ is a normal chord of the parabola $y^2 = 4ax$ at P, A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x-axis in R. Then line length of AR is

a. equal to the length of the latus rectum

b. equal to the focal distance of the point P

c. equal to twice the focal distance of the point P

d. equal to the distance of the point P from the directrix

91. Tangent and normal drawn to parabola at $A(at^2, 2at)$, $t \neq 0$ meet the x-axis at point B and D respectively. If the rectangle $ABCD$ is completed, then locus of 'C' is

a. $y = 2a$ **b.** $y + 2a = c$

b. $x = 2a$ **d.** $x + 2a = 0$

92. An equation for the line that passes through $(10, -1)$ and is perpendicular to $y = \frac{x^2}{4} - 2$ is

a. $4x + y = 39$ **b.** $2x + y = 19$

c. $x + y = 9$ **d.** $x + 2y = 8$

93. P, Q, R are the feet of the normals drawn to a parabola $(y-3)^2 = 8(x-2)$. A circle cuts the above parabola in points P, Q, R and S. Then this circle always passes through the point

a. $(2, 3)$ **b.** $(3, 2)$ **c.** $(0, 3)$ **d.** $(2, 0)$

94. Normals at two point (x_1, y_1) and (x_2, y_2) of parabola $y^2 = 4x$ meet again on the parabola where $x_1 + x_2 = 4$, then $|y_1 + y_2|$ equals to

a. $\sqrt{2}$ **b.** $2\sqrt{2}$

c. $4\sqrt{2}$ **d.** none of these

95. The end points of two normal chords of a parabola are concyclic, then the tangents at the feet of the normals will intersect at

a. tangent at vertex of the parabola

b. axis of the parabola

c. directrix of the parabola

d. none of these

96. The set of points on the axis of the parabola $(x-1)^2 = 8(y+2)$, from where three distinct normals can be drawn to the parabola is the set (h, k) of points satisfying

a. $h > 2$ **b.** $h > 1$

c. $k > 2$ **d.** none of these

97. The shortest distance between the parabola $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ is

a. $2\sqrt{2}$ **b.** $\frac{1}{2\sqrt{2}}$ **c.** 4 **d.** $\sqrt{\frac{36}{5}}$

98. A tangent and normal is drawn at the point $P \equiv (16, 16)$ of the parabola $y^2 = 16x$ which cut the axis of the parabola at the points A and B, respectively. If the centre of the circle through P, A and B is C, then the angle between PC and the axis of x is

a. $\tan^{-1}\frac{1}{2}$ **b.** $\tan^{-1} 2$ **c.** $\tan^{-1}\frac{3}{4}$ **d.** $\tan^{-1}\frac{4}{3}$

99. Length of the shortest normal chord of the parabola $y^2 = 4ax$ is

a. $2a\sqrt{27}$ **b.** $9a$

c. $a\sqrt{54}$ **d.** None of these

100. From the point (15, 12), three normals are drawn to the parabola $y^2 = 4x$, then centroid of triangle formed by three co-normals points is

a. $\left(\frac{16}{3}, 0\right)$ **b.** $(4, 0)$ **c.** $\left(\frac{26}{3}, 0\right)$ **d.** $(6, 0)$

101. 't_1' and 't_2' are two points on the parabola $y^2 = 4ax$. If the focal chord joining them coincides with the normal chord, then

a. $t_1(t_1 + t_2) + 2 = 0$ **b.** $t_1 + t_2 = 0$

c. $t_1 t_2 = -1$ **d.** none of these

102. The line $x - y = 1$ intersects the parabola $y^2 = 4x$ at A and B. Normals at A and B intersect at C. If D is the point at which line CD is normal to the parabola, then coordinates of D are

a. $(4, -4)$ **b.** $(4, 4)$

c. $(-4, -4)$ **d.** none of these

103. If normals are drawn from a point $P(h, k)$ to the parabola $y^2 = 4ax$, then the sum of the intercepts which the normals cutoff from the axis of the parabola is

a. $(h + a)$ **b.** $3(h + a)$

c. $2(h + a)$ **d.** none of these

104. The radius of circle touching parabola $y^2 = x$ at (1, 1) and having directrix of $y^2 = x$ as its normal is

a. $\frac{5\sqrt{5}}{8}$ b. $\frac{10\sqrt{5}}{3}$

c. $\frac{5\sqrt{5}}{4}$ d. none of these

105. Normals drawn to $y^2 = 4ax$ at the points where it is intersected by the line $y = mx + c$ intersect at P. Foot of the another normal drawn to the parabola from the point 'P' is

a. $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$ b. $\left(\frac{9a}{m}, \frac{-6a}{m}\right)$

c. $(am^2, -2am)$ d. $\left(\frac{4a}{m^2}, -\frac{4a}{m}\right)$

106. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$ then

a. $|b| > \frac{1}{2\sqrt{2}}$ b. $|b| < \frac{1}{2\sqrt{2}}$

c. $|b| > \frac{1}{\sqrt{2}}$ d. $|b| < \frac{1}{\sqrt{2}}$

107. Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ is equal to

a. 3 b. 4 c. 6 d. 5

108. A ray of light travels along a line $y = 4$ and strikes the surface of a curves $y^2 = 4(x + y)$, then equations of the line along which reflected ray travel is

a. $x = 0$ b. $x = 2$

c. $x + y = 4$ d. $2x + y = 4$

109. The largest value of a for which the circle $x^2 + y^2 = a^2$ falls totally in the interior of the parabola $y^2 = 4(x + 4)$ is

a. $4\sqrt{3}$ b. 4 c. $4\frac{\sqrt{6}}{7}$ d. $2\sqrt{3}$

110. If two chords drawn from the point $A(4, 4)$ to the parabola $x^2 = 4y$ are bisected by line $y = mx$, the interval in which m lies is

a. $(-2\sqrt{2}, 2\sqrt{2})$

b. $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

c. $(-\infty, -2\sqrt{2}-2) \cup (2\sqrt{2}-2, \infty)$

d. none of these

111. The point of intersection of the tangents of the parabola $y^2 = 4x$, drawn at end points of the chord $x + y = 2$ lies on

a. $x - 2y = 0$ b. $x + 2y = 0$

c. $y - x = 0$ d. $x + y = 0$

112. The number of common chords of the parabolas $x = y^2 - 6y + 11$ and $y = x^2 - 6x + 1$ are

a. 1 b. 2 c. 4 d. 6

113. Two parabola have the focus $(3, -2)$. Their directrices are the x-axis, and the y-axis, respectively. Then the slope of their common chord is

a. -1 b. $-\frac{1}{2}$

c. $-\frac{\sqrt{3}}{2}$ d. none of these

114. If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then the mid-point of QR is

a. (4, 2) b. (2, 4) c. (7, 9) d. None of these

Multiple Correct Answers Type

Solutions on page 3.71

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. The equation of the directrix of the parabola with vertex at the origin and having the axis along the x-axis and a common tangent of slope 2 with the circle $x^2 + y^2 = 5$ is/are

a. $x = 10$ b. $x = 20$ c. $x = -10$ d. $x = -20$

2. Tangent is drawn at any point (x_1, y_1) other than vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2), then

a. x_1, a, x_2 are in G.P.

b. $\frac{y_1}{2}, a, y_2$ are in G.P.

c. $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in G.P.

d. $x_1x_2 + y_1y_2 = a^2$

3. If the focus of the parabola $x^2 - ky + 3 = 0$ is $(0, 2)$, then a value of k is/are

a. 4 b. 6 c. 3 d. 2

4. Let P be a point whose coordinates differ by unity and the point does not lie on any of the axes of reference. If the parabola $y^2 = 4x + 1$ passes through P, then the ordinate of P may be

a. 3 b. -1 c. 5 d. 1

5. If $y = 2$ be the directrix and $(0, 1)$ be the vertex of the parabola $x^2 + \lambda y + \mu = 0$ then

a. $\lambda = 4$ b. $\mu = 8$ c. $\lambda = -8$ d. $\mu = 4$

6. The extremities of latus rectum of a parabola are $(1, 1)$ and $(1, -1)$, then the equation of the parabola can be

a. $y^2 = 2x - 1$ b. $y^2 = 1 - 2x$

c. $y^2 = 2x - 3$ d. $y^2 = 2x - 4$

7. Parabola $y^2 = 4x$ and the circle having it's centre at (6, 5) intersect at right angle. Possible point of intersection of these curves can be

   a. (9, 6) b. $(2, \sqrt{8})$ c. (4, 4) d. $(3, 2\sqrt{3})$

8. A normal drawn to parabola $y^2 = 4ax$ meet the curve again at Q such that angle subtended by PQ at vertex is 90°, then coordinates of P can be

   a. $(8a, 4\sqrt{2}a)$ b. $(8a, 4a)$
   c. $(2a, -2\sqrt{2}a)$ d. $(2a, 2\sqrt{2}a)$

9. A quadrilateral is inscribed in parabola, then
   a. quadrilateral may be cyclic.
   b. diagonal of the quadrilateral may be equal.
   c. all possible pairs of adjacent sides may be perpendicular.
   d. none of these

10. The locus of the midpoint of the focal distance of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose
    a. latus rectum is half the latus rectum of the original parabola.
    b. vertex is $(\frac{a}{2}, 0)$.
    c. directrix is y-axis.
    d. focus has the co-ordinates $(a, 0)$.

11. Which of the following line can be tangent to parabola $y^2 = 8x$?

    a. $x - y + 2 = 0$ b. $9x - 3y + 2 = 0$
    c. $x + 2y + 8 = 0$ d. $x + 3y + 12 = 0$

12. Which of the following line can be normal to parabola $y^2 = 12x$?

    a. $x + y - 9 = 0$ b. $2x - y - 32 = 0$
    c. $2x + y - 36 = 0$ d. $3x - y - 72 = 0$

13. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, the coordinates of the vertices of the square are

    a. $(4a, 4a)$ b. $(4a, -4a)$
    c. $(0, 0)$ d. $(8a, 0)$

Reasoning Type

Solutions on page 3.72

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2 .

a. Both the statements are true and Statement 1 is the correct explanation of Statement 2
b. Both the statements are true but Statement 1 is not the correct explanation of Statement 2
c. Statement 1 is true and Statement 2 is False
d. Statement 1 is false and Statement 2 is True

1. **Statement 1:** Slope of tangents drawn from (4, 10) to parabola $y^2 = 9x$ are $\frac{1}{4}, \frac{9}{4}$.

   Statement 2: Two tangents can be drawn to parabola from any point lying outside parabola.

2. **Statement 1:** Through $(\lambda, \lambda + 1)$ there can't be more than one normal to the parabola $y^2 = 4x$, if $\lambda < 2$.

   Statement 2: The point $(\lambda, \lambda + 1)$ lies outside the parabola for all $\lambda \neq 1$.

3. **Statement 1:** In parabola $y^2 = 4ax$, the circle drawn taking focal radii as diameter touches y-axis.

   Statement 2: The portion of the tangent intercepted between point of contact and directix subtends 90° angle at focus.

4. **Statement 1:** The line joining points (8, – 8) and $(\frac{1}{2}, 2)$, which are on parabola $y^2 = 8x$, passes through focus of parabola.

   Statement 2: Tangents drawn at (8, – 8) and $(\frac{1}{2}, 2)$ on the parabola $y^2 = 4ax$ are perpendicular.

5. **Statement 1:** The normals at the point (4, 4) and $\left(\frac{1}{4}, -1\right)$ of the parabola $y^2 = 4x$ are perpendicular.

   Statement 2: The tangents to the parabola at the end of a focal chord are perpendicular.

6. **Statement 1:** The line $y = x + 2a$ touches the parabola $y^2 = 4a(x + a)$.

   Statement 2: The line $y = mx + am + a/m$ touches $y^2 = 4a(x + a)$ for all real values of m.

7. **Statement 1:** Equation $(5x - 5)^2 + (5y + 10)^2 = (3x + 4y + 5)^2$ is parabola.

   Statement 2: If distance of the point from the given line and from the given point (not lying on the given line) is equal, then locus of variable point is parabola.

8. **Statement 1:** Length of focal chord of a parabola $y^2 = 8x$ making on angle of 60° with x-axis is 32.

   Statement 2: Length of focal chord of a parabola $y^2 = 4ax$ making an angle a with x-axis is $4a \operatorname{cosec}^2 \alpha$.

9. **Statement 1:** Circumcircle of a triangle formed by the lines $x = 0$, $x + y + 1 = 0$ and $x - y + 1 = 0$ also passes through the point (1, 0).

   Statement 2: Circumcircle of a triangle formed by three tangents of a parabola passes through its focus.

10. **Statement 1 :** The point of intersection of the tangents at three distinct points A, B, C on the parabola $y^2 = 4x$ can be collinear.

Statement 2: If a line L does not intersect the parabola $y^2 = 4x$, then from every point of the line two tangents can be drawn to the parabola.

11. **Statement 1:** There are no common tangents between circle $x^2 + y^2 - 4x + 3 = 0$ and parabola $y^2 = 2x$.

    Statement 2: Given circle and parabola do not intersect.

12. **Statement 1:** The line $ax + by + c = 0$ is a normal to the parabola $y^2 = 4ax$, then the equation of tangent at the foot of this normal is $y = (b/a)x + (a^2/b)$.

    Statement 2: Equation of normal at any point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is $y = -tx + 2at + at^3$.

13. **Statement 1:** The values of α for which the point (α, α^2) lies inside the triangle formed by the lines $x = 0, x + y = 2$ and $3y = x$ is $(0, 1)$.

    Statement 2: Parabola $y = x^2$ meets the line $x + y = 2$ at $(1, 1)$.

14. **Statement 1:** If there exists points on the circle $x^2 + y^2 = a^2$ from which two perpendicular tangents can be drawn to parabola $y^2 = 2x$, then $a \geq 1/2$.

    Statement 2: Perpendicular tangents to parabola meet on the directrix.

15. **Statement 1:** If straight line $x = 8$ meets the parabola $y^2 = 8x$ at P and Q, then PQ substends a right angle at the origin.

    Statement 2: Double ordinate equal to twice of latus rectum of a parabola subtends a right angle at the vertex.

16. **Statement 1:** Normal chord drawn at the point $(8, 8)$ of the parabola $y^2 = 8x$ subtends a right angle at the vertex of the parabola.

    Statement 2: Every chord of the parabola $y^2 = 4ax$ passing through the point $(4a, 0)$ subtends a right angle at the vertex of the parabola.

17. **Statement 1:** If end points of two normal chords AB and CD (normal at A and C) of a parabola $y^2 = 4ax$ are concyclic, then the tangents at A and C will intersect on the axis of the parabola.

    Statement 2: If four point on the parabola $y^2 = 4ax$ are concyclic, then sum of their ordinates is zero.

18. **Statement 1:** If parabola $y^2 = 4ax$ and circle $x^2 + y^2 + 2bx = 0$ touch each other externally, then roots of the equation, $f(x) = x^2 - (b + a + 1)x + a = 0$ has real roots.

    Statement 2: For parabola and circle externally touching a and b must have the same sign.

19. **Statement 1:** Line $x - y - 5 = 0$ cannot be normal to parabola $(5x - 15)^2 + (5y + 10)^2 = (3x - 4y + 2)^2$.

    Statement 2: Normal to parabola never passes through its focus.

20. **Statement 1:** AA' and BB' are double ordinates of the parabola. Then points A, A', B, B' are concyclic.

    Statement 2: Circle can cut parabola in maximum four points.

Linked Comprehension Type

Solutions on page 3.75

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

A tangent is drawn at any point $P(t)$ on the parabola $y^2 = 8x$ and on it is taken a point $Q(\alpha, \beta)$ from which pair of tangents QA and QB are drawn to the circle $x^2 + y^2 = 4$. Using this information answer the following questions

1. The locus of the point of concurrency of the chord of contact AB of the circle $x^2 + y^2 = 4$ is

   a. $y^2 - 2x = 0$ **b.** $y^2 - x^2 = 4$
   c. $y^2 + 4x = 0$ **d.** $y^2 - 2x^2 = 4$

2. The point from which perpendicular tangents can be drawn both to the given circle and the parabola is

   a. $(4, \pm\sqrt{3})$ **b.** $(-1, \sqrt{2})$
   c. $(-\sqrt{2}, -\sqrt{2})$ **d.** $(-2, \pm 2\sqrt{3})$

3. The locus of circumcentre of ΔAQB if $t = 2$ is

   a. $x - 2y + 2 = 0$ **b.** $x + 2y - 4 = 0$
   c. $x - 2y - 4 = 0$ **d.** $x + 2y + 4 = 0$

For Problems 4–6

Tangent to the parabola $y = x^2 + ax + 1$, at the point of intersection of y-axis also touches the circle $x^2 + y^2 = r^2$. Also no point of the parabola is below x-axis.

4. The radius of circle when a attains its maximum value

   a. $\frac{1}{\sqrt{10}}$ **b.** $\frac{1}{\sqrt{5}}$ **c.** 1 **d.** $\sqrt{5}$

5. The slope of the tangent when radius of the circle is maximum is

   a. -1 **b.** 1 **c.** 0 **d.** 2

6. The minimum area bounded by the tangent and the coordinate axes is

   a. 1 **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{4}$

For Problems 7–9

If the locus of the circumcentre of a variable triangle having sides y-axis, $y = 2$ and $lx + my = 1$, where (l, m) lies on the parabola $y^2 = 4x$ is a curve C, then

7. Coordinates of the vertex of this curve C is

   a. $\left(-2, \frac{3}{2}\right)$ **b.** $\left(-2, -\frac{3}{2}\right)$
   c. $\left(2, \frac{3}{2}\right)$ **d.** $\left(-2, -\frac{3}{2}\right)$

8. The length of smallest focal chord of this curve C is

   a. $\frac{1}{4}$ **b.** $\frac{1}{12}$ **c.** $\frac{1}{8}$ **d.** $\frac{1}{16}$

9. The curve C is symmetric about the line

a. $x=\frac{3}{2}$ b. $y=-\frac{3}{2}$ c. $x=-\frac{3}{2}$ d. $y=\frac{3}{2}$

For Problems 10–12

$y=x$ is tangent to the parabola $y=ax^2+c$

10. If $a=2$, then the value of c is

a. 1 b. $-\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{8}$

11. If (1, 1) is point of contact, then a is

a. $\frac{1}{4}$ b. $\frac{1}{3}$ c. $\frac{1}{2}$ d. $\frac{1}{6}$

12. If $c=2$, then point of contact is

a. (3, 3) b. (2, 2) c. (6, 6) d. (4, 4)

For Problems 13–15

If l, m are variable real numbers such that $5l^2+6m^2-4lm+3l=0$, then variable line $lx+my=1$ always touches a fixed parabola, whose axes is parallel to x-axis.

13. Vertex of the parabola is

a. $\left(-\frac{5}{3},\frac{4}{3}\right)$ b. $\left(-\frac{7}{4},\frac{3}{4}\right)$

c. $\left(\frac{5}{6},-\frac{7}{6}\right)$ d. $\left(\frac{1}{2},-\frac{3}{4}\right)$

14. Focus of the parabola is

a. $\left(\frac{1}{6},-\frac{7}{6}\right)$ b. $\left(\frac{1}{3},\frac{4}{3}\right)$

c. $\left(\frac{3}{2},-\frac{3}{2}\right)$ d. $\left(-\frac{3}{4},\frac{3}{4}\right)$

15. Directrix of the parabola is

a. $6x+7=0$ b. $4x+11=0$

c. $3x+11=0$ d. none of these

For Problems 16–18

Consider the parabola whose focus is at (0, 0) and tangent at vertex is $x-y+1=0$.

16. The length of latus rectum is

a. $4\sqrt{2}$ b. $2\sqrt{2}$ c. $8\sqrt{2}$ d. $3\sqrt{2}$

17. The length of the chord of parabola on the x-axis is

a. $4\sqrt{2}$ b. $2\sqrt{2}$ c. $8\sqrt{2}$ d. $3\sqrt{2}$

18. Tangents drawn to the parabola at the extremities of the chord $3x+2y=0$ intersect at an angle

a. $\frac{\pi}{6}$ b. $\frac{\pi}{3}$

c. $\frac{\pi}{2}$ d. none of these

For Problems 19–21

Two tangents on a parabola are $x-y=0$ and $x+y=0$. If (2, 3) is focus of the parabola, then

19. The equation of tangent at vertex is

a. $4x-6y+5=0$ b. $4x-6y+3=0$

c. $4x-6y+1=0$ d. $4x-6y+3/2=0$

20. Length of latus rectum of the parabola is

a. $\frac{6}{\sqrt{3}}$ b. $\frac{10}{\sqrt{13}}$

c. $\frac{2}{\sqrt{13}}$ d. none of these

21. If P, Q are ends of focal chord of the parabola, then $\frac{1}{SP}+\frac{1}{SQ}=$

a. $\frac{2\sqrt{13}}{3}$ b. $2\sqrt{13}$

c. $\frac{2\sqrt{13}}{5}$ d. none of these

For Problems 22–24

$y^2=4x$ and $y^2=-8(x-a)$ intersect at points A and C. Points $O(0, 0)$, A, $B(a, 0)$, C are concyclic.

22. The length of common chord of parabolas is

a. $2\sqrt{6}$ b. $4\sqrt{3}$ c. $6\sqrt{5}$ d. $8\sqrt{2}$

23. The area of cyclic quadrilateral $OABC$ is

a. $24\sqrt{3}$ b. $48\sqrt{2}$ c. $12\sqrt{6}$ d. $18\sqrt{5}$

24. Tangents to parabola $y^2=4x$ at A and C intersect at point D and tangents to parabola $y^2=-8(x-a)$ intersect at point E, then the area of quadrilateral $DAEC$ is

a. $96\sqrt{2}$ b. $48\sqrt{3}$ c. $54\sqrt{5}$ d. $36\sqrt{6}$

For Problems 25–27

PQ is double ordinate of the parabola $y^2=4x$ which passes through the focus S. ΔPQA is isosceles right angle triangle, where A is on the axis of the parabola. Line PA meets the parabola at C and QA meets the parabola at B.

25. Area of the trapezium $PBCQ$ is

a. 96 sq. units b. 64 sq. units

c. 72 sq. units d. 48 sq. units

26. Circumradius of trapezium $PBCQ$ is

a. $6\sqrt{5}$ b. $3\sqrt{6}$ c. $2\sqrt{10}$ d. $5\sqrt{3}$

27. Ratio of inradius of ΔABC and that of ΔPAQ is

a. 2 : 1 b. 3 : 2

c. 4 : 3 d. 3 : 1

For Problems 28–30

Consider the inequality, $9^x-a.\,3^x-a+3\le 0$, where 'a' is a real parameter.

28. The given inequality has at least one negative solution for $a \in$

a. $(-\infty, 2)$ **b.** $(3, \infty)$ **c.** $(-2, \infty)$ **d.** $(2, 3)$

29. The given inequality has at least one positive solution for $a \in$

a. $(-\infty, -2)$ **b.** $[3, \infty)$ **c.** $(2, \infty)$ **d.** $[-2, \infty)$

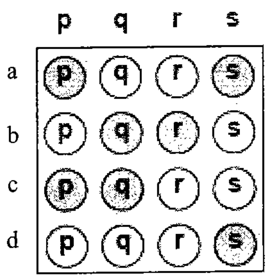
30. The given inequality has at least one real solution for $a \in$

a. $(-\infty, 3)$ **b.** $[2, \infty)$ **c.** $(3, \infty)$ **d.** $[-2, \infty)$

Matrix-Match Type

Solutions on page 3.79

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, , b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Consider the parabola $(x-1)^2 + (y-2)^2 = \dfrac{(12x - 5y + 3)^2}{169}$

Column I	Column II
a. Locus of point of intersection of perpendicular tangent	**p.** $12x - 5y - 2 = 0$
b. Locus of foot of perpendicular from focus upon any tangent	**q.** $5x + 12y - 29 = 0$
c. Line along which minimum length of focal chord occurs	**r.** $12x - 5y + 3 = 0$
d. Line about which parabola is symmetrical	**s.** $24x - 10y + 1 = 0$

2. Consider the parabola $y^2 = 12x$.

Column I	Column II
a. Equation of tangent can be	**p.** $2x + y - 6 = 0$
b. Equation of normal can be	**q.** $3x - y + 1 = 0$
c. Equation of chord of contact w.r.t. any point on the directrix can be	**r.** $x - 2y - 12 = 0$
d. Equation of chord which subtends right angle at the vertex can be	**s.** $2x - y - 36 = 0$

3.

Column I	Column II
a. Tangents are drawn from point (2, 3) to the parabola $y^2 = 4x$, then points of contact are	**p.** $(9, -6)$
b. From a point P on the circle $x^2 + y^2 = 5$, the equation of chord of contact to the parabola $y^2 = 4x$ is $y = 2(x - 2)$, then the coordinate of point P will be	**q.** $(1, 2)$
c. $P(4, -4)$, Q are points on parabola $y^2 = 4x$ such that area of ΔPOQ is 6 sq. units where O is the vertex, then coordinates of Q may be	**r.** $(-2, 1)$
d. The common chord of circle $x^2 + y^2 = 5$ and parabola $6y = 5x^2 + 7x$ will pass through point(s)	**s.** $(4, 4)$

4.

Column I	Column II
a. Points from which perpendicular tangents can be drawn to parabola $y^2 = 4x$	**p.** $(-1, 2)$
b. Points from which only one normal can be drawn to parabola $y^2 = 4x$	**q.** $(3, 2)$
c. Point at which chord $x - y + 1 = 0$ of parabola $y^2 = 4x$ is bisected	**r.** $(-1, -5)$
d. Points from which tangents cannot be drawn to parabola $y^2 = 4x$	**s.** $(5, -2)$

Integer Type

Solutions on page 3.80

1. If the length of the latus rectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x - 12y + 17)^2$ is L then the value of $\dfrac{13L}{4}$ is

2. $y = x + 2$ is any tangent to the parabola $y^2 = 8x$. The ordinate of the point P on this tangent such that the other tangent from it which is perpendicular to it is

3. The focal chord of $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible value of the square of slope of this chord is

4. Two tangents are drawn from the point (–2, –1) to the parabola $y^2 = 4x$. If θ is the angle between these tangents then $\tan\theta =$

5. The equation of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is $ax + by + c = 0$ then the value of $a + b + c$ is

6. If the point $P(4, -2)$ is the one end of the focal chord PQ of the $y^2 = x$, then the slope of the tangent at Q is

7. If the line $x + y = 6$ is a normal to the parabola $y^2 = 8x$ at point (a, b) then the value of $a + b$ is

8. The locus of the mid-points of the portion of the normal to the parabola $y^2 = 16x$ intercepted between the curve and the axis is another parabola whose latus rectum is

9. Consider locus of center of circle which touches circle $x^2 + y^2 = 4$ and line $x = 4$. The distance of the vertex of the locus from origin is

10. If on a given base BC ($B(0, 0)$ and $C(2, 0)$) a triangle be described such that the sum of the tangents of the base angles is 4, then equation of locus of opposite vertex A is parabola whose directrix is $y = k$, then the value of $8k - 9$ is

11. PQ is any focal chord of the parabola $y^2 = 8x$. Then the length of PQ can never be less than

12. If the length of focal chord to the parabola $y^2 = 12x$ drawn from the point (3, 6) on it is L then the value of $L/3$ is

13. From the point (–1, 2) tangent lines are drawn to the parabola $y^2 = 4x$. If the area of the triangle formed by the chord of contact & the tangents is A then the value of $\frac{A}{\sqrt{2}}$ is

14. Line $y = 2x - b$ cuts the parabola $y = x^2 - 4x$ at points A and B. Then the value of b for which the $\angle AOB$ is a right angle is

15. A line through the origin intersects the parabola $5y = 2x^2 - 9x + 10$ at two points whose x-coordinates add up to 17. Then the slope of the line is

16. If circle and $(x - 6)^2 + y^2 = r^2$ and parabola $y^2 = 4x$ have maximum number of common chord then least integral value of r is

Archives

Solutions on page 3.82

Subjective Type

1. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2$. **(IIT-JEE, 1981)**

2. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola. find the slope of AB **(IIT-JEE, 1982)**

3. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2). **(IIT-JEE, 1984)**

4. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. **(IIT-JEE, 1991)**

5. Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. **(IIT-JEE, 1994)**

6. Show that the locus of a point which divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1:2 is a parabola. Find the vertex of this parabola. **(IIT-JEE, 1995)**

7. Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C, taken in pairs, intersect at points P, Q and R. Determine the ratio of the areas of the triangles ABC and PQR. **(IIT-JEE, 1996)**

8. From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola. **(IIT-JEE, 1996)**

9. The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of the point P is a hyperbola. **(IIT-JEE, 1998)**

10. Let C_1 and C_2 be, respectively, the parabola $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line $y = x$. Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min\{PP_1, QQ_1\}$. Hence, or otherwise determine points P_0 and Q_0 on the parabolas C_1 and C_2, respectively, such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2. **(IIT-JEE, 2000)**

11. Normals are drawn from the point P with slopes m_1, m_2, m_3 to that parabola $y^2 = 4x$. If locus of P with $m_1m_2 = \alpha$ is a part of the parabola itself, then find α. **(IIT-JEE, 2003)**

12. Tangent is drawn to parabola $y^2 - 2y - 4x + 5 = 0$ at a point P which cuts the directrix at the point Q. A point R is such that it divides QP externally in the ratio $\frac{1}{2}:1$. Find the locus of point R. **(IIT-JEE, 2004)**

Objective Type

Fill in the blanks

1. The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is____________.

(IIT-JEE, 1994)

Multiple choice questions with one correct answer

1. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and parabola is

a. $\left(\frac{p}{2}, p\right)$ or $\left(\frac{p}{2}, -p\right)$ **b.** $\left(\frac{p}{2}, -\frac{p}{2}\right)$

c. $\left(-\frac{p}{2}, p\right)$ **d.** $\left(-\frac{p}{2}, -\frac{p}{2}\right)$

(IIT-JEE, 1995)

2. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents

a. a pair of straight lines

b. an ellipse

c. a parabola

d. a hyperbola

(IIT-JEE, 1999)

3. If $x + y = k$ is normal to $y^2 = 12x$, then k is

a. 3 **b.** 9 **c.** -9 **d.** -3

(IIT-JEE, 2000)

4. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is

a. $\frac{1}{8}$ **b.** 8 **c.** 4 **d.** $\frac{1}{4}$

(IIT-JEE, 2000)

5. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

a. $\sqrt{3}y = 3x + 1$ **b.** $\sqrt{3}y = -(x + 3)$

c. $\sqrt{2}y = x + 3$ **d.** $\sqrt{3}\,y = -(3x + 1)$

(IIT-JEE, 2001)

6. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

a. $x = -1$ **b.** $x = 1$ **c.** $x = -\frac{3}{2}$ **d.** $x = \frac{3}{2}$

(IIT-JEE, 2001)

7. The locus of the midpoint of the segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directix

a. $y = 0$ **b.** $x = -a$

c. $x = 0$ **d.** none of these

(IIT-JEE, 2002)

8. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible value of the slope of this chord, are

a. $\{-1. 1\}$ **b.** $\{-2, 2\}$

c. $\{-2, \frac{1}{2}\}$ **d.** $\{2, -\frac{1}{2}\}$

(IIT-JEE, 2003)

9. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

(IIT-JEE, 2004)

10. Tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 = 16x + 12y + c = 0$ at a point Q. Then the coordinate of Q are

a. $(-6, -11)$ **b.** $(-9, -13)$

c. $(-10, -15)$ **d.** $(-6, -7)$

(IIT-JEE, 2005)

11. The axis of a parabola is along the line $y = x$ and the distance of its vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively. If vertex and focus both lie in the first quadrant, then the equation of the parabola is

a. $(x + y)^2 = (x - y - 2)$

b. $(x - y)^2 = (x + y - 2)$

c. $(x - y)^2 = 4\,(x + y - 2)$

d. $(x - y)^2 = 8\,(x + y - 2)$

12. Consider the two curves C_1:$y^2 = 4x$, C_2:$x^2 + y^2 - 6x + 1 = 0$. Then

a. C_1 and C_2 touch each other only at one point

b. C_1 and C_2 touch each other exactly at two points

c. C_1 and C_2 intersect (but do not touch) at exactly two points

d. C_1 and C_2 neither intersect nor touch each other

(IIT-JEE, 2008)

13. Let (x, y) by any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is

a. $x^2 = y$ b. $y^2 = 2x$

c. $y^2 = x$ d. $x^2 = 2y$ **(IIT-JEE, 2011)**

Multiple choice questions with one or more than one correct answer

1. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x-2)^2$ is/are
   a. $y = 4(x-1)$ b. $y = 0$
   c. $y = -4(x-1)$ d. $y = -30x - 50$

   (IIT-JEE, 2006)

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are
   a. $x^2 + 2\sqrt{3}\,y = 3 + \sqrt{3}$
   b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
   d. $x^2 - 2\sqrt{3}\,y = 3 - \sqrt{3}$

   (IIT-JEE, 2008)

3. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at point T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose
   a. vertex is $\left(\frac{2a}{3}, 0\right)$
   b. directrix is $x = 0$
   c. latus rectum is $\frac{2a}{3}$
   d. focus is $(a, 0)$

   (IIT-JEE, 2009)

4. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
   a. $-\frac{1}{r}$ b. $\frac{1}{r}$
   c. $\frac{2}{r}$ d. $-\frac{2}{r}$

   (IIT-JEE, 2010)

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by
   a. $y - x + 3 = 0$ b. $y + 3x - 33 = 0$
   c. $y + x - 15 = 0$ d. $y - 2x + 12 = 0$

   (IIT-JEE,2011)

Match the following

1. $(3, 0)$ is the point from which three normals are drawn to the parabola $y^2 = 4x$ which meet the parabola in the points P, Q and R. Then **(IIT-JEE, 2006)**

Column I	Column II
i. Area of ΔPQR	**a.** 2
ii. Radius of circumcircle of ΔPQR	**b.** $\frac{5}{2}$
iii. Centroid of ΔPQR	**c.** $\left(\frac{5}{2}, 0\right)$
iv. Circumcentre of ΔPQR	**d.** $\left(\frac{2}{3}, 0\right)$

Comprehension based questions

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. **(IIT-JEE, 2007)**

1. The ratio of the areas of the triangles PQS and PQR is
   a. $1:\sqrt{2}$ b. 1:2 c. 1:4 d. 1:8
2. The radius of the circumcircle of the triangle PRS is
   a. 5 b. $3\sqrt{3}$ c. $3\sqrt{2}$ d. $2\sqrt{3}$
3. The radius of the incircle of the triangle PQR is
   a. 4 b. 3 c. $\frac{8}{3}$ d. 2

Assertion and Reasoning

1. **Statement 1:** The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$.

   Statement 2 : A parabola is symmetric about its axis.
   a. Statement 2 is true, statement 2 is true; statement 2 is a correct explanation for statement 2
   b. Statement 2 is true, statement 2 is true; statement 2 is NOT a correct explanation for statement 2
   c. Statement 2 is true, statement 2 is false
   d. Statement 2 is false, statement 2 is true.

   (IIT-JEE, 2007)

Integer type

1. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is

   (IIT-JEE, 2011)

ANSWERS AND SOLUTIONS

Subjective Type

1.

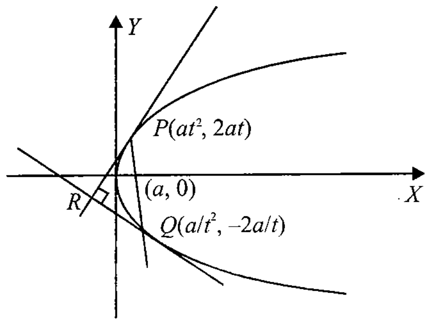


Fig. 3.63

Let the parabola be $y^2 = 4ax$.

ΔPQR is right angled at R as tangents at the extremities of the focal chord meet on the directrix at right angle.

Also coordinates of points P and Q are $P(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ respectively.

Hence, point of intersection of tangents at point $P(t)$ and $Q\left(-\frac{1}{t}\right)$ is $\left(-a, a\left(t-\frac{1}{t}\right)\right)$ and the coordinates of the centroid (G) is $\left(\frac{a}{3}\left(t^2-\frac{1}{t^2}-1\right), a\left(t-\frac{1}{t}\right)\right)$.

Hence, the slope of line $RG = 0$ (R is orthocentre).

2.

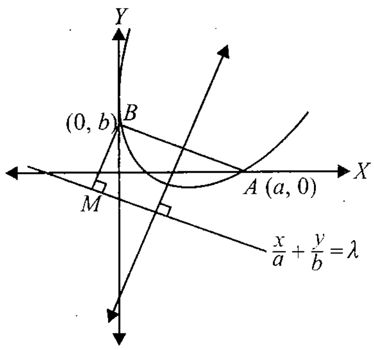


Fig. 3.64

AB is the shortest focal chord of the parabola, i.e., it is the latus rectum.

$\Rightarrow$ Focus of the parabola is the midpoint of AB

$$=\left(\frac{a}{2}, \frac{b}{2}\right)$$

Equation of the AB is $\frac{x}{a}+\frac{y}{b}=1$

Equation of the directrix is $\frac{x}{a}+\frac{y}{b}=\lambda$

By definition of parabola $BD = BM = \frac{\sqrt{a^2+b^2}}{2}$

$$\Rightarrow \left|\frac{\frac{0}{a}+\frac{b}{b}-\lambda}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{\sqrt{a^2+b^2}}{2}$$

$$\Rightarrow \quad 2(1-\lambda)\, ab = \pm(a^2+b^2)$$

$$\Rightarrow \quad \lambda = \frac{-(a-b)^2}{2ab} \text{ or } \frac{(a+b)^2}{2ab}$$

$\Rightarrow$ Directrices are $\frac{x}{a}+\frac{y}{b} = \frac{-(a-b)^2}{2ab}$,

and $\quad \frac{x}{a}+\frac{y}{b} = \frac{(a+b)^2}{2ab}$

$\Rightarrow$ Two parabolas are possible whose equations are given by

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2 = \frac{\left(\frac{x}{a}+\frac{y}{b}+\frac{(a-b)^2}{2ab}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2 = \frac{\left(\frac{x}{a}+\frac{y}{b}-\frac{(a+b)^2}{2ab}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}}$$

3.

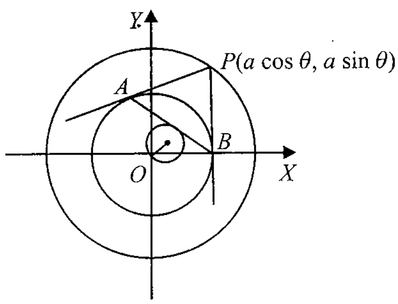


Fig. 3.65

Let centre of the variable circle be (h, k) and the point taken on the first circle $x^2 + y^2 = a^2$ be $(a \cos\theta, a \sin\theta)$.

Equation of the chord of contact AB will be

$$(a\cos\theta)x + (a\sin\theta)y = b^2$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta = \frac{b^2}{a}$$

As this is tangent to the variable circle, centre of variable circle is at a constant distance r from AB and O where r is the radius (variable).

Hence, locus is a parabola.

4.

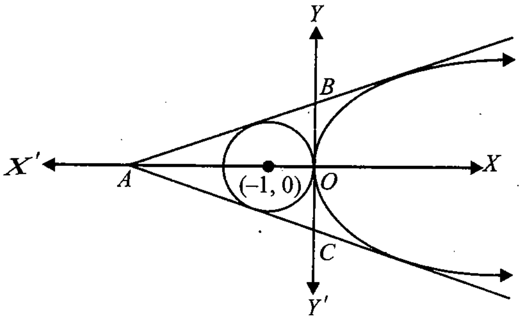


Fig. 3.66

As circle is $(x + 1)^2 + y^2 = 1$, one of the common tangent is along y-axis.

Let the other common tangent has slope m,

Then, its equation is $\quad y = mx + \frac{1}{m}$

Solving it with the equation of circle,

$$x^2 + \left(mx + \frac{1}{m}\right)^2 + 2x = 0$$

$$\Rightarrow \quad (1 + m^2)\,x^2 + 4x + \frac{1}{m^2} = 0$$

As the line touches the circle, $D = 0$

$$\Rightarrow \quad 16 - \frac{4}{m^2}(1 + m^2) = 0 \Rightarrow 4m^2 = 1 + m^2$$

$$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$$

i.e. $\quad \angle BOA = \angle OAC = \frac{\pi}{6}$

Hence, the triangle is equilateral.

5.

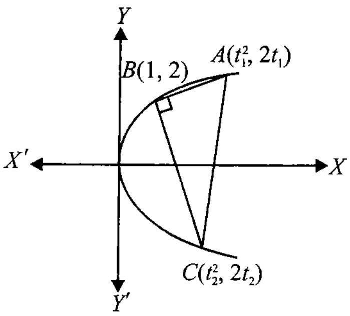


Fig. 3.67

Let points A and B be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$,

then $\quad m_{AB}m_{BC} = -1$

$$\Rightarrow \quad \frac{2}{(t_1 + 1)}\cdot\frac{2}{(t_2 + 1)} = -1$$

$$\Rightarrow \quad t_1 + t_2 + t_1t_2 = -5 \qquad \text{(i)}$$

Let the centroid of the ΔABC be (h, k), then

$$h = \frac{t_1^2 + t_2^2 + 1}{3}$$

and
$$k = \frac{2t_1 + 2t_2 + 2}{3} \qquad \text{(ii)}$$

From Eq. (ii), we get

$$t_1^2 + t_2^2 = 3h - 1 \text{ and } t_1 + t_2 = \frac{3k - 2}{2} \qquad \text{(iii)}$$

or $\quad (t_1 + t_2)^2 - 2t_1t_2 = 3h - 1$

or $\quad \left(\frac{3k - 2}{2}\right)^2 - 2t_1t_2 = 3h - 1$

Hence, from Eq. (i), $\frac{3k - 2}{2} + \frac{\left(\frac{3k - 2}{2}\right)^2 - (3h - 1)}{2} = -5$

Hence, locus is $3y - 2 + \left(\frac{3y - 2}{2}\right)^2 - (3x - 1) + 10 = 0$

6.

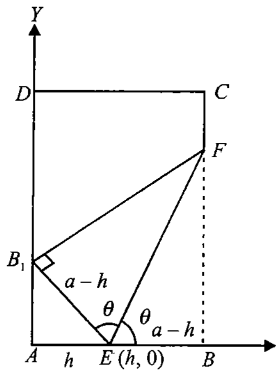


Fig. 3.68

Let $ABCD$ be a page with $AB = a$, $BC = b$.

The corner B of folded leaf is moving along AD.

Let A be origin, $AE = h$, then equation of crease EF is

$$y = (x - h)\tan\theta \qquad \text{(i)}$$

Now $\quad \angle AEB_1 = \pi - 2\theta$

$$\Rightarrow \quad \cos(\pi - 2\theta) = \frac{h}{a - h}$$

$$\Rightarrow \quad \frac{\tan^2\theta - 1}{\tan^2\theta + 1} = \frac{h}{a - h}$$

$$\Rightarrow \quad \tan^2\theta = \frac{a}{a - 2h}$$

$$\Rightarrow \quad \frac{y^2}{(x - h)^2} = \frac{a}{a - 2h} \qquad \text{[from Eq. (i)]}$$

$\Rightarrow \quad (a - 2h)y^2 = a(x - h)^2$

$\Rightarrow ah^2 + 2(y^2 - ax)h + a(x^2 - y^2) = 0$

For two coincident lines, discriminant $D = 0$

$4(y^2 - ax)^2 - 4a^2(x^2 - y^2) = 0$

$\Rightarrow \quad y^2(y^2 - 2ax + a^2) = 0$

Therefore, the crease is tangent to $y^2 - 2ax + a^2 = 0$, which is a parabola.

7. Let the fixed parabola is

$y^2 = 4ax$ (i)

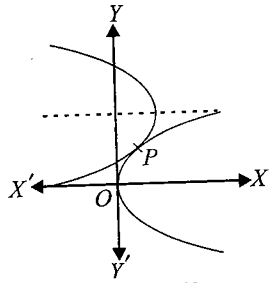


Fig. 3.69

and moving parabola is

$$(y - k)^2 = -4a(x - h) \quad \text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$(y - k)^2 = -4a\left(\frac{y^2}{4a} - h\right)$$

$\Rightarrow \quad y^2 - 2ky + k^2 = -y^2 + 4ah$

$\Rightarrow \quad 2y^2 - 2ky + k^2 - 4ah = 0$

Since the two parabolas touch each other, so $D = 0$

$\Rightarrow \quad 4k^2 - 8(k^2 - 4ah) = 0$

$\Rightarrow \quad -4k^2 + 32ah = 0$

$\Rightarrow \quad k^2 = 8ah$

$\Rightarrow$ Locus of the vertex of the moving parabola is

$$y^2 = 8ax.$$

8. Since the x-axis and the y-axis are two perpendicular tangents to the parabola and both meet at the origin, the directrix passes through the origin.

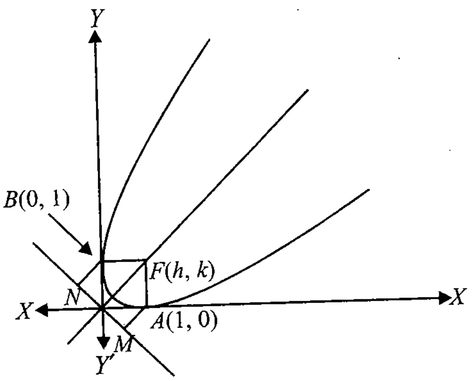


Fig. 3.70

Let $y = mx$ be the directrix and (h, k) be the focus

$\Rightarrow \quad FA = AM$

$\Rightarrow \quad \sqrt{(h-1)^2 + k^2} = \left|\frac{m}{\sqrt{1+m^2}}\right|$ (i)

and $\quad FB = BN$

$\Rightarrow \quad \sqrt{h^2 + (k-1)^2} = \left|\frac{1}{\sqrt{1+m^2}}\right|$ (ii)

Squaring and adding Eqs. (i) and (ii), we get

$(h - 1)^2 + h^2 + k^2 + (k - 1)^2 = 1$

$\Rightarrow \quad 2x^2 - 2x + 2y^2 - 2y + 1 = 0$, which is the required locus.

9. Tangent at the vertex is y-axis or $x = 0$ (i)

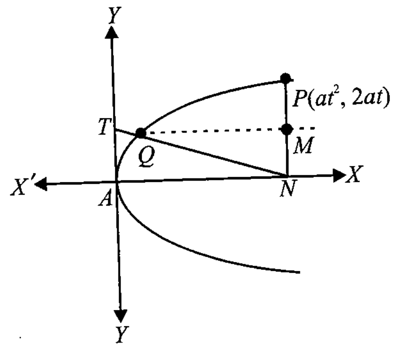


Fig. 3.71

Let P be any point $(at^2, 2at)$ on the parabola.

So, $PN = 2at$ and coordinates of N are $(at^2, 0)$.

If M is the midpoint of PN, then $NM = \frac{1}{2}PN = at$.

As MQ is parallel to x-axis, its equation is

$$y = at \quad \text{(ii)}$$

Solving (ii) with parabola $a^2t^2 = 4ax$

$\Rightarrow \quad x = \frac{1}{4}at^2$

So, the coordinates of Q are $\left(\frac{1}{4}at^2, at\right)$.

Equation of NQ will be

$$(y - 0) = \frac{at - 0}{\frac{1}{4}at^2 - at^2}(x - at^2)$$

$\Rightarrow \quad y = \frac{-4}{3t}(x - at^2)$ (iv)

If NQ meets the tangent at the vertex, i.e., $x = 0$ at T, then on putting $x = 0$ in Eq. (ii), we get $y = \frac{4at}{3}$.

So the coordinates of T are $\left(0, \frac{4at}{3}\right)$ or $AT = \frac{4at}{3}$.

$\Rightarrow \quad AT = \frac{2}{3} \times PN$

10. $y = mx + \frac{a}{m}$ (i)

is a tangent to $y^2 = 4ax$ and,

$x = m_1 y + \frac{b}{m_1}$ (ii)

is a tangent to $x^2 = 4by$.

Lines (i) and (ii) are perpendicular.

$\Rightarrow \quad m \times \frac{1}{m_1} = -1$

$\Rightarrow \quad m_1 = -m$

Let (h, k) be the point of intersection of Eqs. (i) and (ii), then

$$k = mh + \frac{a}{m} \text{ and } h = m_1 k + \frac{b}{m_1}$$

$\Rightarrow \quad k = mh + \frac{a}{m}$ and $h = -mk - \frac{b}{m}$

$\Rightarrow \quad m^2h - mk + a = 0$

and $\quad m^2k + mh + b = 0$

$\Rightarrow \quad \frac{m^2}{-kb - ah} = \frac{m}{ak - bh} = \frac{1}{h^2 + k^2}$

[By cross-multiplication]

Eliminating m, we have

$-(h^2 + k^2)(kb + ah) = (bh - ak)^2$

$\Rightarrow \quad (x^2 + y^2)(ax + by) + (bx - ay)^2 = 0$

11. $AS = 1 + t^2$ (focal distance of any point on the parabola $y^2 = 4ax$ is $a + at^2$)

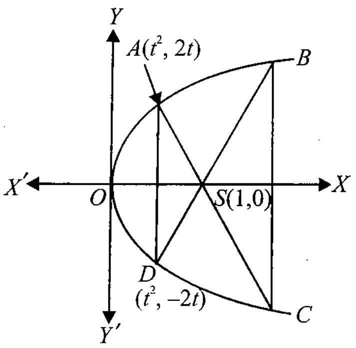


Fig. 3.72

$\therefore \quad CS = AC - AS = \frac{25}{4} - (1 + t^2)$

AC is a focal chord.

$\Rightarrow \quad \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a}$

$\Rightarrow \quad \frac{1}{1+t^2} + \frac{1}{\frac{25}{4} - (1+t^2)} = 1$

$\Rightarrow \quad \frac{1}{1+t^2} + \frac{4}{25 - 4(1+t^2)} = 1$

$\Rightarrow \quad 25 - 4(1+t^2) + 4(1+t^2) = (1+t^2)(25 - 4(1+t^2))$

$\Rightarrow 4(1+t^2)^2 - 25(1+t^2) + 25 = 0$

$\Rightarrow \{(1+t^2) - 5\}\{4(1+t^2) - 5\} = 0$

$\Rightarrow \quad 1 + t^2 = 5, \frac{5}{4}$

$\Rightarrow \quad t^2 = 4, \frac{1}{4}$

$\Rightarrow \quad t = \pm 2, \pm\frac{1}{2}$

Thus, the coordinates of A, B, C, D are

$A \equiv (1/4, 1), B \equiv (4, 4),$

$C \equiv (4, -4)$

and $\quad D \equiv (1/4, -1)$

$\therefore \quad AD = 2, BC = 8$ and distance between AD and BC

$= \frac{15}{4}$.

Therefore, area of trapezium $ABCD$

$= \frac{1}{2}(2 + 8) \times \frac{15}{4}$

$= \frac{75}{4}$ sq. units

12. The given curves are $x^2 - y^2 = a^2$ (i)

and $\quad y = x^2$ (ii)

Solving Eqs. (i) and (ii) for y, we find that

$x^2 - y^2 = a^2$

$\Rightarrow \quad y - y^2 = a^2$

$\Rightarrow \quad y^2 - y + a^2 = 0$

Since y is real, $\quad 1 > 4a^2$

$\Rightarrow \quad -\frac{1}{2} < a < \frac{1}{2}$

The equation of any conic through the point of intersection of hyperbola and parabola is

$(x^2 - y^2 - a^2) + \lambda(y - x^2) = 0$

$\Rightarrow \quad x^2(1 - \lambda) - y^2 + \lambda y - a^2 = 0$ which is circle, then

$1 - \lambda = -1 \Rightarrow \lambda = 2$

Hence, the equation of the circle is

$x^2 + y^2 - 2y + a^2 = 0.$

13.

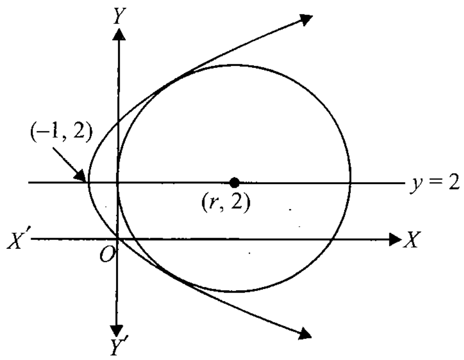


Fig. 3.73

$$y^2 = 4(x + y)$$

$$\Rightarrow \quad (y - 2)^2 = 4(x + 1) \quad \text{(i)}$$

Its focus is (0, 2).

Let radius of the circle is r.

Then, equation of circle is

$$(x - r)^2 + (y - 2)^2 = r^2 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii), we have

$$(x - r)^2 + 4(x + 1) = r^2$$

$$x^2 + (4 - 2r)x + 4 = 0$$

$$(4 - 2r)^2 - 16 = 0 \quad [\because D = 0]$$

$$\Rightarrow \quad 4 - 2r = \pm 4$$

$$\Rightarrow \quad r = 4$$

14. From the property of parabola, R is a focus.

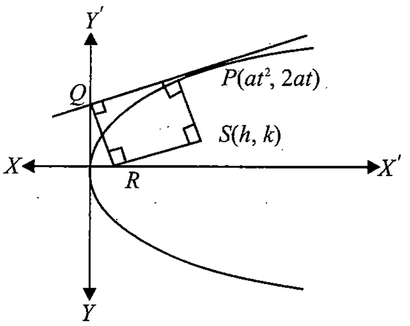


Fig. 3.74

$$\frac{2at - k}{at^2 - h} \times \frac{k - 0}{h - a} = -1$$

and $$\frac{k}{h - a} = \frac{1}{t} \Rightarrow t = \frac{h - a}{k}$$

So the required locus is

$$\left(2a\left(\frac{h-a}{k}\right) - k\right) = (a - h)\,\frac{(a - h)}{k}\left(a\left(\frac{h-a}{k}\right)^2 - h\right)$$

$$\Rightarrow \quad y^4 = (x - a)(a(x - a)^2 - y^2(x^2 + 2a))$$

15. Let the three points be $A\ (at_1^2,\ 2at_1)$, $B\ (at_2^2,\ 2at_2)$, $C\ (at_3^2, 2at_3)$

Tangents at these point are

$$t_1 y = x + at_1^2,\ t_2 y = x + at_2^2$$

$$t_3 y = x + at_3^2$$

Since t_1, t_2, t_3 are distinct, no two tangents are parallel or coincident.

Hence, these tangents will form a triangle.

The vertices of the triangle are $[at_1t_2,\ a(t_1 + t_2)]$, $[at_2t_3,\ a(t_2 + t_3)]$ and $[at_3t_1\ a(t_3 + t_1)]$.

Equation of the two altitudes are

$$[y - a(t_2 + t_3)] = -t_1(x - at_2t_3) \quad \text{(i)}$$

and $$[y - a(t_3 + t_1)] = -t_2(x - at_3\,t_1) \quad \text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get $x = -a$

Hence, the locus of the orthocenter is $x + a = 0$ which is the directrix of the parabola.

16.

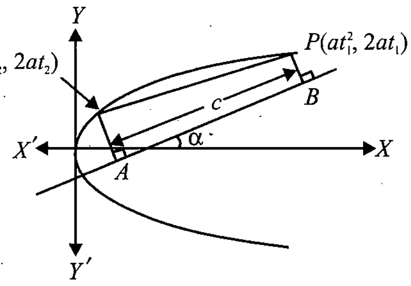


Fig. 3.75

$$\overrightarrow{PQ} = \vec{v} = (at_2^2 - at_1^2)\hat{i} + 2a(t_2 - t_1)\hat{j}$$

Also midpoint of PQ is (h, k).

$$\Rightarrow \quad 2h = a(t_1^2 + t_2^2)\ ;\ a(t_1 + t_2) = k$$

Also projection of $\vec{v}$ on $\overrightarrow{AB} = c$

$$\Rightarrow \quad \left|\frac{\vec{v}\,.\,\overrightarrow{AB}}{|\overrightarrow{AB}|}\right| = c$$

$$\Rightarrow \quad |\,a\,(t_2^2 - t_1^2)\cos\alpha + 2a(t_2 - t_2)\sin\alpha\,| = c$$

$$\Rightarrow \quad a^2\,(t_2 - t_1)^2\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$$

$$\Rightarrow \quad a^2\,[(t_2 + t_1)^2 - 4t_1t_2]\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$$

$$\Rightarrow \quad (y^2 - 4ax)\,(y\cos\alpha + 2a\sin\alpha)^2 + a^2c^2 = 0$$

17.

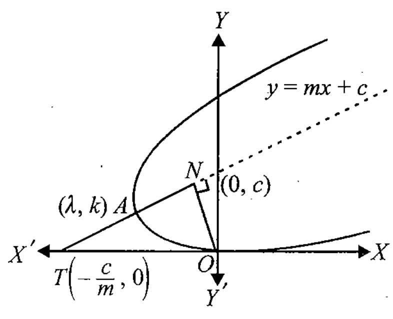


Fig. 3.76

Equation of ON is

$$y = -\frac{1}{m}x \text{ or } x + my = 0$$

Solving it with $\quad y = mx + c,$

coordinates of N are $\left(\dfrac{-cm}{1 + m^2},\ \dfrac{c}{1 + m^2}\right)$

Also $\quad TA = AN$

$\therefore \quad \frac{-cm}{1+m^2} - \frac{c}{m} = 2\lambda$

and $\quad \frac{c}{1+m^2} = 2k$

$\Rightarrow \quad c = 2k(1+m^2)$

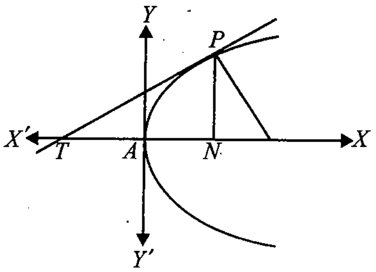


Fig. 3.77

Hence, equation of the axis of symmetry is

$$y = mx + 2k(1+m)^2$$

or $\quad 2km^2 + mx + 2k - y = 0 \qquad$ (i)

This family of lines touch a curve which is obtained by equating to zero the discriminant of Eq. (i).

Hence, $\quad x^2 - 8k(2k - y) = 0$

or $\quad x^2 = -8k(y - 2k)$

18. Let the point P be $(p, 0)$ and equation of the chord through P be

$$\frac{x-p}{\cos\theta} = \frac{y-0}{\sin\theta} = r \ (r \in R) \qquad \text{(i)}$$

$\therefore \quad (r\cos\theta + p, r\sin\theta)$ lies on the parabola $y^2 = 4ax$

$\therefore \quad r^2 \sin^2\theta - 4ar\cos\theta - 4ap = 0 \qquad$ (ii)

If $AP = r_1$, $BP = -r_2$, then r_1 and r_2 are roots of Eq. (ii)

$$\therefore \quad r_1 + r_2 = \frac{4a\cos\theta}{\sin^2\theta}, \ r_1r_2 = \frac{-4ap}{\sin^2\theta}$$

Now $\quad \frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{r_2^2} + \frac{1}{r_2^2}$

$$= \frac{(r_1+r_2)^2 - 2r_1r_2}{r_1^2r_2^2}$$

$$= \frac{\cos^2\theta}{p^2} + \frac{\sin^2\theta}{2ap}$$

Since $\frac{1}{AP^2} + \frac{1}{BP^2}$ should be independent of θ, we take $p = 2a$.

Then $\quad \frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{4a^2}(\cos^2\theta + \sin^2\theta)$

$$= \frac{1}{4a^2}$$

Hence, $\frac{1}{AP^2} + \frac{1}{BP^2}$ is independent of θ for all positions of the chord if $P \equiv (2a, 0)$.

Objective Type

1. b. The coordinates of the focus and vertex of the required parabola are $S(a_1, 0)$ and $A(a, 0)$, respectively. Therefore, the distance between the vertex and focus is $AS = a_1 - a$ and so the length of the latus rectum $= 4(a_1 - a)$.

Thus, the equation of the parabola is

$$y^2 = 4(a_1 - a)(x - a).$$

2. d. Parabola having axis parallel to y-axis is

$$(x-a)^2 = 4A(y-b).$$

According to question, length of latus rectum $4A = l$.

Hence, equation of parabola is

$$(x-a)^2 = \frac{l}{4}(y-b)$$

or $\quad (x-a)^2 = \frac{l}{8}(2y - 2b)$

3. b. $x = 3\cos t$, $y = 4\sin t$

Eliminating t, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \text{ which is ellipse}$$

$\therefore \quad x^2 - 2 = 2\cos t; \ y = 4\cos^2\frac{t}{2}$

$\Rightarrow \quad y = 2(1 + 2\cos t)$

$$y = 2\left(1 + \frac{x^2-2}{2}\right),$$

which is parabola

$$\sqrt{x} = \tan t; \sqrt{y} = \sec t$$

Eliminating t, we have

$$y - x = 1, \text{ which is straight line}$$

$$x = \sqrt{1 - \sin t}; y$$

$$y = \sin\frac{t}{2} + \cos\frac{t}{2}$$

Eliminating t, we have

$x^2 + y^2 = 1 - \sin t + 1 + \sin t = 2$, which is circle.

4. c. Tangent at the vertex is

$x - y + 1 = 0 \qquad$ (i)

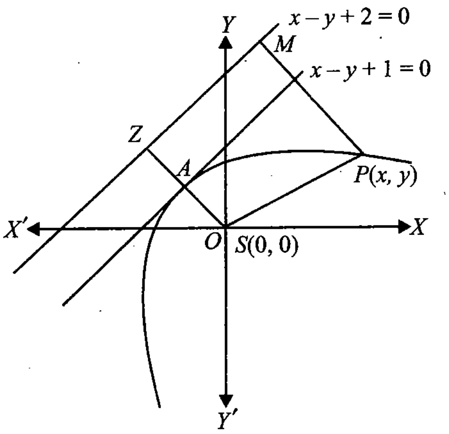


Fig. 3.78

Therefore, equation of axis of the parabola is

$$x + y = 0 \qquad \text{(ii)}$$

Now solving Eqs. (i) and (ii), we get $A\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\therefore$ Z is $(-1, 1)$

Now directrix is

$$x - y + k = 0$$

But this passes through $Z(-1, 1)$

$\Rightarrow \quad k = 2$

$\Rightarrow$ Directrix is $\quad x - y + 2 = 0$

Therefore, by definition equation of parabola is given by

$$OP = PM$$

$$\Rightarrow \quad OP^2 = PM^2$$

$$\left(\frac{x-y+2}{\sqrt{2}}\right)^2 = x^2 + y^2$$

$$\Rightarrow \quad (x - y + 2)^2 = 2x^2 + 2y^2$$

$$\Rightarrow x^2 + y^2 + 4 - 2xy + 4x - 4y = 2x^2 + 2y^2$$

$$\Rightarrow x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$$

5. b. We have, $\sqrt{px} + \sqrt{qy} = 1$

$$\Rightarrow \quad (\sqrt{px} + \sqrt{qy})^2 = 1$$

$$\Rightarrow \quad px + qy + 2\sqrt{(pq)(xy)} = 1$$

$$\Rightarrow \quad (px + qy - 1)^2 = 4(pq)(xy)$$

$$\Rightarrow \quad p^2x^2 - 2(pq)(xy) + q^2y^2 - 2px - 2qy + 1 = 0$$

On comparing this equation with the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = p^2, b = q^2, c = 1, g = -p,$$

$$f = -q \text{ and } h = -pq$$

$$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= p^2q^2 - 2p^2q^2 - p^2q^2 - p^2q^2 - p^2q^2$$

$$= -4p^2q^2 \neq 0$$

and, $\quad h^2 - ab = p^2q^2 - p^2q^2 = 0$

Thus, we have $\quad \Delta \neq 0$ and $h^2 = ab$

Hence, the given curve is parabola.

6. d. Two parabolas are equal if the length of their latus rectum are equal.

Length of the latus rectum of $y^2 = \lambda x$ is λ.

The equation of the second parabola is

$$25\{(x-3)^2 + (y+2)^2\} = (3x - 4y - 2)^2$$

$$\Rightarrow \quad \sqrt{(x-3)^2 + (y+2)^2} = \frac{|3x - 4y - 2|}{\sqrt{3^2 + 4^2}}$$

Here focus is $(3, -2)$ and equation of the directrix is

$$3x - 4y - 2 = 0.$$

Therefore, length of the latus rectum = 2 × distance between focus and directrix

$$= 2\left|\frac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}}\right| = 6$$

Thus, the two parabolas are equal, if $\lambda = 6$.

7. d. Length of latus rectum

= 2 × distance of focus form directrix

$$= 2 \times \left|\frac{-\frac{u^2}{2g}\cos 2\alpha - \frac{u^2}{2g}}{\sqrt{1}}\right|$$

$$= \frac{2u^2}{g}\cos^2\alpha$$

8. a. Make homogeneous and put $A + B = 0$

$$y^2 = 4ax\left(\frac{lx + my}{-n}\right)$$

$$\therefore \quad 4al + n = 0$$

9. a. $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$

$$\Rightarrow \quad (x - y)^2 = 8(x + y - 4)$$

is a parabola whose axis is $x - y = 0$ and the tangent at the vertex is $x + y - 4 = 0$.

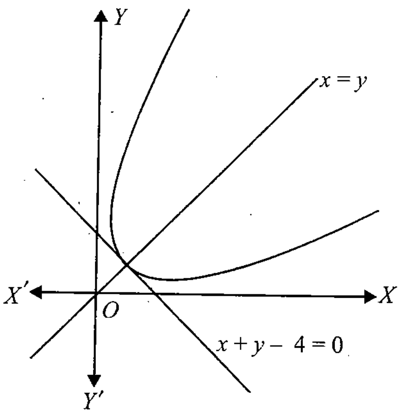


Fig. 3.79

Also, when $y = 0$, we have

$$x^2 - 8x + 32 = 0$$

which gives no real values of x.

when $x = 0$, we have $y^2 - 8y + 32 = 0$ which gives no real values of y.

So, the parabola does not intersect the axes. Hence, the graph falls in the first quadrant.

10. d. Eliminating θ from the given equations, we get

$$y^2 = -4a(x - a);$$

which is a parabola but $0 \le \cos^2\theta \le 1$

$\Rightarrow$ $0 \le x \le a$

and $-1 \le \sin\theta \le 1$

$\Rightarrow$ $-2a \le y \le 2a$

Hence, the locus of the point P is not exactly the parabola, rather it is a part of the parabola.

11. c. $(\sqrt{3h}, \sqrt{3k+2})$ lies on the line $x - y - 1 = 0$

$\Rightarrow$ $(\sqrt{3h})^2 = (\sqrt{3k+2} + 1)^2$

$\Rightarrow$ $3h = 3k + 2 + 1 + 2\sqrt{3k+2}$

$\Rightarrow$ $3^2(h - k - 1)^2 = 2^2(\sqrt{3k+2})^2$

$\Rightarrow$ $9(h^2 + k^2 + 1 - 2hk - 2h + 2k) = 4(3k + 2)$

$\Rightarrow$ $9(x^2 + y^2) - 18xy - 18x + 6y + 1 = 0$

Now $h^2 = ab$ and $\Delta \ne 0$

Therefore, locus is a parabola.

12. c.

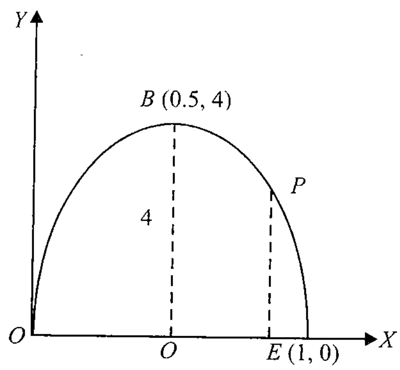


Fig. 3.80

The path of the water jet is a parabola.

Let its equation be

$$y = ax^2 + bx + c$$

It should pass through (0, 0), (0.5, 4), (1, 0)

$\Rightarrow$ $c = 0, a = -16, b = 16$

$\Rightarrow$ $y = -16x^2 + 16x$

If $x = 0.75$, we get $y = 3$.

13. a. $x = t^2 - t + 1, y = t^2 + t + 1$

$\Rightarrow$ $x + y = 2(t^2 + 1)$ and $y - x = 2t$

$\Rightarrow$ $\frac{x+y}{2} = 1 + \left(\frac{y-x}{2}\right)^2$

$\Rightarrow$ $(y - x)^2 = 2(x + y) - 4$

$\Rightarrow$ $(y - x)^2 = 2(x + y - 2)$

Vertex will be the point where lines $y - x = 0$ and $x + y - 2 = 0$ meet, i.e., the point (1, 1).

14. c. Let the point $P(h, k)$ on the parabola divides the line joining $A(4, -6)$ and $B(3, 1)$ in ratio λ.

Then, we have $(h, k) \equiv \left(\frac{3\lambda+4}{\lambda+1}, \frac{\lambda-6}{\lambda+1}\right)$

This point lies on the parabola,

$\therefore$ $\left(\frac{\lambda-6}{\lambda+1}\right)^2 = 4\left(\frac{3\lambda+4}{\lambda+1}\right)$

$\Rightarrow$ $(\lambda - 6)^2 = 4(3\lambda + 4)(\lambda + 1)$

$\Rightarrow$ $11\lambda^2 + 40\lambda - 20 = 0$

$\Rightarrow$ $\lambda = \frac{-20 \pm 2\sqrt{155}}{11} : 1$

15. d. Let $P(x, y)$ be the coordinates of the other end of the chord. OP.

Then $\frac{x+0}{2} = a, \frac{y+0}{2} = b$

But (x, y) lies on the parabola.

$\therefore$ $y^2 = 4x$

$\Rightarrow$ $(2b)^2 = 4(2a)$

$\Rightarrow$ $b^2 = 2a$

16. c. Let points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$.

Here points P and Q are variable but slope of the chord PQ,

$$m_{PQ} = \frac{2}{t_1 + t_2}$$

Now let midpoint PQ be $R(h, k)$,

$$k = \frac{2at_1 + 2at_2}{2}$$

or $k = a(t_1 + t_2) = \frac{2}{m}$

$\Rightarrow$ $y = \frac{2}{m}$,

which is a line parallel to the axis of parabola.

17. d. Any line passing through focus other than axis always meets parabola in two distinct points.

Hence, $m \in R - \{0\}$.

18. c. Since the semi-latus rectum of a parabola is the harmonic mean between the segments of any focal chord of a parabola, therefore SP, 4, SQ are in HP.

$$\Rightarrow \quad 4 = \frac{2SP.SQ}{SP + SQ}$$

$$\Rightarrow \quad 4 = \frac{2(6)(SQ)}{6 + SQ}$$

$$\Rightarrow \quad 24 + 4(SQ) = 12(SQ)$$

$$\Rightarrow \quad SQ = 3$$

19. a. The graph shows $\lambda > 0$.

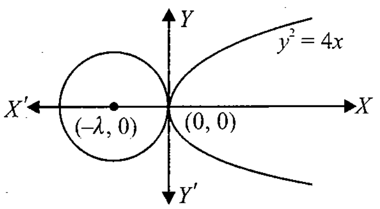


Fig. 3.81

20. c. Let x_1, x_2 and x_3 be the abscissae of the points on the parabola whose ordinates are y_1, y_2 and y_3, respectively. Then $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$. Therefore, the area of the triangle whose vertices are (x_1, y_1) (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

21. d. The given parabolas are

$$y^2 = 4ax \quad \text{(i)}$$

and

$$x^2 = 4ay \quad \text{(ii)}$$

From Eq. (ii),

$$y = \frac{x^2}{4a}$$

Putting in Eq. (i),

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x = 0 \text{ or } x = 4a$$

When $x = 0, y = 0,$

and when $x = 4a, y = \frac{16a^2}{4a} = 4a$

Thus, Eqs. (i) and (ii) meet at (0, 0) and $(4a, 4a)$.

Now $2bx + 3cy + 4d = 0$

passes through $(4a, 4a)$ and (0, 0).

$$\Rightarrow \quad d = 0$$

and

$$2b(4a) + 3c(4a) = 0$$

$$\Rightarrow \quad 2b + 3c = 0$$

$$\Rightarrow \quad d^2 + (2b + 3c)^2 = a^2$$

22. b. Let $R(h, k)$ be the midpoint of PQ. Therefore, Q is $(2h - 1, 2k)$

Since Q lies on $y^2 = 8x$

$$\therefore \quad (2k)^2 = 8(2h - 1)$$

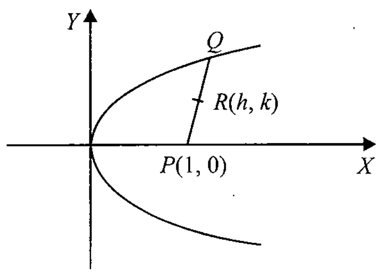


Fig. 3.82

Hence, locus of $Q(h, k)$ is

$$y^2 = 2(2x - 1)$$

or

$$y^2 = 4x - 2$$

$$\Rightarrow \quad y^2 - 4x + 2 = 0$$

23. a. The family of parabolas is $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ and the vertex is $A\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = (h, k)$

$$\Rightarrow \quad h = -\frac{\frac{a^2}{2}}{2\frac{a^3}{3}} = -\frac{3}{4a}$$

and

$$k = \frac{\left(\frac{a^2}{2}\right)^2 - \frac{4a^3(-2a)}{3}}{4\frac{a^3}{3}}$$

$$\Rightarrow \quad h = -\frac{3}{4a} \text{ and } k = -\frac{35a}{16}$$

Eliminating a, we have $hk = 105/64$.

Hence, the required locus is $xy = 105/64$.

24. c. Let $C_1(h, k)$ be the centre of the circle.

Circle touches the x-axis then its radius is $r_1 = k$.

Also circle touches the circle with centre $C_2(0, 3)$ and radius $r_2 = 2$.

$\therefore \quad |C_1 C_2| = r_1 + r_2$

$\Rightarrow \quad \sqrt{(h-0)^2 + (k-3)^2} = |k+2|$

Squaring

$$h^2 - 10k + 5 = 0$$

$\Rightarrow$ Locus is $x^2 - 10y + 5 = 0$, which is parabola.

25. b. Let $P(x, y)$ be the point of contact. At 'P' both of them must have same slope. We have,

$$2y\frac{dy}{dx} = 4a,\ 2x = 4a\frac{dy}{dx}$$

Eliminating $\frac{dy}{dx}$, we get $xy = 4a^2$.

26. a. Let the point be $P(at^2, 2at)$.

Then according to question, $SP = at^2 + a = k$ (i)

Let (α, β) is the moving point, then $\alpha = at^2, \beta = 2at$

$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{t}{2}$

and $\quad a = \frac{\beta^2}{4\alpha}$

($\because$ point (α, β) lies on $y^2 = 4ax$)

On substituting these values in Eq. (i),

$$\frac{\beta^2}{4\alpha}\left(1 + \frac{4\alpha^2}{\beta^2}\right) = k$$

$\Rightarrow \quad \beta^2 + 4\alpha^2 = 4k\alpha$

$\Rightarrow \quad 4x^2 + y^2 - 4kx = 0$ is the required locus.

27. d. $y - \sqrt{3}\,x + 3 = 0$ can be rewritten as

$$\frac{y-0}{\sqrt{3}/2} = \frac{x-\sqrt{3}}{1/2} = r \quad \text{(i)}$$

On solving Eq. (i) with the parabola $y^2 = x + 2$

$$\frac{3r^2}{4} = \frac{r}{2} + \sqrt{3} + 2$$

$\Rightarrow \quad 3r^2 - 2r - (4\sqrt{3} + 8) = 0$

$\Rightarrow \quad AP \cdot AQ = |r_1 r_2|$

$$= \frac{4(\sqrt{3}+2)}{3}$$

(product of roots)

28. a. Let $P(-2 + r\cos\theta, r\sin\theta)$ and Q lies on parabola

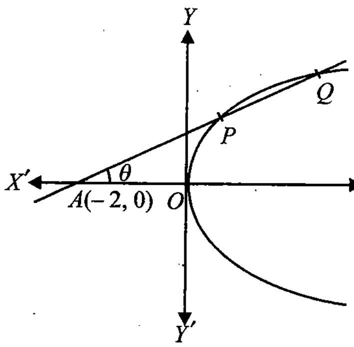


Fig. 3.83

$\Rightarrow \quad r^2\sin^2\theta - 4(-2 + r\cos\theta) = 0$

$\Rightarrow \quad r_1 + r_2 = \frac{4\cos\theta}{\sin^2\theta},\ r_1 r_2 = \frac{8}{\sin^2\theta}$

Now $\quad \frac{1}{AP} + \frac{1}{AQ} = \frac{r_1 + r_2}{r_1 r_2} = \frac{\cos\theta}{2}$

Given that $\quad \frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$

$\Rightarrow \quad \cos\theta < \frac{1}{2}$

$\Rightarrow \quad \tan\theta > \sqrt{3}$

($\because \cos\theta$ is decreasing and $\tan\theta$ is increasing in $(0, \pi/2)$)

$\Rightarrow \quad m > \sqrt{3}$

29. a. The general equation of a parabola having its axis parallel to y-axis is

$$y = ax^2 + bx + c \quad \text{(i)}$$

This is touched by the line $y = x$ at $x = 1$.

Therefore, slope of the tangent at $(1, 1)$ is 1 and, $x = ax^2 + bx + c$ must have equal roots.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 1$ and $(b-1)^2 = 4ac$

$\Rightarrow \quad 2a + b = 1$ and $(b-1)^2 = 4ac$

Also, $(1, 1)$ lies on Eq. (i)

$\Rightarrow \quad a + b + c = 1$

From $\quad 2a + b = 1$ and $a + b + c = 1,\ a - c = 0$

$\Rightarrow \quad a = c$

Then from $\quad a + b + c = 1,\ 2c + b = 1$

$\Rightarrow \quad 2f(0) + f'(0) = 1$

$[\because f(0) = c$ and $f'(0) = b]$

30. b.

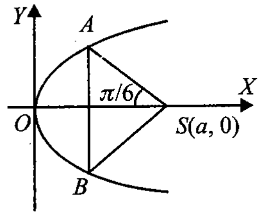


Fig. 3.84

Let $A = (at_1^2, 2at_1)$, $B \equiv (at_2^2, -2at_1)$.

We have

$$m_{AS} = \tan\left(\frac{5\pi}{6}\right)$$

$$\Rightarrow \quad \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad t_1^2 + 2\sqrt{3}t_1 - 1 = 0$$

$$\Rightarrow \quad t_1 = -\sqrt{3} \pm 2$$

Clearly $t_1 = -\sqrt{3} - 2$ is rejected.

Thus, $t_1 = (2 - \sqrt{3})$

Hence, $AB = 4at_1 = 4a(2 - \sqrt{3})$

31. c. $\vec{V} = (T^2 - 1)\hat{i} + 2T\hat{j}$

$$\vec{n} = \hat{j} - \hat{i}$$

Projection of $\vec{V}$ on $\vec{n}$

$$y = \frac{\vec{V}.\vec{n}}{|\vec{n}|} = \frac{(1 - T^2) + 2T}{\sqrt{2}}$$

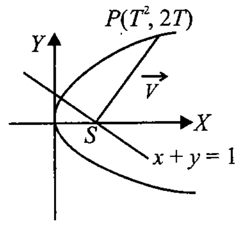


Fig. 3.85

Given $\frac{dx}{dt} = 4$; but $x = T^2$

$$\Rightarrow \quad \frac{dx}{dt} = 2T\frac{dT}{dt}$$

When $P(4, 4)$ then $T = 2$

$$\Rightarrow \quad 4 = 2(2)\frac{dT}{dt}$$

$$\Rightarrow \quad \frac{dT}{dt} = 1$$

Now $\frac{dy}{dt} = \left(\frac{-2T + 2}{\sqrt{2}}\right)\frac{dT}{dt}$

Therefore, at $T = 2$,

$$\sqrt{2}\frac{dy}{dt} = -4 + 2 = -2$$

$$\Rightarrow \quad \frac{dy}{dt} = -\sqrt{2}$$

32. a. Let focus be (a, b).

Equations are

$$S_1 : (x - a)^2 + (y - b)^2 = x^2$$

and $S_2 : (x - a)^2 + (y - b)^2 = y^2$

Common chord $S_1 - S_2 = 0$ gives $x^2 - y^2 = 0$

$$\Rightarrow \quad y = \pm x.$$

33. d. Put $x^2 = \frac{y}{a}$ in circle, $x^2 + (y - 1)^2 = 1$, we get (Note that for $a < 0$ they cannot intersect other than origin) $\frac{y}{a} + y^2 - 2y = 0$. Hence, we get $y = 0$ or $y = 2 - \frac{1}{a}$

Substituting $y = 2 - \frac{1}{a}$ in $y = ax^2$, we get

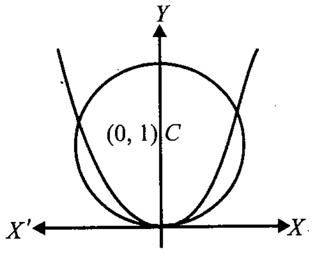


Fig. 3.86

$$ax^2 = 2 - \frac{1}{a}$$

$$\Rightarrow \quad x^2 = \frac{2a - 1}{a^2} > 0$$

$$\Rightarrow \quad a > \frac{1}{2}$$

34. d. Chord through (2, 1) is $\frac{x - 2}{\cos\theta} = \frac{y - 1}{\sin\theta} = r$ (i)

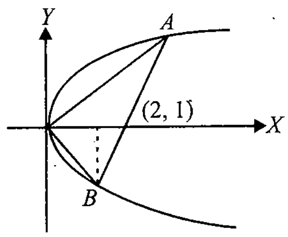


Fig. 3.87

Solving Eq. (i) with parabola $y^2 = x$, we have

$$(1 + r\sin\theta)^2 = 2 + r\cos\theta$$

$$\Rightarrow \sin^2\theta\, r^2 + (2\sin\theta - \cos\theta)r - 1 = 0$$

This equation has two roots $r_1 = AC$ and $r_2 = -BC$

Then, sum of roots $r_1 + r_2 = 0$

$$\Rightarrow \quad 2\sin\theta - \cos\theta = 0 \Rightarrow \tan\theta = \frac{1}{2}$$

$$AB = |r_1 - r_2|$$

$$= \sqrt{(r_1 + r_2)^2 - 4r_1 r_2}$$

$$= \sqrt{4\frac{1}{\sin^2\theta}} = 2\sqrt{5}$$

35. c. Solving circle $x^2 + y^2 = 5$ and parabola $y^2 = 4x$, we have

$$x^2 + 4x - 5 = 0$$

$$\Rightarrow \quad x = 1$$

or $\quad x = -5$ (not possible)

$$\Rightarrow \quad y^2 = 4 \text{ or } y = \pm 2$$

$\Rightarrow$ Points of intersection are $P(1, 2)$; $Q(1, -2)$

Hence, $\quad PQ = 4$

36. c. Difference of the ordinate

$$d = \left|2at + \frac{2a}{t}\right| = 2a\left|t + \frac{1}{t}\right|$$

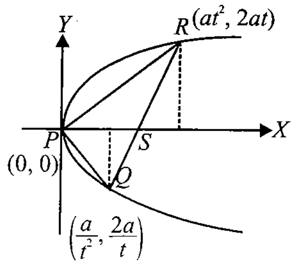


Fig. 3.88

Now area $\quad A = \frac{1}{2}\begin{vmatrix} at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2\left(t + \frac{1}{t}\right)$

$$\Rightarrow \quad 2a\left(t + \frac{1}{t}\right) = \frac{2A}{a}$$

37. a. A_1B_1 is a focal chord, then $A_1(at_1^2, 2at_1)$ and $B_1\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$.

A_2B_2 is a focal chord, then $A_2(at_2^2, 2at_2)$ and $B_2\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

Now equation of chord A_1A_2 is

$$y(t_1 + t_2) - 2x - 2at_1t_2 = 0 \qquad \text{(i)}$$

Chord B_1B_2 is

$$y\left(-\frac{1}{t_1} - \frac{1}{t_2}\right) - 2x - 2a\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right) = 0$$

or $\quad y(t_1 + t_2) + 2x\,t_1t_2 + 2a = 0 \qquad$ (ii)

For their intersection, we subtract them and get

$$2x(t_1t_2 + 1) + 2a(t_1t_2 + 1) = 0$$

or $\quad (x + a)(1 + t_1t_2) = 0$

$$\Rightarrow \quad x + a = 0$$

Hence, they interset on directrix.

38. b. Joint equation of OA and OB is

$$x^2 - 4x(y - 3x) - 4y(y - 3x) + 20(y - 3x)^2 = 0$$

Making equation of parabola homogeneous using straight line.

$$\Rightarrow \quad x^2(1 + 12 + 180) - y^2(4 - 20) - xy(4 - 12 + 120) = 0$$

$$\Rightarrow \quad 193x^2 + 16y^2 - 112xy = 0$$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{56^2 - 193 \times 16}}{193 + 16} = \frac{8\sqrt{3}}{209}$$

39. a. Let the midpoint of PQ be (α, β)

$$\Rightarrow \quad \alpha = x + \frac{c}{2} \text{ and } \beta = y + \frac{c}{2}$$

$$\Rightarrow \quad \left(\beta - \frac{c}{2}\right)^2 = 4a\left(\alpha - \frac{c}{2}\right)$$

$$\Rightarrow \quad \left(y - \frac{c}{2}\right)^2 = 4a\left(x - \frac{c}{2}\right)$$

which is required locus.

40. c. Latus rectum of $y^2 = 2bx$ is $2b$.

Semi-latus rectum is b.

We know that semi-latus rectum is H.M. of segments of focal chord.

Then $\quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

$$\Rightarrow \quad b = \frac{2ac}{a + c}$$

Now for $\quad ax^2 + bx + c = 0$,

$$D = b^2 - 4ac$$

$$= \left(\frac{2ac}{a + c}\right)^2 - 4ac$$

$$= -4ac\left(\frac{a^2 + c^2 - ac}{(a + c)^2}\right) < 0$$

Hence, roots are imaginary.

41. c. $\tan\theta = \frac{y}{x}$

Projection of BC on the x-axis

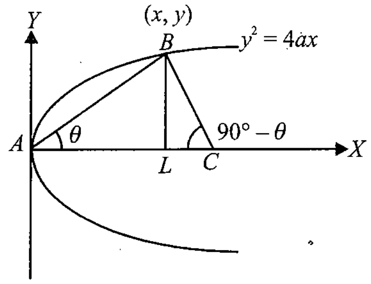


Fig. 3.89

$$LC = \frac{y}{\tan(90° - \theta)} = y\tan\theta$$

$$\frac{y^2}{x} = 4a$$

42. c. $\alpha^2 + 1 - 4 < 0$

$\Rightarrow \quad \alpha^2 < 3, |\alpha| < \sqrt{3}$

$\Rightarrow \quad 1 - 4\alpha < 0$

$\Rightarrow \quad \alpha > \frac{1}{4}$

43. c. $y^2 = 6\left(x - \frac{3}{2}\right)$

Equation of directrix is

$$x - \frac{3}{2} = -\frac{3}{2}, \text{ i.e, } x = 0$$

Let coordinates of P be $\left(\frac{3}{2} + \frac{3}{2}t^2, 3t\right)$

Therefore, coordinate of M are $(0, 3t)$

$\Rightarrow \quad MS = \sqrt{9 + 9t^2}$

$$MP = \frac{3}{2} + \frac{3}{2}t^2$$

$\therefore \quad 9 + 9t^2 = \left(\frac{3}{2} + \frac{3}{2}t^2\right)^2 = \frac{9}{4}(1 + t^2)^2$

$\therefore \quad 4 = 1 + t^2$

$\therefore \quad$ Length of side $= 6$

44. d. Equation of tangent to given parabola having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or $\quad y = mx + am + \frac{a}{m}$

Comparing Eq. (i) with $y = mx + c$, we have

$$c = am + \frac{a}{m}$$

45. d. The coordinates of the focus of the parabola $y^2 = 4ax$ are $(a, 0)$. The line $y - x - a = 0$ pass through this point. Therefore, it is a focal chord of the parabola. Hence, the tangent intersect at right angle.

46. c.

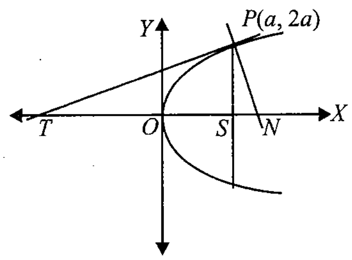


Fig. 3.90

One end of the latus rectum, $P(a, 2a)$.

The equation of the tangent PT at $P(a, 2a)$ is

$$2ya = 2a(x + a), \text{ i.e., } y = x + a$$

The equation of normal PN at $(a, 2a)$ is

$$y + x = 2a + a, \text{ i.e., } y + x = 3a$$

Solving $\quad y = 0$ and $y = x + a$, we get

$$x = -a, y = 0.$$

Solving $\quad y = 0, y + x = 3a$, we get

$$x = 3a, y = 0.$$

The area of the triangle with vertices $P(a, 2a)$, $T(-a, 0)$, $N(3a, 0)$ is $4a^2$.

47. a. Let $A \equiv (at^2, 2at)$, $B \equiv (at^2, -2at)$.

$$m_{OA} = \frac{2}{t}, m_{OB} = \frac{-2}{t}$$

Thus, $\quad \left(\frac{2}{t}\right)\left(\frac{-2}{t}\right) = -1$

$\Rightarrow \quad t^2 = 4.$

Thus, tangents will intersect at $(-4a, 0)$.

48. b. Clearly P is the point of intersection of two perpendicular tangents to the parabola $y^2 = 8x$.

Hence, P must lie on the directrix $x + a = 0$ or $x + 2 = 0$

$\therefore \quad x = -2.$

Hence, the point is $(-2, 0)$.

49. c.

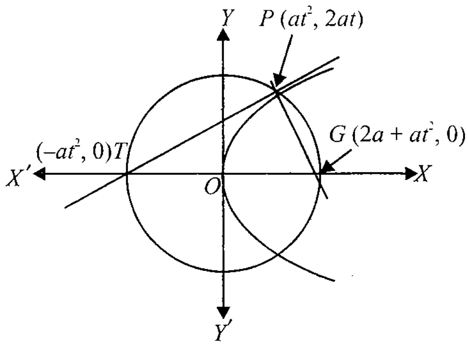


Fig. 3.91

Tangent and normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$ty = x + at^2 \quad \text{(i)}$$

and $$y = -tx + 2at + at^3 \quad \text{(ii)}$$

Equations (i) and (ii) meet the x-axis where $y = 0$

From Eq. (i), $x = -at^2$

$\Rightarrow T$ is $(-at^2, 0)$

From Eq. (ii), $tx = 2at + at^3$

$\Rightarrow G$ is $(2a + at^2, 0)$

Midpoint of $TG = \left(\frac{2a + at^2 - at^2}{2}, 0\right)$

$= O(a, 0)$

Since $\angle TPG = 90°$, therefore centre of the circle through PTG is $(a, 0)$.

If θ is the angle between tangents at P to the parabola and circle through P, T, G, then $(90° - \theta)$ is the angle between PT and OP.

Slope of $PT = \frac{2at}{2at^2} = \frac{1}{t}$

Slope of $OP = \frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$

$\therefore \quad \tan(90° - \theta) = \left|\frac{\frac{1}{t} - \frac{2t}{t^2-1}}{1 + \frac{1}{t}\left(\frac{2t}{t^2-1}\right)}\right| = \frac{1}{t}$

$\therefore \quad \cot\theta = \frac{1}{t} \Rightarrow \tan\theta = t$

$\Rightarrow \quad \theta = \tan^{-1}(t)$

50. a. $\frac{dy}{dx} = 2x - 5$

$\therefore \quad m_1 = \left(\frac{dy}{dx}\right)_{(2,0)} = 4 - 5 = -1$ and

$m_2 = \left(\frac{dy}{dx}\right)_{(3,0)} = 6 - 5 = 1$

$\Rightarrow m_1 m_2 = -1 \Rightarrow$ angle between tangents $= \frac{\pi}{2}$

51. a.

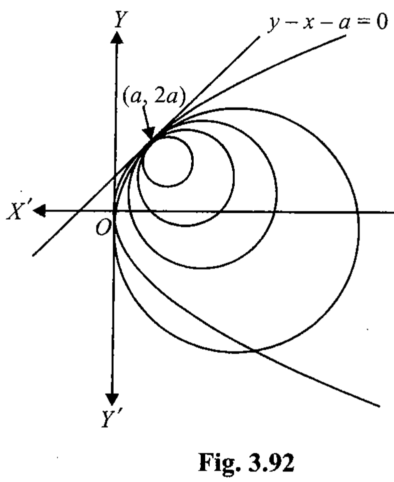


Fig. 3.92

Equation of tangent of parabola at $(a, 2a)$ is $2ya = 2a(x + a)$, i.e., $y - x - a = 0$.

Equation of circle touching the parabola at $(a, 2a)$ is

$(x - a)^2 + (y - 2a)^2 + \lambda(y - x - a) = 0$

It passes through $(0, 0)$

$\Rightarrow \quad a^2 + 4a^2 + \lambda(-a) = 0 \Rightarrow \lambda = 5a$

Thus, required circle is $x^2 + y^2 - 7ax - ay = 0$

it's radius is $\sqrt{\frac{49}{4}a^2 + \frac{a^2}{4}} = \frac{5}{\sqrt{2}}a$

52. c. The required point is obtained by solving $x + y = 1$ and $y^2 - y + x = 0$.

53. c. Any tangent to $y^2 = 4a(x + a)$ is

$$y = m(x + a) + \frac{a}{m} \quad \text{(i)}$$

Any tangent to $y^2 = 4b(x + b)$ which is perpendicular to Eq. (i) is

$$y = -\frac{1}{m}x + (x + b) - bm \quad \text{(ii)}$$

Subtracting, we get

$$\left(m + \frac{1}{m}\right)x + (a + b)\left(m + \frac{1}{m}\right) = 0$$

or $x + a + b = 0$ which is a locus of their point of intersection.

54. c. Any point on the given parabola is $(t^2, 2t)$.

The equation of the tangent at $(1, 2)$ is $x - y + 1 = 0$

The image (h, k) of the point $(t^2, 2t)$ in $x - y + 1 = 0$ is given by

$$\frac{h - t^2}{1} = \frac{k - 2t}{-1} = -\frac{2(t^2 - 2t + 1)}{1 + 1}$$

$\therefore \quad h = t^2 - t^2 + 2t - 1 = 2t - 1$

and $\quad k = 2t + t^2 - 2t + 1 = t^2 + 1$

Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$, we get

$$(h + 1)^2 = 4(k - 1)$$

The required equation of reflection is $(x + 1)^2 = 4(y - 1)$.

55. a.

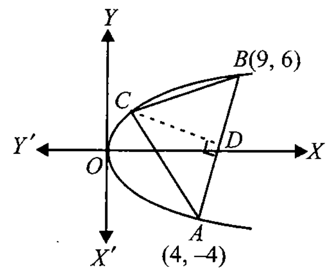


Fig. 3.93

Area of triangle ABC is maximum if CD is maximum, because AB is fixed.

That means tangent drawn to parabola at 'C' should be parallel to AB.

Slope of $\qquad AB = \frac{6+4}{9-4} = 2$

For $\qquad y^2 = 4x, \frac{dy}{dx} = \frac{2}{y} = 2$

$\Rightarrow \qquad y = 1$

$\Rightarrow \qquad x = \frac{1}{4}$

56. b. Let $y_1 = 5 + \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4x_2}$ or

$x_1^2 + (y_1 - 5)^2 = 1$ and $y_2^2 = 4x_2$

Thus (x_1, y_1) lies on the circle $x^2 + (y - 5)^2 = 1$

and (x_2, y_2) lies on the parabola $y^2 = 4x$.

Thus, given expression is the shortest distance between the curves $x^2 + (y - 5)^2 = 1$ and $y^2 = 4x$.

Now the shortest distance always occur along common normal to the curves and normal to circle passes through the centre of the circle.

Normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ passes through (0, 5) gives

$m^3 + 2m + 5 = 0$, which has only one root $m = -2$.

Hence, corresponding point on the parabola is (4, 4).

Thus, required minimum distance $= \sqrt{4^2 + 8^2} - 1 = 4\sqrt{5} - 1$.

57. c. $SP_1 = a(1 + t_1^2)$; $SP_2 = a(1 + t_2^2)$

$\Rightarrow \qquad t_1 t_2 = -1$

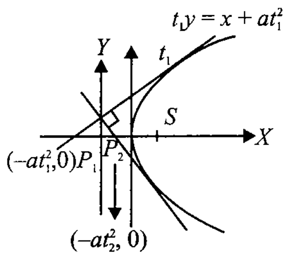


Fig. 3.94

$$\frac{1}{SP_1} = \frac{1}{a(1 + t_1^2)}$$

$$\frac{1}{SP_2} = \frac{1}{a(1 + t_2^2)} = \frac{t_1^2}{a(t_1^2 + 1)}$$

$$\therefore \qquad \frac{1}{SP_1} + \frac{1}{SP_2} = \frac{1}{a}$$

58. b. Tangent at point P is $ty = x + t^2$, where slope of tangent is $\tan\theta = \frac{1}{t}$.

Now required area is $A = \frac{1}{2}(AN)(PN) = \frac{1}{2}(2t^2)(2t)$

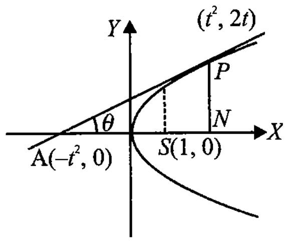


Fig. 3.95

$$A = 2t^3 = 2(t^2)^{3/2}$$

Now $t^2 \in [1, 4]$, then A_{max} occurs when $t^2 = 4$

$\Rightarrow \qquad A_{max} = 16$

59. d. $OT^2 = OA \cdot OB = \alpha\beta = \frac{c}{a} \Rightarrow OT = \sqrt{\frac{c}{a}}$

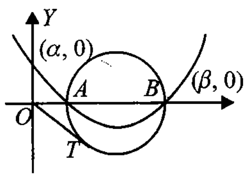


Fig. 3.96

60. b. Tangent at point P is $ty = x + at^2$. (i)

Line perpendicular to Eq. (i) passes through $(a, 0)$

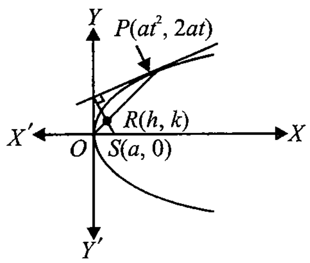


Fig. 3.97

$\therefore\ y - 0 = -t(x - a)$ or $tx + y = ta$ or $y = t(a - x)$ (ii)

Equation of OP

$y - \frac{2}{t}x = 0$ or $y = \frac{2}{t}x$ (iii)

From Eqs. (ii) and (iii), eliminating t, we get

$$y^2 = 2x(a - x)$$

or $\qquad 2x^2 + y^2 - 2ax = 0$

61. a. Slope of tangent at P is $\frac{1}{t_1}$ and at Q is $\frac{1}{t_2}$.

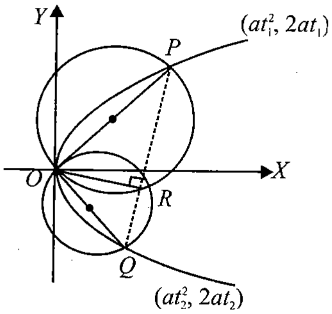


Fig. 3.98

$\Rightarrow \quad \cot\theta_1 = t_1$ and $\cot\theta_2 = t_2$

Slope of $\quad PQ = \dfrac{2}{t_1 + t_2}$

$\Rightarrow$ Slope of OR is $-\dfrac{t_1 + t_2}{2} = \tan\phi$

(Note angle in a semicircle is 90°)

$\Rightarrow \quad \tan\phi = -\frac{1}{2}(\cot\theta_1 + \cot\theta_2)$

$\Rightarrow \quad \cot\theta_1 + \cot\theta_2 = -2\tan\phi$

62. b.

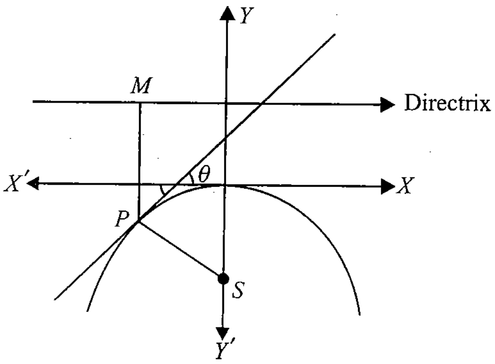


Fig. 3.99

Slope of line $\quad \lambda = \tan\theta$

$\Rightarrow \tan(\angle MPS) = \tan 2\left(\frac{\pi}{2} - \theta\right) = \tan(\pi - 2\theta) = -\tan 2\theta$

$= \dfrac{2\lambda}{\lambda^2 - 1}$

63. c. Solving $y = 2x - 3$ and $y^2 = 4a\left(x - \frac{1}{3}\right)$, we have

$$(2x-3)^2 = 4a\left(x - \frac{1}{3}\right)$$

$$\Rightarrow \quad 4x^2 + 9 - 12x = 4ax - \frac{4a}{3}$$

$$\Rightarrow 4x^2 - 4(3+a)x + 9 + \frac{4a}{3} = 0$$

This equation must have equal roots $\Rightarrow D = 0$

$$\Rightarrow 16(3+a)^2 - 16\left(9 + \frac{4a}{3}\right) = 0$$

$$\Rightarrow \quad 9 + a^2 + 6a = 9 + \frac{4a}{3}$$

$$\Rightarrow \quad a^2 + \frac{14a}{3} = 0$$

$$\Rightarrow \quad a = 0 \text{ or } a = \frac{14}{3}$$

64. c.

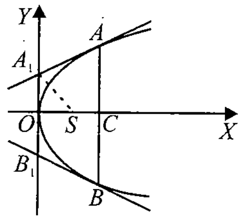


Fig. 3.100

Let $\quad A \equiv (at_1^2, 2at_1)$,

$B \equiv (at_1^2, -2at_1)$

Equation of tangents at A and B are,

$t_1 y = x + at_1^2$

and $\quad -t_1 y = x + at_1^2$, respectively.

These tangents meet y-axis at

$A_1 \equiv (0, at_1)$

and $\quad B_1 \equiv (0, -at_1)$.

Area of trapezium $\quad AA_1B_1B = \frac{1}{2}(AB + A_1B_1) \times OC$

$$\Rightarrow \quad 24a^2 = \frac{1}{2}(4at_1 + 2at_1)(at_1^2)$$

$$\Rightarrow \quad t_1^3 = 8 \Rightarrow t_1 = 2$$

$$\Rightarrow \quad A_1 \equiv (0, 2a)$$

If $\quad \angle OSA_1 = \theta \Rightarrow \tan\theta = \frac{2a}{a} = 2$

$$\Rightarrow \quad \theta = \tan^{-1}(2)$$

Thus, required angle is $2\tan^{-1}(2)$.

65. b. Here $a = 2$ for parabola and the two tangents pass through the points $(-2, -3)$, which lie on the directrix, then tangents are perpendicular or $m_1 m_2 = -1$.

66. b.

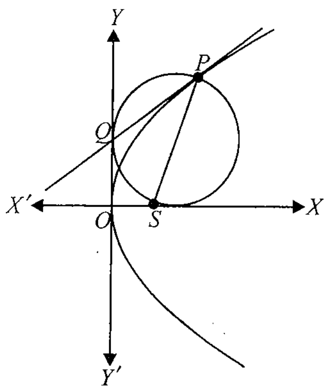


Fig. 3.101

Tangent at P intersects y-axis at $Q \equiv (0, at)$.

Also circle with PS as diameter touches the y-axis at $(0, at)$.

$\Rightarrow$ y-axis is the tangent to circumcircle of ΔPQS at Q.

67. d. Tangents $y = m_1x + c$ and $y = m_2x + c$ intersect at $(0, c)$ which lies on the directrix of the given parabola.

Hence, tangents are perpendicular for which, $m_1m_2 = -1$.

68. b. $y = ax^2 - 6x + b$ passes through $(0, 2)$

Here, $2 = a(0^2) - 6(0) + b$

$\therefore$ $b = 2$

Also, $\dfrac{dy}{dx} = 2ax - 6$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{x=\frac{3}{2}} = 2a\left(\dfrac{3}{2}\right) - 6$

$= 3a - 6 = 0$

$\therefore$ $a = 2$

69. b. Since tangents are perpendicular, they intersect on the directrix.

$\Rightarrow$ $(\lambda, 1)$ lies on the line $x = -4$

$\Rightarrow$ $\lambda = -4$

70. b. Parabolas $y = x^2 + 1$ and $x = y^2 + 1$ are symmetrical about $y = x$.

Therefore, tangent at point A is parallel to $y = x$.

$\Rightarrow$ $\dfrac{dy}{dx} = 2x \Rightarrow 2x = 1$

$\Rightarrow$ $x = \dfrac{1}{2}$ and $y = \dfrac{5}{4}$

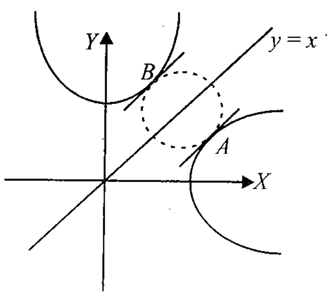


Fig. 3.102

$A\left(\dfrac{1}{2}, \dfrac{5}{4}\right)$ and $B\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$

Therefore, radius $= \dfrac{1}{2}\sqrt{\left(\dfrac{1}{2} - \dfrac{5}{4}\right)^2 + \left(\dfrac{5}{4} - \dfrac{1}{2}\right)^2} = \dfrac{1}{2}\sqrt{\dfrac{9}{16} + \dfrac{9}{16}}$

$= \dfrac{3}{8}\sqrt{2}$

Therefore, area $= \dfrac{9\pi}{32}$

71. b.

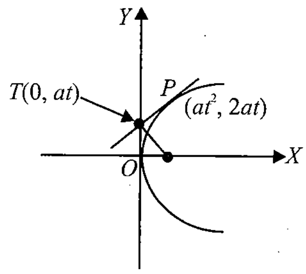


Fig. 3.103

Let middle point of P and T be (h, k)

$\therefore$ $2h = at^2$

and $2k = 3at$

$\therefore$ $2h = a \cdot \dfrac{4k^2}{9a^2}$

Locus of (h, k) is $2y^2 = 9ax$

As $a = 2 \therefore y^2 = 9x$

72. c. Tangent to parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

Tangent to circle $(x - 1)^2 + (y + 2)^2 = 16$ having slope m is

$$(y + 2) = m(x - 1) + 4\sqrt{1 + m^2}$$

Distance between tangents

$$= \left|\frac{4\sqrt{1 + m^2} - m - 2 - 1/m}{\sqrt{1 + m^2}}\right|$$

$$= \left|4 - \frac{2}{\sqrt{1 + m^2}} - \frac{\sqrt{m^2 + 1}}{m}\right|$$

As $m > 0 \Rightarrow d < 4$

73. d.

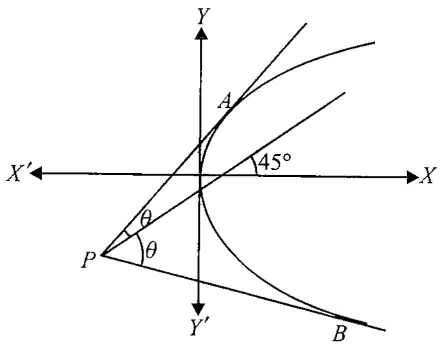


Fig. 3.104

Here $\dfrac{1}{t_1} = \tan\left(\dfrac{\pi}{4} + \theta\right)$

and $\frac{1}{t_2} = \tan\left(\frac{\pi}{4} - \theta\right)$

So, $t_1 t_2 = 1$

$\Rightarrow$ the x-coordinate of $P = at_1t_2 = a$.

74. a.

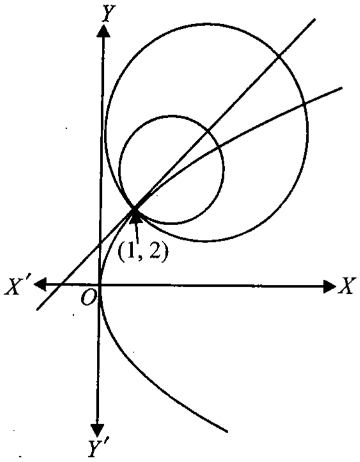


Fig. 3.105

Tangent to parabola $y^2 = 4x$ at (1, 2) will be the locus

i.e., $2y = 2(x + 1)$

$\Rightarrow$ $y = x + 1$

75. c.

Circle S_2, taking focal chord AB as diameter will touch directrix at point P and circle S_1, taking AP as diameter will pass through focus S (since AP subtends angle 90° at focus of parabola).

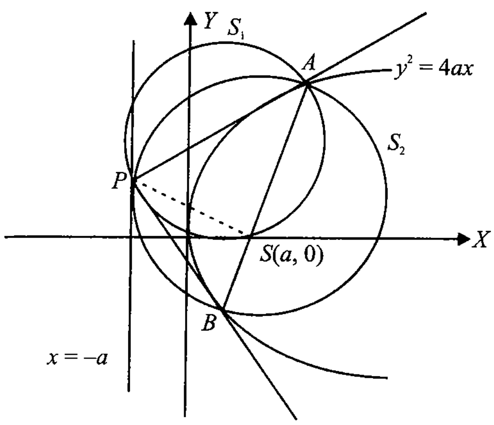


Fig. 3.106

Hence, common chord of given circles is line AP (which is intercept of tangent at point 'A' between point A and directrix).

76. a. Since the normal at $(ap^2, 2ap)$ to $y^2 = 4ax$ meets the parabola at $(aq^2, 2aq)$,

$\therefore$ $$q = -p - \frac{2}{p} \quad \text{(i)}$$

Since $OP \perp OQ$,

$\therefore$ $$\frac{2ap - 0}{ap^2 - 0} \times \frac{2aq - 0}{aq^2 - 0} = -1 \Rightarrow pq = -4.$$

$\Rightarrow$ $$p\left(-p - \frac{2}{p}\right) = -4 \quad \text{[Using (i)]}$$

$\Rightarrow$ $$p^2 = 2$$

77. d. Given parabola is $y^2 = 4x + 8$ or $y^2 = 4(x + 2)$.

Equation of normal to parabola at any point $P(t)$ is $y = -t(x + 2) + 2t + t^3$.

It passes through $(k, 0)$ if $tk = t^3 \Rightarrow t(t^2 - k) = 0$

Hence, it has three real values of t if $k > 0$.

78. c. Equations of tangent and normal at $P(at^2, 2at)$ are $ty = x + at^2$ and $y = -tx + 2at + at^3$, respectively.

Thus, $T \equiv (-at^2, 0)$,

$N \equiv (2a + at^2, 0)$

Also, $S \equiv (a, 0)$

Hence, $SP = a + at^2$, $ST = a + at^2$

and $SN = a + at^2$

Thus, $SP = ST = SN$

79. b. Let AB be a normal chord where $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$. If it's midpoint is $P(h, k)$, then

$$2h = a\left(t_1^2 + t_2^2\right)$$

$$= a\left[(t_1 + t_2)^2 - 2t_1t_2\right]$$

and $$2k = 2a(t_1 + t_2)$$

We also have

$$t_2 = -t_1 - \frac{2}{t_1}$$

$\Rightarrow$ $$t_1 + t_2 = \frac{-2}{t_1} \text{ and } t_1t_2 = -t_1^2 - 2$$

$\Rightarrow$ $$t_1 = -\frac{2a}{k} \text{ and } h = a\left(t_1^2 + 2 + \frac{2}{t_1^2}\right)$$

Thus, required locus is $x = a\left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2}\right)$.

80. c.

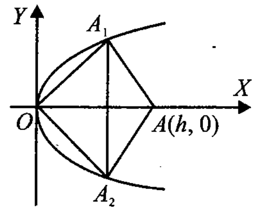


Fig. 3.107

Let $\quad A_1 \equiv (2at_1^2, 4t_1), A_2 \equiv (2t_1^2, -4t_1)$

Clearly, $\quad \angle A_1OA = \dfrac{\pi}{6}$

$\Rightarrow \quad \dfrac{2}{t_1} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad t_1 = 2\sqrt{3}$

Equation of normal at A_1 is $y = -t_1x + 4t_1 + 2t_1^3$

$\Rightarrow \quad h = 4 + 2t_1^2 = 4 + 2\,(12) = 28$

81. c. $y = mx + c$ is a normal to $y^2 = 4ax$ if $c = -2am - am^3$, $y = -2x - \lambda$

$\Rightarrow \quad m = -2, a = -2$

$\Rightarrow -\lambda = -2am - am^3 = -2(-2)(-2) - (-2)(-2)^3 = -24$

$\Rightarrow \quad \lambda = 24$

82. d. We have $a = 1$

Normal at $(m^2, -2m)$ is $y = mx - 2m - m^3$

Given that normal makes equal angle with axes, then its slope $m = \pm 1$

Therefore, point P is $(m^2, -2m) = (1, \pm 2)$.

83. c. Let $A = (\alpha, \beta)$

The normal at $(at^2, 2at)$ is $y = -tx + 2at + at^3$

$\therefore \quad at^3 + (2a - \alpha)t - \beta = 0 \quad$ (i)

Let t_1, t_2, t_3 be roots of Eq. (i), then

$at^3 + (2a - \alpha)t - \beta = a(t - t_1)(t - t_2)(t - t_3) \quad$ (ii)

Let $P = (at_1^2, 2at_1)$, $Q = (at_2^2, 2at_2)$, and $R = (at_3^2, 2at_3)$.

Since the focus S is $(a, 0)$

$\therefore \quad SP = a(t_1^2 + 1)$

Similarly, $\quad SQ = a(t_2^2 + 1)$,

and $\quad SR = a(t_3^2 + 1)$

Put $\quad t = i = \sqrt{-1}$ in Eq. (ii), we have

$-ai + (2a - \alpha)\,i - \beta = a(i - t_1)(i - t_2)(i - t_3)$

$\Rightarrow \quad |(a - \alpha)\,i - \beta| = a|(i - t_1)(i - t_2)(i - t_3)|$

$\Rightarrow \quad \sqrt{(a - \alpha)^2 + \beta^2} = a\sqrt{1 + t_1^2}\,\sqrt{1 + t_2^2}\,\sqrt{1 + t_3^2}$

$\Rightarrow \quad a\sqrt{(a - \alpha)^2 + \beta^2} = \sqrt{a + at_1^2}\,\sqrt{a + at_2^2}\,\sqrt{a + at_3^2}$

$\Rightarrow \quad aSA^2 = SP \cdot SQ \cdot SR$

84. d. Ends of latus rectum are $P(a, 2a)$ and $P'(a, -2a)$.

Point P has parameter $t_1 = 1$ and point P' has parameter $t_2 = -1$.

Normal at point P meets the curve again at point Q whose parameter $t_1' = -t_1 - \dfrac{2}{t_1} = -3$.

Normal at point P' meets the curve again at point Q' whose parameter $t_2' = -t_2 - \dfrac{2}{t_2} = 3$.

Hence, point Q and Q' have coordinates $(9a, -6a)$ and $(9a, 6a)$, respectively.

Hence, $\quad QQ' = 12a$

85. d. Point $(\sin\theta, \cos\theta)$ lies on the circle $x^2 + y^2 = 1$ for $\forall\, \theta \in R$.

Now three normals can be drawn to the parabola $y^2 = 4ax$ if $x = |2a|$ meets this circle.

Hence, we must have $\cos\theta > |2a|$.

$\Rightarrow 0 < |2a| < 1$

$\Rightarrow 0 < |a| < \dfrac{1}{2}$

$\Rightarrow a \in \left(-\dfrac{1}{2}, 0\right) \cup \left(0, \dfrac{1}{2}\right)$

86. c. Normal at point $P(t_1)$ meets the parabola again at point $R(t_3)$, then

$$t_3 = -t_1 - \frac{2}{t_1}$$

Also normal at point $Q(t_2)$ meets the parabola at the same point $R(t_3)$, then

$$t_3 = -t_2 - \frac{2}{t_2}$$

Comparing these values of t_3, we have

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \text{ or } t_1t_2 = 2$$

87. b. Normal at point $(t^2, 2t)$ is $y = -tx + 2t + t^3$

Slope of the tangent is 1.

Hence, $\quad -t = 1 \Rightarrow t = -1$

$\Rightarrow$ Coordinates of P are $(1, -2)$.

Hence, parameter at Q is $\quad t_2 = -t_1 - 2/t_1 = 1 + 2$
$= 3$

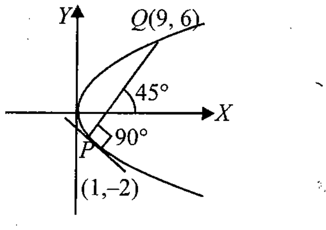


Fig. 3.108

Therefore, coordinates at Q are (9, 6).

$\therefore \quad l(PQ) = \sqrt{64+64} = 8\sqrt{2}$

88. b. $\tan\alpha = -t_1$ and $\tan\beta = -t_2$

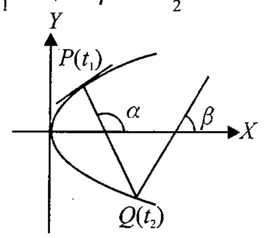


Fig. 3.109

also $\quad t_2 = -t_1 - \frac{2}{t_1}$

$$t_1t_2 + t_1^2 = -2$$

$$\tan\alpha\tan\beta + \tan^2\alpha = -2$$

89. b. Slope of normal at point $P(t_1)$ and $Q(t_2)$ is $-t_1$ and $-t_2$, respectively.

Equation of chord joining $P(t_1)$ and $Q(t_2)$ is

$$y - 2at_1 = \frac{2}{t_1+t_2}(x - at_1^2)$$

or $\quad 2x - (t_1+t_2)y + 2at_1t_2 = 0$

But $\quad t_1t_2 = -1$

Chord PQ is $2x - (t_1+t_2)y - 2a = 0$

or $\quad (2x - 2a) - (t_1+t_2)y = 0$

which passes through the fixed point $(a, 0)$.

90. c. $t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_1t_2 = -t_1^2 - 2$

Equation of the line through P parallel to AQ

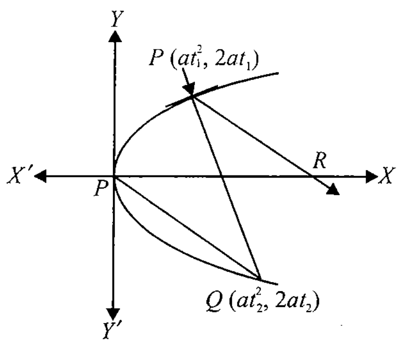


Fig. 3.110

$$y - 2at_1 = \frac{2}{t_2}(x - at_1^2)$$

Put $\quad y = 0 \Rightarrow x = at_1^2 - at_1t_2$

$$= at_1^2 - a(-2 - t_1^2)$$

$$= 2a + 2at_1^2$$

$$= 2(a + at_1^2)$$

= twice the focal distance of P.

91. b. Equations of tangent and normal at A are $yt = x + at^2$ and $y = -tx + 2at + at^3$

$\Rightarrow B \equiv (-at^2, 0), D \equiv (2a + at^2, 0)$. If $ABCD$ is a rectangle, then midpoints of BD and AC will be coincident.

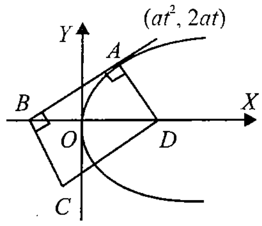


Fig. 3.111

$\Rightarrow \quad h + at^2 = 2a + at^2 - at^2, k + 2at = 0$

$\Rightarrow \quad h = 2a, t = -\frac{k}{2a}$

92. d. $4y = x^2 - 8$

$$4\frac{dy}{dx} = 2x$$

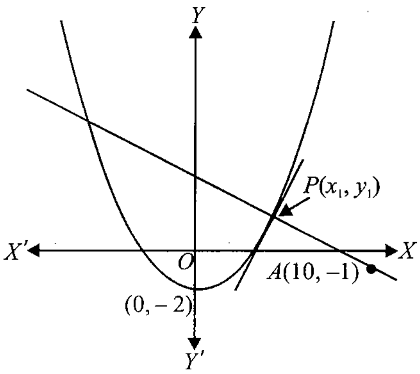


Fig. 3. 112

Therefore, slope of normal $= -\frac{2}{x_1}$; but slope of normal

$$= \frac{y_1+1}{x_1-10}$$

$\therefore \quad \frac{y_1+1}{x_1-10} = -\frac{2}{x_1}$

$\Rightarrow \quad x_1y_1 + x_1 = -2x_1 + 20$

$\Rightarrow \quad x_1y_1 + 3x_1 = 20$

Substituting $y_1 = \frac{x_1^2 - 8}{4}$

(from the given equation)

$$x_1\left(\frac{x_1^2-8}{4}+3\right) = 20$$

$\Rightarrow$ $x_1(x_1^2+4) = 80$

$\Rightarrow$ $x_1^3 + 4x_1 - 80 = 0,$

which has one root $x_1 = 4$

Hence, $x_1 = 4; y_1 = 2$

$\therefore$ $P = (4, 2)$

Therefore, equation of PA is

$$y + 1 = -\frac{1}{2}(x - 10)$$

$\Rightarrow$ $2y + 2 = -x + 10$

$\Rightarrow$ $x + 2y - 8 = 0$

93. a. A circle through three co-normal points of a parabola always passes through the vertex of the parabola. Hence, the circle through P, Q, R, S out of which P, Q, R are co-normals points will always pass through vertex (2, 3) of parabola.

94. c. Normal at point $P(x_1, y_1) \equiv (at_1^2, 2at_1)$ meets the parabola at $R(at^2, 2at)$

$\Rightarrow$ $t = -t_1 - \frac{2}{t_1}$ (i)

Normal at point $Q(x_2, y_2) \equiv (at_2^2, 2at_2)$ meets the parabola at $R(at^2, 2at)$

$\Rightarrow$ $t = -t_2 - \frac{2}{t_2}$ (ii)

From Eqs. (i) and (ii)

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$\Rightarrow$ $t_1t_2 = 2$

Now given that $x_1 + x_2 = 4$

$\Rightarrow$ $t_1^2 + t_2^2 = 4$

$\Rightarrow$ $(t_1 + t_2)^2 = 4 + 4 = 8$

$\Rightarrow$ $|t_1 + t_2| = 2\sqrt{2}$

$\Rightarrow$ $|y_1 + y_2| = 4\sqrt{2}$

95. b. Let the concyclic points be t_1, t_2, t_3 and t_4

$\Rightarrow$ $t_1 + t_2 + t_3 + t_4 = 0$

Here, t_1 and t_3 are feet of the normals.

$\Rightarrow$ $t_2 = -t_1 - \frac{2}{t_1}$ and $t_4 = -t_3 - \frac{2}{t_3}$

$\Rightarrow$ $t_1 + t_2 = -\frac{2}{t_1}$ and $t_4 + t_3 = -\frac{2}{t_3}$

Adding,

$$-2\left(\frac{1}{t_1} + \frac{1}{t_3}\right) = 0$$

$\Rightarrow$ $t_1 + t_3 = 0$

$\Rightarrow$ Point of intersection of tangents at t_1 and t_3 $(at_1t_3, a(t_1 + t_3)) \equiv (at_1t_3, 0)$.

$\Rightarrow$ This point lies on the axis of the parabola.

96. c. Axis of the parabola is $x = 1$. Any point on it is $(1, k)$. Now distance of $(1, k)$ from $(1, -2)$ should be more than the semi-latus rectum and $(1, k)$ should be inside the parabola, hence $k > 2$.

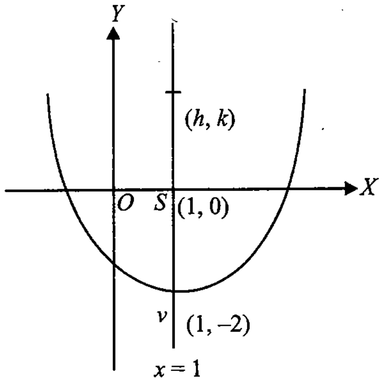


Fig. 3.113

97. b.

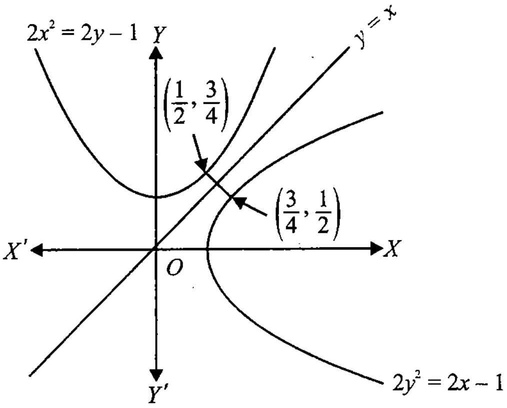


Fig. 3.114

Given parabolas $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ are symmetrical about the line $y = x$.

Also shortest distance occurs along the common normal which perpendicular to the line $y = x$.

Differentiating $2y^2 = 2x - 1$ w.r.t. x,

$$2y\frac{dy}{dx} = 1$$

we have

$$\frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$$

Hence, points are as shown in the figure.

Then, the shortest distance, $d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$

98. d.

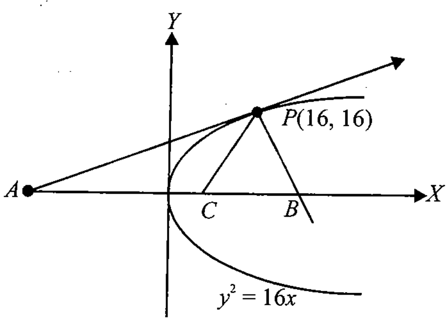


Fig. 3.115

By property centre of circle coincides with focus of parabola

$\Rightarrow$ $C \equiv (4, 0)$

$$\tan\alpha = \text{slope of } PC = \frac{16}{12}$$

$\Rightarrow$ $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

99. a. Let AB be a normal chord where $A \equiv (at^2, 2at)$, $B \equiv (at_1^2, 2at_1)$.

We have $t_1 = -t - \frac{2}{t}$.

Now,
$$AB^2 = [a^2(t^2 - t_1^2)]^2 + 4a^2(t - t_1)^2$$
$$= a^2(t - t_1)^2 [(t + t_1)^2 + 4]$$
$$= a^2\left(t + t + \frac{2}{t}\right)^2 \left(\frac{4}{t^2} + 4\right)$$
$$= \frac{16a^2(1 + t^2)^3}{t^4}$$

$$\Rightarrow \frac{d(AB)^2}{dt} = 16a^2\left(\frac{t^4[3(1 + t^2)^2 . 2t] - (1 + t^2)^3 . 4t^3}{t^8}\right)$$

$$= 32a^2(1 + t^2)^2\left(\frac{3t^2 - 2 - 2t^2}{t^5}\right)$$

$$= \frac{a^2 \times 32(1 + t^2)^2}{t^5}(t^2 - 2)$$

For $\frac{d(AB^2)}{dt} = 0 \Rightarrow t = \sqrt{2}$ for which AB^2 is minimum.

Thus, $AB_{min} = \frac{4a}{2}(1 + 2)^{3/2} = 2a\sqrt{27}$ units

100. c. The equation of any normal be $y = -tx + 2t + t^3$

Since it passes through the points (15, 12)

$\therefore$ $12 = -15t + 2t + t^3$

$\Rightarrow$ $t^3 - 13t - 12 = 0$

One root is -1, then

$$(t + 1)(t^2 + t - 12) = 0$$

$\Rightarrow$ $t = 1, 3, 4$

Therefore, the co-normal points are $(1, -2)$, $(9, -6)$, $(16, 8)$.

Therefore, centroid is $\left(\frac{26}{3}, 0\right)$.

101. d. For a focal chord $t_1 t_2 = -1$ and for the normal $t_1(t_1 + t_2) + 2 = 0$.

$\therefore$ $t_1^2 + t_1 t_2 + 2 = 0 \Rightarrow t_1^2 = -1$

Therefore, t_1 is imaginary.

102. b. Solving the line $y = x - 1$ and parabola $y^2 = 4x$, we have

$$(x - 1)^2 = x$$

$\Rightarrow$ $x^2 - 6x + 1 = 0$

$\Rightarrow$ $x = 3 \pm \sqrt{8}$

$\therefore$ $y = 2 \pm \sqrt{8}$

Suppose point D is (x_3, y_3), then

$$y_1 + y_2 + y_3 = 0$$

$\Rightarrow$ $2 + \sqrt{8} + 2 - \sqrt{8} + y_3 = 0$

$\Rightarrow$ $y_3 - 4$, then $x_3 = 4$

Therefore, the point is (4, 4).

103. c. Equation of normal $y = mx - 2am - am^3$

Put $y = 0$, we get

$$x_1 = 2a + am_1^2$$
$$x_2 = 2a + am_2^2$$
$$x_3 = 2a + am_3^2$$

where x_1, x_2, x_3 are the intercepts on the axis of the parabola.

The normal passes through (h, k).

$\Rightarrow$ $am^3 + (2a - h)m + k = 0$,

which has roots m_1, m_2, m_3 which are slopes of the normals.

$\Rightarrow$ $m_1 + m_2 + m_3 = 0$

and $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$

$\Rightarrow$ $m_1^2 + m_2^2 + m_3^2 = (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)$

$$= -\frac{2(2a - h)}{a}$$

$\Rightarrow$ $x_1 + x_2 + x_3 = 6a - 2(2a - h) = 2(h + a)$

104.c.

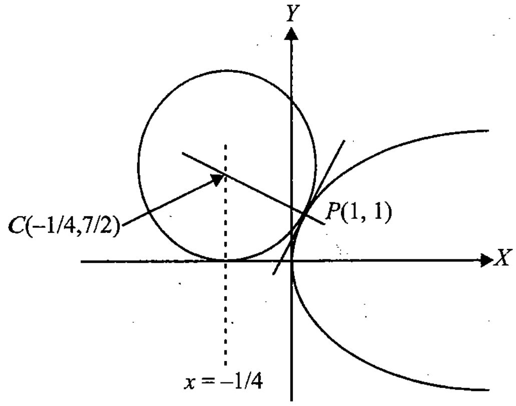


Fig. 3.116

Equation of normal at $P(1, 1)$ is

$$y - 1 = -2(x - 1)$$

or $$y + 2x = 3 \quad \text{(i)}$$

Directrix of the parabola $y^2 = x$ is

$$x = -\frac{1}{4}. \quad \text{(ii)}$$

Centre of the circle is intersection of two normals to the circle, i.e., Eqs. (i) and (ii) which is $\left(-\frac{1}{4}, \frac{7}{2}\right)$.

Hence, radius of the circle is

$$\sqrt{\left(1+\frac{1}{4}\right)^2 + \left(1-\frac{7}{2}\right)^2} = \sqrt{\frac{25}{16}+\frac{25}{4}} = \frac{5\sqrt{5}}{4}$$

105.d. Let $y = mx + c$, intersect $y^2 = 4ax$ at $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$.

Then, $$\frac{2}{t_1 + t_2} = m$$

$$\Rightarrow \quad t_1 + t_2 = \frac{2}{m}$$

Let the foot of another normal be $C(at_3^2, 2at_3)$.

Then,

$$t_1 + t_2 + t_3 = 0$$

$$\Rightarrow \quad t_3 = -(t_1 + t_2) = -\frac{2}{m}$$

Thus, other foot is $\left(\frac{4a}{m^2}, \frac{-4a}{m}\right)$.

106.b. Tangent to $y^2 = 4x$ in terms of 'm' is

$$y = mx + \frac{1}{m}$$

Normal to $x^2 = 4by$ in terms of 'm' is

$$y = mx + 2b + \frac{b}{m^2}$$

If these are same lines, then

$$\frac{1}{m} = 2b + \frac{b}{m^2}$$

$$\Rightarrow \quad 2bm^2 - m + b = 0$$

For two different tangents

$$1 - 8b^2 > 0$$

$$\Rightarrow \quad |b| < \frac{1}{\sqrt{8}}$$

107.d. Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of 'm' are

$$y = mx - 2am - am^3$$

and $$y = mx + 2b + \frac{b}{m^2}.$$

For a common normal,

$$2b + b/m^2 + 2am + am^3 = 0$$

$$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$$

This means there can be most '5' common normals.

108. a.

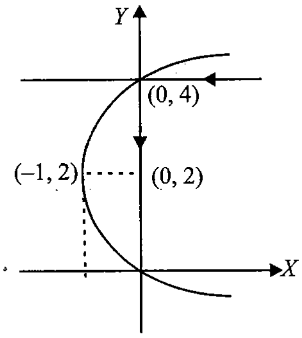


Fig. 3.117

Given curve is $(y - 2)^2 = 4(x + 1)$

Focus is (0, 2).

Point of intersection of the curve and $y = 4$ is (0, 4).

From the reflection property of parabola, reflected ray passes through the focus.

$\therefore$ $x = 0$ is required line.

109.d. Solving the equations,

$$x^2 + 4(x + 4) = a^2$$

If circle and parabola touch each other, then

$$D = 0$$

$$\Rightarrow \quad 16 - 4(16 - a^2) = 0$$

$$\Rightarrow \quad a = 2\sqrt{3}$$

110.c.

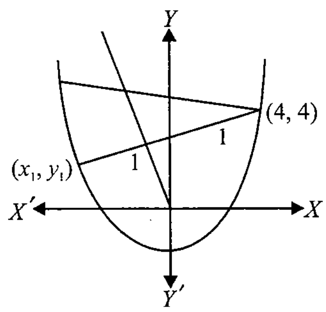


Fig. 3.118

Point (4, 4) lies on the parabola.

Let the point of intersection of the line $y = mx$ with the chords be $(\alpha, m\alpha)$, then

$$\alpha = \frac{4+x_1}{2}$$

$$\Rightarrow \quad x_1 = 2\alpha - 4$$

and

$$m\alpha = \frac{4+y_1}{2}$$

$$\Rightarrow \quad y_1 = 2m\alpha - 4$$

as (x_1, y_1) lies on the curve

$$\therefore \quad (2\alpha - 4)^2 = 4(2m\alpha - 4)$$

$$\Rightarrow \quad 4\alpha^2 + 16 - 16\alpha = 8(m\alpha - 2)$$

$$\Rightarrow \quad 4\alpha^2 - 8\alpha(2+m) + 32 = 0$$

For two distinct chords

$$\therefore \quad D > 0$$

$$(8(2+m))^2 - 4(4)(32) > 0$$

$$\Rightarrow \quad (2+m)^2 - 8 > 0$$

$$2 + m > 2\sqrt{2}$$

or

$$2 + m < -2\sqrt{2}$$

$$\Rightarrow \quad m > 2\sqrt{2} - 2$$

or

$$m < -2\sqrt{2} - 2$$

111. c. Let point of intersection be (α, β).

Therefore, chord of contact w.r.t. this point is

$$\beta y = 2x + 2\alpha$$

which is same as $\quad x + y = 2.$

$$\Rightarrow \quad \alpha = \beta = -2$$

which satisfy $\quad y - x = 0.$

112. d. The given parabolas are symmetrical about the line $y = x$ as shown in the figure.

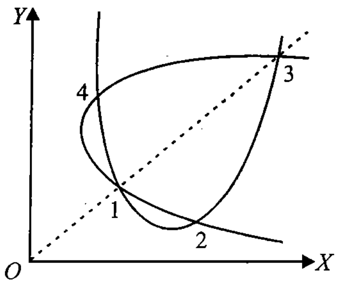


Fig. 3.119

They intersect to each other at four distinct points.

Hence, the number of common chords $= 4C_2 = \frac{4 \times 3}{2} = 6$

113. a. Let the focus be F. The parabolas are open down and open right, respectively. Let the parabolas intersect at points P and Q. From P perpendiculars are drawn on the x-axis and y-axis at A and B, respectively, then

$$PA = PF = PB$$

$$\Rightarrow \quad P \text{ lies on the line } y = -x.$$

Similarly, Q lies on the line $y = -x$

$\Rightarrow$ slope of $PQ = -1$

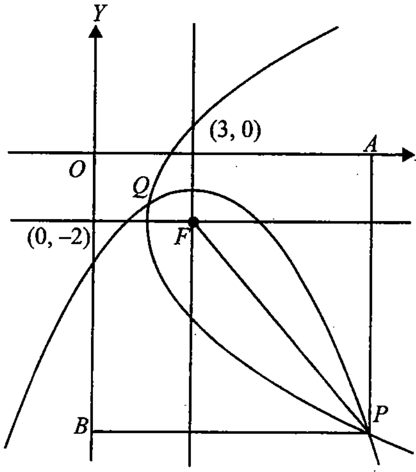


Fig. 3.120

114. b.

Equation of tangent to the parabola $y^2 = 8x$ at $P(2, 4)$ is

$$4y = 4(x + 2)$$

or

$$x - y + 2 = 0 \quad \text{(i)}$$

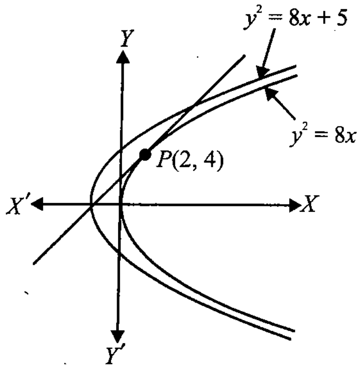


Fig. 3.121

Equation of chord of parabola $y^2 = 8x + 5$ whose middle point is (h, k) is $T = S_1$.

i.e., $\quad ky - 4(x + h) - 5 = k^2 - 8h - 5$

or $\quad 4x - ky + k^2 - 4h = 0 \quad \text{(ii)}$

Equations (i) and (ii) must be identical

$$\therefore \quad \frac{4}{1} = \frac{k}{1} = \frac{k^2 - 4h}{2}$$

By comparing Eqs. (i), (ii)

$$k = 4$$

and

$$8 = k^2 - 4h$$

Hence, the required point is (2, 4).

Multiple Correct Answers Type

1. a., c. The line $y = 2x + c$ is a tangent to $x^2 + y^2 = 5$.

If $$c^2 = 25$$

$$\Rightarrow \quad c = \pm 5$$

Let the equation of parabola be $y^2 = 4ax$. Then

$$\frac{a}{2} = \pm 5$$

$$\Rightarrow \quad a = \pm 10$$

$\Rightarrow$ Equation of the parabola is $y^2 = \pm 40x$.

$\Rightarrow$ Equations of the directrix are $x = \pm 10$.

2. b., c., d. Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $\left(h, \left(\frac{h + at^2}{t}\right)\right)$.

Chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t}\right)y = a^2$$

or $$(aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of straight lines passing through point of intersection of

$$ty - a = 0 \text{ and } x + \frac{y}{t} = 0$$

So, the fixed point is $\left(-\frac{a}{t^2}, \frac{a}{t}\right)$.

$$\therefore \quad x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$$

Clearly, $$x_1x_2 = -a^2, y_1y_2 = 2a^2$$

Also, $$\frac{x_1}{x_2} = -t^4$$

$$\frac{y_1}{y_2} = 2t^2$$

$$\Rightarrow \quad 4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$$

3. b., d. Given parabola is

$$x^2 - ky + 3 = 0$$

or $$x^2 = k\left(y - \frac{3}{k}\right)$$

Le $$x = Y, y - \frac{3}{k} = X$$

then the parabola is

$$Y^2 = kX$$

whose focus is $\left(0, \frac{k}{4}\right)$.

Therefore, the focus of $x^2 = k\left(y - \frac{3}{k}\right)$ is

$$\left(0, \frac{3}{k} + \frac{k}{4}\right) \equiv (0, 2)$$

$$\therefore \quad \frac{3}{k} + \frac{k}{4} = 2$$

$$\Rightarrow \quad 12 + k^2 = 8k$$

$$\Rightarrow \quad k^2 - 8k + 12 = 0$$

$$\Rightarrow \quad (k - 6)(k - 2) = 0$$

$$\Rightarrow \quad k = 2, 6$$

4. a., c. $P = (\alpha, \alpha + 1)$ where $\alpha \neq 0, -1$

or $P = (\alpha, \alpha - 1)$ where $\alpha \neq 0, 1$

$(\alpha, \alpha + 1)$ is on $y^2 = 4x + 1$

$$\Rightarrow \quad (\alpha + 1)^2 = 4\alpha + 1$$

$$\Rightarrow \quad \alpha^2 - 2\alpha = 0$$

$$\Rightarrow \quad \alpha = 2 \qquad (\because \alpha \neq 0)$$

Therefore, ordinate of P is 3

$(\alpha, \alpha - 1)$ is on $y^2 = 4x + 1$

$$\Rightarrow \quad (\alpha - 1)^2 = 4\alpha + 1$$

$$\Rightarrow \quad \alpha^2 - 6\alpha = 0$$

$$\Rightarrow \quad \alpha = 6 \qquad (\because \alpha \neq 0)$$

Therefore, ordinate of P is 5

5. a., d. Here $x^2 = -\lambda\left(y + \frac{\mu}{\lambda}\right)$

Therefore, $$\text{vertex} = \left(0, -\frac{\mu}{\lambda}\right)$$

And the directrix is

$$\left(y + \frac{\mu}{\lambda}\right) + \frac{-\lambda}{4} = 0.$$

Comparing with the given data, $-\frac{\mu}{\lambda} = 1$ and $\frac{\mu}{\lambda} - \frac{\lambda}{4} = -2$

$$\therefore \quad -1 - \frac{\lambda}{4} = -2$$

or $$\lambda = 4 \Rightarrow \mu = 4.$$

6. a., c. Given that the extremities of the latus rectum are (1, 1) and (1, −1)

$$\Rightarrow \quad 4a = 2 \Rightarrow a = \frac{1}{2}$$

$\Rightarrow$ The focus of the parabola is (1, 0).

$\Rightarrow$ The vertex can be $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

$\Rightarrow$ The equations of the parabola can be

$$y^2 = 2\left(x - \frac{1}{2}\right)$$

or $$y^2 = 2\left(x - \frac{3}{2}\right)$$

$$\Rightarrow \quad y^2 = 2x - 1$$

or $$y^2 = 2x - 3$$

7. a., c. Let the possible point be $(t^2, 2t)$. Equation of tangent at this point is

$$yt = x + t^2.$$

It must pass through (6, 5). (Since normal to circle always passes through its centre)

$$\Rightarrow \quad t^2 - 5t + 6 = 0$$

$$\Rightarrow \quad t = 2, 3$$

$\Rightarrow$ Possible points are (4, 4), (9, 6).

8. c., d.

$$t_2 = -t_1 - \frac{2}{t_1}$$

Also,
$$\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$$

$$\Rightarrow \quad t_1t_2 = -4$$

$$\therefore \quad \frac{-4}{t_1} = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_1^2 + 2 = 4 \text{ and } t_1 = \pm\sqrt{2}$$

So point can be $(2a, \pm 2\sqrt{2}a)$.

9. a., b. As a circle can intersect a parabola in four points, so quadrilateral may be cyclic.

The diagonals of the quadrilateral may be equal as the quadrilateral may be an isosceles trapezium.

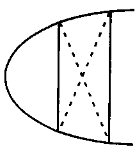


Fig. 3.122

A rectangle cannot be inscribed in a parabola. So (**C**) is wrong.

10. a., b., c., d. Any point on the parabola is $P(at^2, 2at)$.

Therefore, midpoint of $S(a, 0)$ and $P(at^2, 2at)$ is

$$R\left(\frac{a + at^2}{2}, at\right) \equiv (h, k).$$

$$\therefore \quad h = \frac{a + at^2}{2}, \; k = at$$

Eliminate 't'

i.e.,
$$2x = a\left(1 + \frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

i.e.,
$$2ax = a^2 + y^2$$

i.e.,
$$y^2 = 2a\left(x - \frac{a}{2}\right)$$

It's a parabola with vertex at $\left(\frac{a}{2}, 0\right)$, latus rectum = $2a$

Directrix is

$$x - \frac{a}{2} = -\frac{a}{2},$$

i.e.,
$$x = 0$$

Focus is

$$x - \frac{a}{2} = \frac{a}{2},$$

i.e.,
$$x = a$$

i.e., $(a, 0)$

11. a., b., c. Equation of tangent to parabola $y^2 = 8x$ having slope m is $y = mx + \frac{2}{m}$

Options (a), (b), (c) are tangents for $m = 1, 3, -\frac{1}{2}$ respectively.

12. a., c., d. Equation of normal to parabola $y^2 = 12x$ having slope m is $y = mx - 6m - 3m^3$. Options (a), (c), (d) are normal for $m - 1, -2$ and 3 respectively.

13. a., b., c., d.

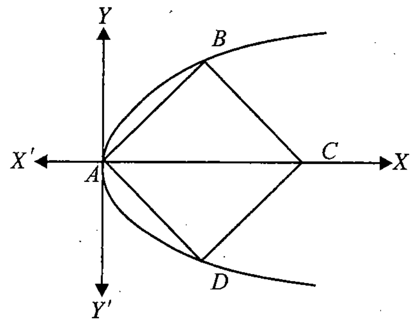


Fig. 3.123

AC is one diagonal along x-axis, then the other diagonal is BD where both B and D lie on parabola. Also slope of AB is $\tan\frac{\pi}{4} = 1$. If B is $(at^2, 2at)$ then the slope of AB

$$= \frac{2at}{at^2} = \frac{2}{t} = 1$$

$$\therefore \quad t = 2$$

Therefore, B is $(4a, 4a)$ and hence D is $(4a, -4a)$.

Clearly, C is $(8a, 0)$.

Reasoning Type

1. b. Any tangent having slope m is

$$y = mx + \frac{a}{m}$$

or
$$y = mx + \frac{9/4}{m}$$

It passes through the point (4, 10), then

$$10 = 4m + \frac{9/4}{m}$$

$$\Rightarrow \quad 16m^2 - 40m + 9 = 0$$

$$\Rightarrow \quad m_1 = \frac{1}{4}, m_2 = \frac{9}{4}$$

$\Rightarrow$ Statement 1 is correct.

Also statement 2 is correct but it does not say anything about slope of the tangents, hence it is not correct explanation of statement 1.

2. **b.** Any normal to $y^2 = 4x$ is

$$y = -tx + 2t + t^3$$

If only one normal can be drawn to parabola from $(\lambda, \lambda + 1)$, then $\lambda < 2$.

Hence, statement 1 is true.

Statement 2 is also true as $(\lambda + 1)^2 > 4\lambda$ is true $\forall \lambda \in R - \{1\}$, but does not explain statement 1, as it is not necessary that from every outside points only one normal can be drawn.

3. **b.** Both the statements are true (see properties of focal chord), but statement 2 is not correct explanation of statement 1.

4. **a.** Differentiating $y^2 = 8x$ w.r.t. x, we have

$$2y\frac{dy}{dx} = 8$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{y}$$

Now slopes of tangents at $(8, -8)$ and $(\frac{1}{2}, 2)$ are $-\frac{1}{2}$ and 2. Hence, tangents are perpendicular.

Also tangents at the extremities of the focal chord are perpendicular and meet on the directrix. Hence, both the statements are true and statement 2 is correct explanation of statement 1.

5. **a.** For parabola $y^2 = 4x$, $(4, 4)$ and $\left(\frac{1}{4}, -1\right)$ are extremities of the focal chord. Hence, tangents are perpendicular.

Then obviously normals at these points are also perpendicular.

6. **c.** Any tangent having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or

$$y = mx + am + \frac{a}{m}$$

is tangent to the given parabola for all $m \in R - \{0\}$.

Hence, statement 2 is false.

However, statement 1 is true as when $m = 1$, tangent is

$$y = x + 2a.$$

7. **d.** Statement 2 is true as it is the definition of parabola.

From statement 1, we have

$$\sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x + 4y + 5|}{5},$$

which is not parabola as point $(1, -2)$ lies on the line $3x + 4y + 5 = 0$. Hence, statement 1 is false.

8. **d.** Statement 2 is correct (see properties of the focal chord).

Then length of the focal chord according to the statement 1 is

$$4(2)\left(\frac{4}{3}\right) = \frac{32}{3}.$$

9. **a** Let parabola be $y^2 = 4x$

Clearly $x = 0$ is tangent to the parabola at $(0, 0)$.

And lines $y = -x - 1$ and $y = x + 1$ are tangents to the parabola at $(1, 2)$ and $(1, -2)$ which are extremities of the latus rectum. These tangents meet on the directrix at right angle at $(-a, 0)$. Hence, circle passing through the point A, B, C also passes through its focus, as shown in the figure.

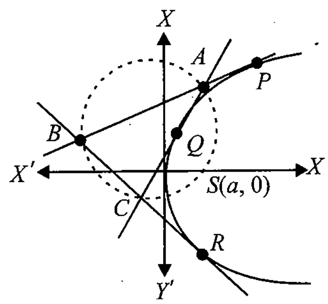


Fig. 3.124

Now consider a parabola $y^2 = 4ax$

Let $P(t_1)$, $Q(t_2)$ and $R(t_3)$ be three points on it.

Tangents are drawn at these points which intersect at

$$A \equiv (at_1t_2, a(t_1 + t_2))$$

$$B \equiv (at_1t_3, a(t_1 + t_3))$$

$$C \equiv (at_2t_3, a(t_2 + t_3))$$

Let $\quad \angle SAC = \alpha$ and $\angle SBC = \beta$

$$\Rightarrow \quad \tan\alpha = \left|\frac{\frac{1}{t_2} - \frac{t_1 + t_2}{t_1t_2 - 1}}{1 + \frac{1}{t_2}\left(\frac{t_1 + t_2}{t_1t_2 - 1}\right)}\right| = \left|\frac{1}{t_1}\right|$$

Similarly $\quad \tan\beta = \left|\frac{1}{t_1}\right|$

$$\Rightarrow \quad \alpha = \beta \text{ or } \alpha + \beta = \pi$$

$\Rightarrow A, B, C$ and S are concyclic.

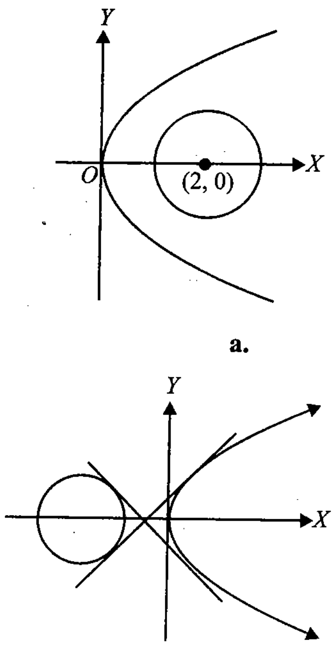
10. d. Area of the triangle formed by the intersection points of tangents at point $A(t_1)$, $B(t_2)$ and $C(t_3)$ is

$$\frac{1}{2}|t_1 - t_2|\,|t_2 - t_3|\,|t_3 - t_1| \neq 0.$$

Hence, statement 1 is wrong. However, statement 2 is correct.

11. b. Statement 2 is true as circle is lying inside parabola without intersecting it.

But this cannot be considered the explanation of the statement 1, as even if they are not intersecting we can have common tangents as shown in the figure.



a.

b.

Fig. 3.125

12. a. Let the foot of normal be $P(at^2, 2at)$, then

$$ax + by + c = 0$$

and $\quad y = -tx + 2at + at^3$

are identical line,

$$\Rightarrow \quad \frac{1}{b} = \frac{t}{a} = \frac{2at + at^3}{-c}$$

$$\Rightarrow \quad t = \frac{a}{b}.$$

Thus 'P' is $\left(\frac{a^3}{b^2}, \frac{2a^2}{b}\right)$.

Hence, equation of required tangent is

$$ty = x + at^2$$

or $\quad \frac{a}{b}y = x + a\left(\frac{a}{b}\right)^2$

or $\quad y = \frac{b}{a}x + \frac{a^2}{b}$

13. a.

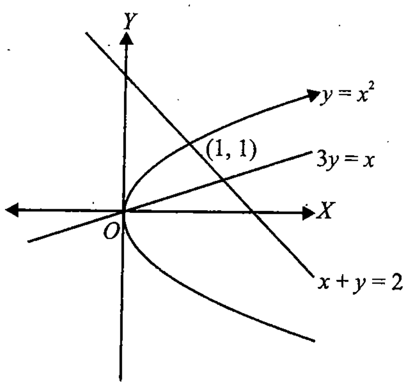


Fig. 3.126

Point (α, α^2) lies on the parabola $y^2 = x$.

As shown in the figure, we have to find the value of α for which the part of the parabola lies inside the triangle formed by three lines.

Now line $x + y = 2$ meets the parabola at point (0, 0) and (1, 1).

Hence, $\alpha \in (0, 1)$

14. a.

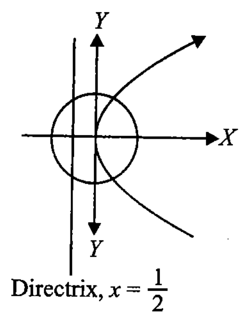


Fig. 3.127

Statement 2 is true as it is the property of the parabola.

Now such points exists on the circle $x^2 + y^2 = a^2$ if it meets the directrix at least one point, for which radius of the circle $a \geq 1/2$.

15. a. Let $y^2 = 4ax$ be a parabola. Consider a line $x = 4a$ (this is a double ordinate which is twice of latus rectum), which cuts the parabola at $A(4a, 4a)$ and $B(4a, -4a)$.

Slope of $\quad OA = 1$,

Slope of $\quad OB = -1$, where O is given.

Therefore, AB subtends 90° at the origin.

$\Rightarrow$ Statement 2 is correct and it clearly explains statement 1.

16. d. Statement 1 is false

Since here $t^2 = 4$

Therefore, the normal chord subtends a right angle at the focus (not at the vertex).

However, statement 2 true (a standard result).

17. a. Let normals at points $A(at_1^2, 2at_1)$ and $C(at_3^2, 2at_3)$ meets the parabola again at points $B(at_2^2, 2at_2)$ and $D(at_4^2, 2at_4)$, then $t_2 = -t_1 - \frac{2}{t_1}$ and $t_4 = -t_3 - \frac{2}{t_3}$

Adding $\quad t_2 + t_4 = -t_1 - t_3 - \frac{2}{t_1} - \frac{2}{t_3}$

$\Rightarrow \quad t_1 + t_2 + t_3 + t_4 = -\frac{2}{t_1} - \frac{2}{t_3}$

$\Rightarrow \quad \frac{1}{t_1} + \frac{1}{t_3} = 0$

$\Rightarrow \quad t_1 + t_3 = 0$

Now, point of intersection of tangent at A and C will be $(at_1 t_3, a(t_1 + t_3))$

Since $t_1 + t_3 = 0$, so this point will lie on x-axis, which is axis of parabola.

18. a.

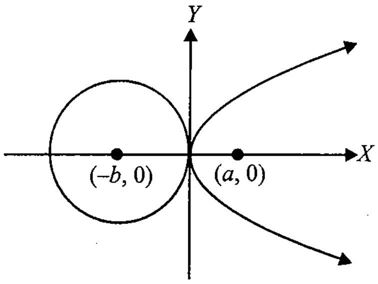


Fig. 3.128

As shown in the figure, circle and parabola touch when a and b have same sign.

Now for

$$f(x) = x^2 - (b + a + 1)x + a,$$

$\Rightarrow \quad f(0) = a$

and $\quad f(1) = 1 - (b + a + 1) + a = -b$

$\Rightarrow \quad f(0) \cdot f(1) = -ab < 0$

Hence, one root lies in (0, 1).

$\Rightarrow$ Both the statements are true and statement 2 is correct explanation of statement 1.

19. c. Statement 2 is false, as axis of parabola is normal to parabola which passes through the focus. However, normal other than axis never passes through focus. Statement 1 is correct as $x - y - 5 = 0$ passes through focus (3, –2), hence it cannot be normal.

20. b. Obviously, statement 2 is true, but it is not the correct explanation of statement 1 as A, A', B, B' form an isosceles trapezium, hence points are concyclic.

Linked Comprehension Type

For Problems 1–3

1. c., 2. d., 3. a.

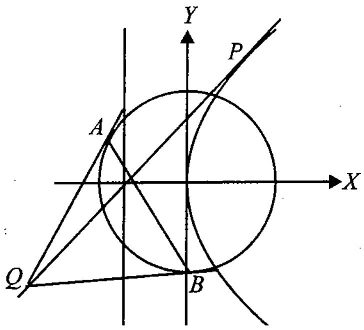


Fig. 3.129

Sol. 1. c. Equation of the tangent at point P of the parabola $y^2 = 8x$ is

$$yt = x + 2t^2 \quad \text{(i)}$$

Equation of the chord of contact of the circle $x^2 + y^2 = 8$ w.r.t. $Q(\alpha, \beta)$ is

$$x\alpha + y\beta = 8 \quad \text{(ii)}$$

$Q(\alpha, \beta)$ lies in Eq. (i)

Hence, $\quad \beta t = \alpha + 2t^2 \quad$ (iii)

$$x\alpha + y\left(\frac{\alpha}{t} + 2t\right) - 8 = 0 \text{ [from Eqs. (ii) and (iii)]}$$

$$2(ty - 4) + \alpha\left(x + \frac{y}{t}\right) = 0$$

For point of concurrency

$$x = -\frac{y}{t} \text{ and } y = \frac{4}{t}$$

Therefore, locus is $y^2 + 4x = 0$

2. d. Required point will lie on the director circle of the given circle as well as on the directrix of parabola.

$\Rightarrow \quad x_1^2 + y_1^2 = 16$ and $x_1 + 2 = 0$

$\Rightarrow \quad 4 + y_1^2 = 16$

$\Rightarrow \quad y = \pm 2\sqrt{3}$

Therefore, point is $(-2, \pm 2\sqrt{3})$.

3. a. Equation of circumcentre of ΔAQB is

$$x^2 + y^2 - 4 + \lambda(x\alpha + y\beta - 8) = 0$$

Because it passes through (0, 0), i.e., centre of circle

$$\Rightarrow \qquad \lambda = -\frac{1}{2}$$

Let circumcentre be (h, k).

$$\therefore \qquad h = \frac{\alpha}{4}, k = \frac{\beta}{4}$$

$$\Rightarrow \qquad \alpha = 4h, \beta = 4k$$

Also $\qquad \beta t = \alpha + 2t^2$

or $\qquad \alpha - 2\beta + 8 = 0 \qquad (\because t = 2)$

Substituting $\alpha = 4h$ and $\beta = 4k$, we get

$$h - 2k + 2 = 0$$

Therefore, locus is $x - 2y + 2 = 0$.

For Problems 4–6

4. b., 5. c., 6. d.

Sol. 4. b. Since no point of the parabola is below x-axis

$$\therefore \qquad D = a^2 - 4 \le 0$$

Therefore, maximum value of a is 2.

Equation of the parabola, when $a = 2$, is

$$y = x^2 + 2x + 1$$

It intersects y-axis at (0, 1).

Equation of the tangent at (0, 1) is

$$y = 2x + 1$$

Since $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$

$$\therefore \qquad r = \frac{1}{\sqrt{5}}$$

5. c. Equation of the tangent at (0, 1) to the parabola $y = x^2 + ax + 1$ is

$$y - 1 = a(x - 0)$$

or $\qquad ax - y + 1 = 0$

As it touches the circle,

$$\therefore \qquad r = \frac{1}{\sqrt{a^2 + 1}}$$

Radius is maximum when $a = 0$

Therefore, equation of the tangent is $y = 1$.

Therefore, slope of the tangent is 0.

6. d. Equation of tangent is $y = ax + 1$.

Intercepts are $-\frac{1}{a}$ and 1.

Therefore, area of the triangle bounded by tangent and the axes $= \frac{1}{2}\left|-\frac{1}{a}\cdot 1\right| = \frac{1}{2|a|}$

It is minimum when $a = 2$

Therefore, minimum area $= \frac{1}{4}$

For Problems 7–9

7. a., 8. c., 9. d.

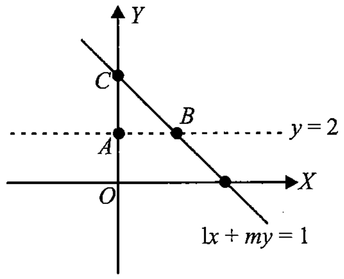


Fig. 3.130

Sol. $C \equiv \left(0, \frac{1}{m}\right)$, $B \equiv \left(\frac{1-2m}{l}, 2\right)$, $A \equiv (0, 2)$

Let (h, k) be the circumcentre of $\Delta\, ABC$ which is mid-point of BC

$$\Rightarrow \qquad h = \frac{1-2m}{2l};\ k = \frac{1+2m}{2m},$$

$$\Rightarrow \qquad m = \frac{1}{2k-2};\ l = \frac{k-2}{2h(k-1)}$$

Given that (l, m) lies on $\quad y^2 = 4x$

$$\therefore \qquad m^2 = 4l$$

$$\Rightarrow \qquad \left(\frac{1}{2k-2}\right)^2 = 4\left\{\frac{k-2}{2h\,(k-1)}\right\}$$

$$\Rightarrow \qquad h = 8(k^2 - 3k + 2)$$

Therefore, locus of (h, k) is

$$x = 8(y^2 - 3y + 2)$$

or $\qquad \left(y - \frac{3}{2}\right)^2 = \frac{1}{8}(x + 2)$

Therefore, vertex is $\left(-2, \frac{3}{2}\right)$.

Length of smallest focal chord = length of latus rectum $= \frac{1}{8}$.

From the equation of curve C, it is clear that it is symmetric about line $y = \frac{3}{2}$.

For Problems 10–12

10. d., 11. c., 12. d.

Sol. 10. d. $y = ax^2 + c$

$$\therefore \qquad \frac{dy}{dx} = 2ax = 1$$

Therefore, point of contact of the tangent is $\left(\frac{1}{2a}, \frac{1}{4a} + c\right)$

Since it lies on $y = x$

$$\therefore \qquad c = \frac{1}{4a}, \text{ thus } c = \frac{1}{8} \text{ for } a = 2.$$

11. c. If (1, 1) is point of contact, then $a = \frac{1}{2}$

12. d. If $c = 2$, then point of contact is $\left(\frac{1}{2a}, \frac{1}{4a} + 2\right)$.

Since it lies on line $y = x$,

$$\therefore \quad \frac{1}{2a} = \frac{1}{4a} + 2,$$

i.e.,

$$a = \frac{1}{8}$$

Therefore, point of contact is (4, 4).

For Problems 13–15

13. a., 14. b., 15. c.

Sol. Any parabola whose axes is parallel to x-axis will be of the form

$$(y - a)^2 = 4b(x - c) \quad \text{(i)}$$

Now, $lx + my = 1$, can be rewritten as

$$y - a = -\frac{l}{m}(x - c) + \frac{1 - am - lc}{m} \quad \text{(ii)}$$

Equation (ii) will touch Eq. (i) if

$$\frac{1 - am - lc}{m} = \frac{b}{-l/m}$$

$$\Rightarrow \quad -\frac{l}{m} = \frac{bm}{1 - am - lc}$$

$$\Rightarrow \quad cl^2 - bm^2 + alm - l = 0 \quad \text{(iii)}$$

But given that $5l^2 + 6m^2 - 4lm + 3l = 0$ (iv)

Comparing Eqs. (iii) and (iv), we get

$$\frac{c}{5} = \frac{-b}{6} = \frac{a}{-4} = \frac{-1}{3}$$

$$\Rightarrow \quad c = \frac{-5}{3}, b = 2 \text{ and } a = \frac{4}{3}$$

So parabola is $\left(y - \frac{4}{3}\right)^2 = 8\left(x + \frac{5}{3}\right)$ whose focus is $\left(\frac{1}{3}, \frac{4}{3}\right)$ and directrix is $3x + 11 = 0$.

For Problems 16–18

16. b., 17. a., 18. c.

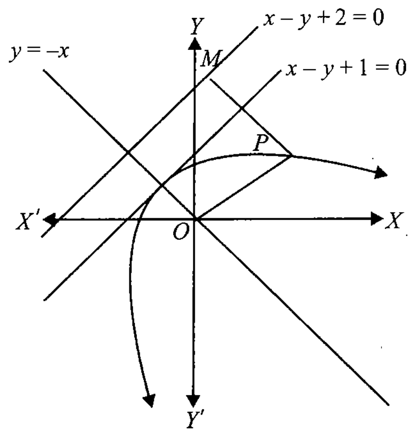


Fig 3.131

Sol. The distance between the focus and the tangent at the vertex $= \frac{|0 - 0 + 1|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$.

The directrix is the line parallel to the tangent at vertex and at a distance $2 \times \frac{1}{\sqrt{2}}$ from the focus.

Let equation of directrix is

$$x - y + \lambda = 0,$$

where

$$\frac{\lambda}{\sqrt{1^2 + 1^2}} = \frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad \lambda = 2$$

Let $P(x, y)$ be any moving point on the parabola, then

$$OP = PM$$

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{x - y + 2}{\sqrt{1^2 + 1^2}}\right)^2$$

$$\Rightarrow \quad 2x^2 + 2y^2 = (x - y + 2)^2$$

$$\Rightarrow x^2 + y^2 + 2xy - 4x + 4y - 4 = 0.$$

Latus rectum length = 2 × (distance of focus from directrix)

$$= 2\left|\frac{0 - 0 + 2}{\sqrt{1^2 + 1^2}}\right|$$

$$= 2\sqrt{2}$$

Solving parabola with x-axis,

$$x^2 - 4x - 4 = 0$$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{32}}{2} = 2 \pm 2\sqrt{2}$$

$\Rightarrow$ Length of chord on x-axis is $4\sqrt{2}$.

Since the chord $3x + 2y = 0$ passes through the focus, it is focal chord.

Hence, tangents at the extremities of chord are perpendicular.

For Problems 19–21

19. a., 20. b., 21. c.

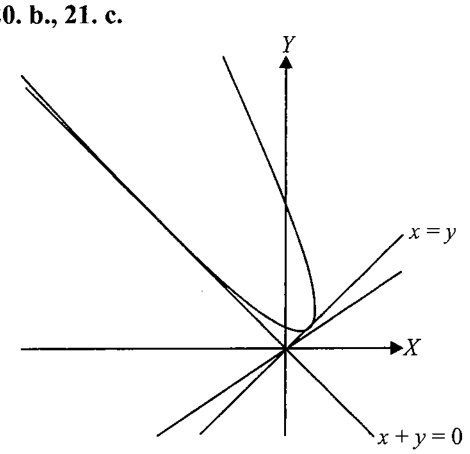


Fig. 3.132

Sol. We know that foot of perpendicular from focus upon tangent lies on the tangent at vertex of the parabola.

Now, if foot of perpendicular of (2, 3) on the line $x - y = 0$ is (x_1, y_1), then

$$\frac{x_1 - 2}{1} = \frac{y_1 - 3}{-1} = \frac{2-3}{2}$$

$$\Rightarrow \quad x_1 = \frac{5}{2} \text{ and } y_1 = \frac{5}{2}$$

If foot of perpendicular of (2, 3) on the line $x + y = 0$ is (x_2, y_2), then

$$\frac{x_2 - 2}{1} = \frac{y_2 - 3}{1} = -\frac{2+3}{2}$$

$$\Rightarrow \quad x_2 = -\frac{1}{2} \text{ and } y_2 = \frac{1}{2}$$

Now tangent at vertex passes through the points $\left(\frac{5}{2}, \frac{5}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Then, its equation is

$$y - \frac{1}{2} = \frac{2}{3}\left(x + \frac{1}{2}\right)$$

or $\quad 4x - 6y + 5 = 0$

Latus rectum of the parabola

$$= 4\times \text{(distance of focus from tangent at vertex)}$$

$$= 4\times \left|\frac{8 - 18 + 5}{\sqrt{52}}\right| = \frac{10}{\sqrt{13}}$$

Also, distance between the focus and tangent at vertex $= \frac{5}{\sqrt{13}}$

Since tangents $x + y = 0$ and $x - y = 0$ are perpendicular, they meet at (0, 0) which lies on the directrix.

Also, it is parallel to the tangent at vertex, hence its equation is $4x - 6y = 0$.

We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$, where a is $\left(\frac{1}{4}\right)$th of latus rectum.

$$\Rightarrow \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2\sqrt{13}}{5}$$

For Problems 22–24

22. d., 23. b., 24. a.

Sol. Solving given parabolas, we have

$$-8(x - a) = 4x$$

$$\Rightarrow \quad x = \frac{2a}{3}$$

$\Rightarrow$ Points of intersection are $\left(\frac{2a}{3}, \pm\sqrt{\frac{8a}{3}}\right)$

Now $OABC$ is concyclic.

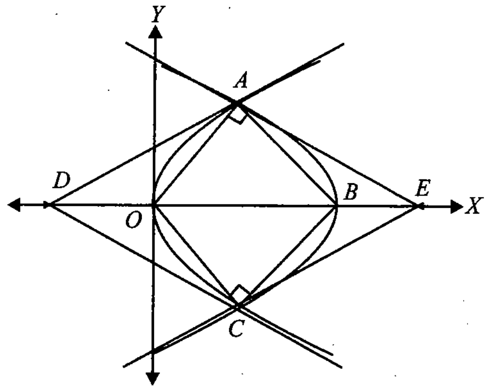


Fig. 3.133

Hence, $\angle OAB$ must be right angle.

$\Rightarrow$ Slope of OA × Slope of $AB = -1$

$$\Rightarrow \quad \frac{\sqrt{\frac{8a}{3}}}{\frac{2a}{3}} \times \frac{\sqrt{\frac{8a}{3}}}{a - \frac{2a}{3}} = -1$$

$$\Rightarrow \quad a = 12$$

$\Rightarrow$ Coordinates of A and B are $(8, 4\sqrt{2})$ and $(8, -4\sqrt{2})$ respectively

$\Rightarrow$ Length of common chord $= 8\sqrt{2}$.

$$\text{Area of quadrilateral} = \frac{1}{2} OB \times AC$$

$$= \frac{1}{2} \times 12 \times 8\sqrt{2}$$

$$= 48\sqrt{2}$$

Tangent to parabola $y^2 = 4x$ at point $(8, 4\sqrt{2})$ is $4\sqrt{2}y = 2(x + 8)$ or $x - 2\sqrt{2}y + 8 = 0$ which meets the x-axis at $D(-8, 0)$.

Tangent to parabola $y^2 = -8(x - 12)$ at point $(8, 4\sqrt{2})$ is $4\sqrt{2}y = -4(x + 8) + 96$ or $x + \sqrt{2}y - 16 = 0$, which meets the x-axis at $E(16, 0)$.

Hence, area of quadrilateral $DAEC = \frac{1}{2} DE \times AC$

$$= \frac{1}{2} \times 24 \times 8\sqrt{2}$$

$$= 96\sqrt{2}$$

For Problems 25–27

25.b., 26. c., 27. d.

Sol. For $y^2 = 4x$, coordinates of end of latus rectum are $P(1, 2)$ and $Q(1, -2)$.

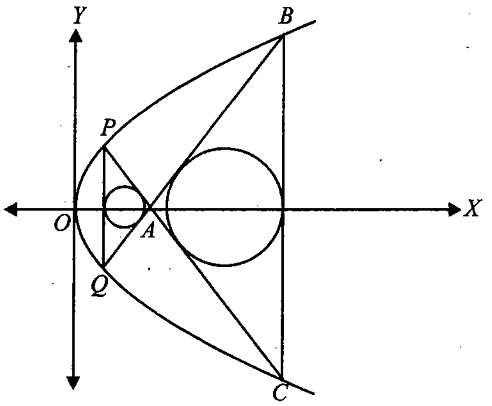


Fig. 3.134

ΔPAQ is isosceles right angled. Therefore, slope of PA is – 1 and its equation is $y - 2 = -(x - 1)$ or $x + y - 3 = 0$.

Similarly, equation of line QB is $x - y - 3 = 0$.

Solving $x + y - 3 = 0$ with the parabola $y^2 = 4x$, we have

$$(3 - x)^2 = 4x \text{ or } x^2 - 10x + 9 = 0$$

$$\therefore \quad x = 1, 9$$

Therefore, coordinates of B and C are (9, – 6) and (9, 6) respectively.

Area of trapezium $PBCQ = \frac{1}{2} \times (12 + 4) \times 8$

$= 64$ sq. units

Let the circumcentre of trapezium $PBCQ$ is $T(h, 0)$

Then $\quad PT = PB$

$$\Rightarrow \quad \sqrt{(h-1)^2 + 4} = \sqrt{(h-9)^2 + 36}$$

$$\Rightarrow \quad -2h + 5 = -18h + 81 + 36$$

$$\Rightarrow \quad 16h = 112$$

$$\Rightarrow \quad h = 7$$

Hence, radius is $\sqrt{40} = 2\sqrt{10}$

Let inradius of ΔAPQ be r_1, then $r_1 = \frac{\Delta_1}{s_1}$

$$= \frac{\frac{1}{2} \times 4 \times 2}{4 + 2\sqrt{4 + 4}}$$

$$= \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$$

Let inradius of ΔABC be r_2, then

$$r_2 = \frac{\Delta_2}{s_2}$$

$$= \frac{\frac{1}{2} \times 12 \times 6}{12 + 2\sqrt{36 + 36}}$$

$$= \frac{3}{1 + \sqrt{2}} = 3(\sqrt{2} - 1)$$

$$\Rightarrow \quad \frac{r_2}{r_1} = 3$$

For Problems 28–30

28.d., 29. c., 30. b.

Sol. $9x - a.\ 3^x - a + 3 \le 0$

Let $t = 3^x$

$\Rightarrow t^2 - at - a + 3 \le 0$

or $t^2 + 3 \le a(t + 1)$

where $t \in R^+$ for $\forall\ x \in R$

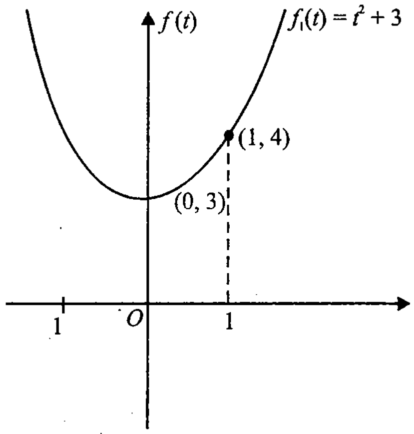


Fig. 3.135

Let $f_1(t)$ be $t^2 + 3$ and $f_2(t)$ be $a(t + 1)$.

28. d. From $x < 0$, $t \in (0, 1)$. That means (1) should have at least one solution in $t \in (0, 1)$.

From (1), it is obvious that $a \in R^+$.

Now $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve.

$f_1(t) = t^2 + 3$, at least once in $t \in (0, 1)$.

$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a$.

If $f_1(0) = f_2(0) \Rightarrow a = 3$; if $f_1(1) = f_2(1) = a = 2$

Hence, required $a \in (2, 3)$.

29. c. For at least one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$.

If $a = 2$, both curve touch each other at (1, 4).

Hence, the required $a \in (2, \infty)$.

30 b. In this case, both graphs should meet at least once in $t \in (0, \infty)$.

For $a = 2$, both curves touch; hence, the required $a \in [2, \infty)$.

Matrix-Match Type

1. a. → r., b. → s., c. → p., d. → q.

Sol. Locus of point of intersection of perpendicular tangent is directrix which is $12x - 5y + 3 = 0$.

Parabola is symmetrical about its axis, which is a line passing through the focus (1, 2) and perpendicular to the directrix, which has equation $5x + 12y - 29 = 0$.

Minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to directrix, i.e,. $12x - 5y - 2 = 0$.

Locus of foot of perpendicular from foucs upon any tangent is tangent at the vertex, which is parallel to directrix and equidistant from directrix and latus rectum line, i.e., $12x - 5y + \lambda = 0$

where $\frac{|\lambda - 3|}{\sqrt{12^2 + 5^2}} = \frac{|\lambda + 2|}{\sqrt{12^2 + 5^2}} \Rightarrow \lambda = \frac{1}{2}$.

Hence, equation of tangent at vertex is $24x - 10y + 1 = 0$.

2. a. → q., b. → s., c. → p., d. → r.

Equation of tangent having slope m is $y = mx + \frac{3}{m}$.

Line $3x - y + 1 = 0$ is tangent for $m = 3$.

Equation of normal having slope m is $y = mx - 6m - 3m^3$.

Line $2x - y - 36 = 0$ is normal for $m = 2$.

Chord of contact w.r.t. any point on the directrix is the focal chord which passes through the focus (3, 0).

Line $2x - y - 36 = 0$ passes through the focus.

Chords which subtends right angle at the vertex are concurrent at point $(4 \times 3, 0)$ or (12, 0).

Line $x - 2y - 12 = 0$ passes through the point (12, 0).

3. a. → q., s; b. → r.; c. → p., q.; d. → q., r

a. Tangent to parabola having slope m is $ty = x + t^2$, it passes through point (2, 3) then $3t = 2 + t^2 \Rightarrow t = 1$ or $2 \Rightarrow$ point of contact $(t^2 + 2t) \equiv (1, 2)$ or (4, 4)

b. Let point on the circle be $P(x_1, y_1)$, then chord of contact of parabola w.r.t. P is $yy_1 = 2(x + x_1)$. Comparing with $y = 2(x - 2)$, we have $y_1 = 1$ and $x_1 = -2$, which also satisfy the circle.

c. Point Q on the parabola $Q(t^2, 2t)$

Now area of triangle OPQ is $\left|\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 4 & -4 \\ t^2 & 2t \\ 0 & 0 \end{vmatrix}\right| = 6 \Rightarrow 8t + 4t^2 = \pm 12$

For $t^2 + 2t - 3 = 0$, $(t - 1)(t + 3) = 0$, then $t = 1$ or $t = -3$.

Then point Q are (1, 2) or (9, – 6).

d. Points (1, 2) and (– 2, 1) satisfy both the curves.

4. a. → p, r.; b. → p., r.; c. → q; d. → q., s.

Points through which perpendicular tangent can be drawn to the parabola $y^2 = 4x$ lie on the directrix. Points (–1, 2) and (–1, –5) lie on the directirx. Also from these points only one normal can be drawn.

Integer type

1. **(7)** Here, $(x - 1)^2 + (y - 3)^2 = \left\{\frac{5x - 12y + 17}{\sqrt{5^2 + (-12)^2}}\right\}^2$

$\therefore$ the focus = (1, 3) and the directrix is $5x - 12y + 17 = 0$

The distance of the focus from the directrix

$= \left|\frac{5 \times 1 - 12 \times 3 + 17}{\sqrt{5^2 + (-12)^2}}\right| = \frac{14}{13}$

$\therefore$ latus rectum $= 2 \times \frac{14}{13} = \frac{28}{13}$

2. **(0)** Clearly P is the point of intersection of two perpendicular tangents to the parabola $y^2 = 8x$, $4a = 8$ or $a = 2$. Hence P must lie on the directrix $x + a = 0$ or $x + 2 = 0$ $\therefore x = -2$. hence the point is (–2, 0)

3. **(1)** Focus of $y^2 = 16x$ is (4, 0).

Any focal chord is $y - 0 = m(x - 4)$

or $mx - y - 4m = 0$

This focal chord touches the circle $(x - 6)^2 + y^2 = 2$

Then distance from the center of circle to this chord is equal to radius of the circle

or $\frac{|6m - 4m|}{\sqrt{m^2 + 1}} = \sqrt{2}$ or $2m = \sqrt{2} \cdot \sqrt{m^2 + 1}$

or $2m^2 = m^2 + 1 \Rightarrow m^2 = 1$

$\therefore$ $m = \pm 1$

4. **(3)** Any tangent to parabola $y^2 = 4x$, $(a = 1)$ is

$y = mx + \frac{1}{m}$. It passes through (–2, – 1)

$\therefore -1 = -2m + \frac{1}{m}$ or $2m^2 - m - 1 = 0$

Or $(2m + 1)(m - 1) = 0$

Or $m = 1/2$ and $m = 1$

Then angle between lines is

$\tan\theta = \left|\frac{m_1 + m_2}{1 - m_1 m_2}\right| = 3$

5. **(3)** Equation of tangent in terms of slope of parabola $y^2 = 4x$ is

$y = mx + \frac{1}{m}$

$\because$ Eq. (i) is also tangent of $x^2 = -32y$

then $x^2 = -32\left(mx + \frac{1}{m}\right)$

$\Rightarrow x^2 + 32mx + \frac{32}{m} = 0$

Above equation must have equal roots,

Hence its discriminant must be zero

$\Longrightarrow (32\,m)^2 = 4.1 \cdot \dfrac{32}{m}$

$\Longrightarrow m^3 = \dfrac{1}{8}$ or $m = \dfrac{1}{2}$

From Eq. (i), $y = \dfrac{x}{2} + 2$

$\Longrightarrow x - 2y + 4 = 0$

6. **(4)** $y^2 = x \quad \therefore 4a = 1,$

$P\ (at_1^2, 2at_1) = (4, -2)$

$\therefore t_1 = -4$

Also $t_1 t_2 = -1$ as PQ is a focal chord.

Slope of tangent at t_2 is $\dfrac{1}{t_2} = -t_1 = 4$

7. **(6)** Slope of the line – 1

From the curve, $\dfrac{dy}{dx} = \dfrac{4}{y}$

Hence slope of normal $= -\dfrac{y}{4} = -1$ or $y = 4$.

Putting $y = 4$ in equation of curve we have $x = 2$

Hence point is (4, 2)

8. **(4)**

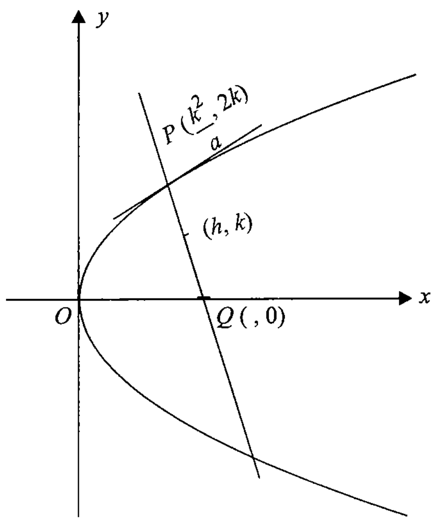


Fig. 3.136

Consider the parabola $y^2 = 4ax$

We have to find the locus of $R(h, k)$, since Q has ordinate 'O', ordinate of P is $2k$.

Also P is on the curve, then abscissa of P is k^2/a.

Now PQ is normal to curve

Slope of tangent to curve at any point $\dfrac{dy}{dx} = \dfrac{2a}{y}$

Hence slope of normal at point P is $-\dfrac{k}{a}$

Also slope of normal joining P and R(h, k) is $\dfrac{2k-k}{\dfrac{k^2}{a} - h}$

Hence comparing slopes $\dfrac{2k-k}{\dfrac{k^2}{a} - h} = -\dfrac{k}{a}$

Or $y^2 = a\,(x - a)$

For $y^2 = 16x$, $a = 4$, hence locus us $y^2 = 4(x - 4)$

9. **(3)**

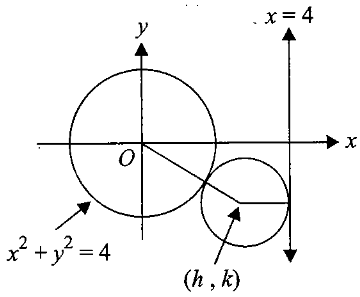


Fig. 3.137

Radius of variable circle is $4 - h$

It touches $x^2 + y^2 = 4$

$\therefore 2 + 4 - h = \sqrt{h^2 + k^2}$

or $x^2 + y^2 = x^2 - 12x + 36$

$\Rightarrow \quad y^2 = -12(x - 3)$

The vertex (3, 0)

10. **(8)**

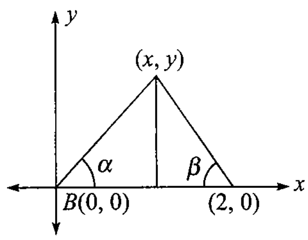


Fig. 3.138

Given $\tan\alpha \cdot \tan\beta = 4$

$\Rightarrow y \cdot \dfrac{y}{x} + \dfrac{y}{2-x} = 4 \Rightarrow y = 2x\,(2 - x)$

$\Rightarrow -\dfrac{y}{2} = x^2 - 2x = (x - 1)^2 - 1$

$\Rightarrow (x - 1)^2 = -\dfrac{1}{2}(y - 2)$

$\Rightarrow$ Directrix $y - 2 = \dfrac{1}{8} \Rightarrow y = \dfrac{17}{8}$

11. **(8)** Length of focal chord having one extremity $(at^2, 2at)$ is

$a\left(t + \dfrac{1}{t}\right)^2$

$\left|r + \dfrac{1}{t}\right| \geq 2 \Rightarrow a\left(1 + \dfrac{1}{t}\right)^2 \geq 4a = 8 \Rightarrow$ length of coal chord $\not< 8$.

12. **(4)** $a = 3$, comparing point (3, 6) with $(3t^2, 6t)$, we have $t = 1$,

then length of chord $= a\left(t+\frac{1}{t}\right)^2 = 3(1+1)^2 = 12$

13. **(8)** Chord of the contact w.r.t. point $O(-1, 2)$ is

$y = (x - 1)$ (using $yy_1 = 2a(x + x_1)$)

solving $y = x - 1$, with parabola, we get the points of intersection as

$$P(3+2\sqrt{2}, 2+2\sqrt{2}) \text{ and } Q(3-2\sqrt{2}, 2-2\sqrt{2})$$

$\therefore PQ^2 = 32 + 32 = 64$

$\therefore PQ = 8$

Also length of perpendicular from $O(-1, 2)$ on PQ

$= \frac{4}{\sqrt{2}}$

Then required area of triangle is

$A = \frac{1}{2}.8.\left(\frac{4}{\sqrt{2}}\right) = 8\sqrt{2}$ sq. units

14. **(7)** Line $y = 2x - b$

$\Rightarrow \quad 1 = \frac{2x-y}{b}$

Homogenising parabola with line

$$x^2 - 4x\left(\frac{2x-y}{b}\right) - y\left(\frac{2x-y}{b}\right) = 0$$

Since $\angle AOB = 90°$

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow 1 - \frac{8}{b} + \frac{1}{b} = 0$

$\Rightarrow \quad b = 7$

15. **(5)** Let the line be $y = mx$ (1)

Solving it with

$5y = 2x^2 - 9x + 10$, we get

$5mx = 2x^2 - 9x + 10$

$2x^2 - (9 + 5m)x + 10 = 0$

sum of the roots $= \frac{9+5m}{2} = 17$

$\Rightarrow 9 + 5m = 34$

$\Rightarrow \quad 5m = 25$

$\Rightarrow \quad m = 5$

16. **(5)** For maximum number of common chord, circle and parabola must intersect in 4 distinct points.

Let us first find the value of r when circle and parabola touch each other.

For that solving the given curves we have $(x - 6)^2 + 4x = r^2$ or $x^2 - 8x + 36 - r^2 = 0$

Curves touch if discriminant $D = 64 - 4(36 - r^2) = 0$ or $r^2 = 20$

Hence least integral value of r for which the curves intersect is 5.

Archives

Subjective Type

1. The equation of a normal to the parabola $y^2 = 4x$ in its slope from is given by

$$y = mx - 2am - am^3$$

Therefore, equation of normal to $y^2 = 4x$ is,

$$y = mx - 2m - m^3 \quad \text{(i)}$$

Since the normal drawn at three different points on the parabola passes through (h, k), it must satisfy Eq. (i).

$\therefore \quad k = mh - 2m - m^3$

$\Rightarrow \quad m^3 - (h - 2)m + k = 0$

This cubic equation in m has three different roots say m_1, m_2, m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0 \quad$ (ii)

$m_1m_2 + m_2m_3 + m_3m_1 = -(h - 2) \quad$ (iii)

Now, $(m_1 + m_2 + m_3)^2 = 0$ [Squaring Eq. (ii)]

$\Rightarrow \quad m_1^2 + m_2^2 + m_3^2 = -2(m_1m_2 + m_2m_3 + m_3m_1)$

$\Rightarrow \quad m_1^2 + m_2^2 + m_3^2 = 2(h - 2)$ [Using Eq. (iii)]

Since LHS of this equation is the sum of perfect squares, therefore, it is positive.

$\therefore \quad h - 2 > 0$

$\Rightarrow \quad h > 2$

2. Normal at point $A(at_1^2, 2at_1)$ meets the parabola at point $B(at_2^2, 2at_2)$.

$\Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1} \quad$ (i)

Also AB subtends right angle at vertex,

$\Rightarrow \quad t_1t_2 = -4 \quad$ (ii)

Eliminating t_2 from Eqs. (i) and (ii),

$\Rightarrow \quad -\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$

$\Rightarrow \quad \frac{2}{t_1} = t_1$

Then slope of AB = slope of normal at $A = \pm\sqrt{2}$

3. Equation of normal to the parabola $x^2 = 4y$ having slope m is

$$x = my - 2m - m^3$$

Since it passes through the point (1, 2), we have

$$1 = 2m - 2m - m^3$$

$$\Rightarrow \quad m = -1$$

Hence, equation of normal is $x = -y + 2 + 1$ or $x + y - 3 = 0$.

4. Using the result of problem 1,

We have $h \geq 2a$, if three normals can be drawn to $y^2 = 4ax$ from (h, k)

Hence for the given question $c \geq 2(1/4)$ or $c \geq \frac{1}{2}$

Alternative Method:

We know that from any point normal to $y^2 = 4ax$ is given by $y = mx - 2am - am^3$, here $a = \frac{1}{4}$,

$\therefore$ Normal is $y = mx - \frac{m}{2} - \frac{m^3}{4}$

This normal passes through $(c, 0)$

$$\Rightarrow \quad mc - \frac{m}{2} - \frac{m^3}{4} = 0 \quad \text{(i)}$$

$$\Rightarrow \quad m\left[c - \frac{1}{2} - \frac{m^2}{4}\right] = 0$$

$$\Rightarrow \quad m = 0$$

or $$m^2 = 4\left(c - \frac{1}{2}\right)$$

$m = 0$ shows normal is $y = 0$, i.e., x-axis is always a normal

Also $$m^2 \geq 0$$

$$\Rightarrow \quad 4\left(c - \frac{1}{2}\right) \geq 0$$

$$\Rightarrow \quad c \geq \frac{1}{2}$$

At $c = \frac{1}{2}$, from Eq. (i), $m = 0$

Therefore, for other real value of m, $c > \frac{1}{2}$.

Now for other two normals to be perpendicular to each other, we must have $m_1 \cdot m_2 = -1$.

$\Rightarrow$ Product of roots of the equation

$$\frac{m^2}{4} + \frac{1}{2} - c = 0 \text{ is } -1.$$

$$\Rightarrow \quad \frac{\left(\frac{1}{2} - c\right)}{\frac{1}{4}} = -1$$

$$\Rightarrow \quad \frac{1}{2} - c = -\frac{1}{4}$$

$$\Rightarrow \quad c = \frac{3}{4}$$

5.

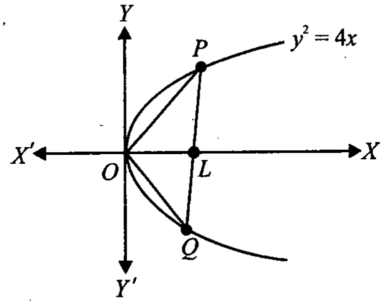


Fig. 3.139

Chord joining $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ subtends right angle at origin.

$$\Rightarrow \quad t_1 t_2 = -4$$

Also slope of the chord $= \frac{2}{t_1 + t_2}$

$\Rightarrow$ Equation of chord PQ is

$$y - 2t_1 = \frac{2}{t_1 + t_2}(x - t_1^2)$$

or $$(t_1 + t_2)y - 2t_1^2 - 2t_1 t_2 = 2(x - t_1^2)$$

or $$(t_1 + t_2)y - 2t_1^2 + 8 = 2(y - t_1^2)$$

or $$(t_1 + t_2)y + 8 = 2x$$

or $$2(x - 4) = (t_1 + t_2)y$$

which always passes through point $(4, 0)$.

If (h, k) is midpoint of PQ, then

$$h = \frac{t_1^2 + t_2^2}{2} \text{ and } k = t_1 + t_2$$

$$\Rightarrow \quad h = \frac{(t_1 + t_2)^2 - 2(t_1 t_2)}{2}$$

$$= \frac{k^2 + 8}{2}$$

or $$y^2 = 2(x - 4)$$

6. Let $P\left(t_1^2, 2t_1\right)$ and $Q\left(t_2^2, 2t_2\right)$ be the ends of the chord PQ of the parabola

$$y^2 = 4x. \quad \text{(i)}$$

Therefore, slope of chord $PQ = \frac{2}{t_2 + t_1} = 2$

$$\Rightarrow \quad t_2 + t_1 = 1 \quad \text{(ii)}$$

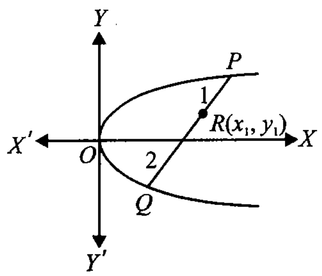


Fig. 3.140

If $R(x_1, y_1)$ is a point dividing PQ internally in the ratio 1 :2, then

$$x_1 = \frac{t_2^2 + 2t_1^2}{1+2},$$

$$y_1 = \frac{1(2t_2) + 2(2t_1)}{1+2}$$

$$\Rightarrow \quad t_2^2 + 2t_1^2 = 3x_1 \qquad \text{(iii)}$$

and $$t_2 + 2t_1 = \frac{(3y_1)}{2} \qquad \text{(iv)}$$

From Eqs. (ii) and (iv), we get

$$t_1 = \frac{3}{2}y_1 - 1, t_2 = 2 - \frac{3}{2}y_1$$

Substituting in Eq. (iii), we get

$$\left(2 - \frac{3}{2}y_1\right)^2 + 2\left(\frac{3}{2}y_1 - 1\right)^2 = 3x_1$$

$$\Rightarrow \quad \left(\frac{9}{4}\right)y_1^2 - 4y_1 = x_1 - 2$$

$$\left(y_1 - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x_1 - \frac{2}{9}\right)$$

Therefore, locus of the point $R(x_1, y_1)$ is

$$\left(y - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x - \frac{2}{9}\right)$$

which is a parabola having vertex at the point $\left(\frac{2}{9}, \frac{8}{9}\right)$.

7. Let the three points on the parabola $y^2 = 4ax$ be $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$.
Then equations of tangents at A, B and C are

$$y = \frac{x}{t_1} + at_1 \qquad \text{(i)}$$

$$y = \frac{x}{t_2} + at_2 \qquad \text{(ii)}$$

$$y = \frac{x}{t_3} + at_3 \qquad \text{(iii)}$$

Solving the above equations pairwise, we get the points $P(at_1t_2, a(t_1 + t_2))$, $Q(at_2t_3, a(t_2 + t_3))$, $R(at_3t_1, a(t_3 + t_1))$

$$\text{Now area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

$$= \left|a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right| \qquad \text{(iv)}$$

$$\text{Also area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} 1 & a_1t_2 & a(t_1 + t_2) \\ 1 & a_2t_3 & a(t_2 + t_3) \\ 1 & a_3t_1 & a(t_3 + t_1) \end{vmatrix}$$

$$= \frac{a^2}{2}\begin{vmatrix} 1 & t_1t_2 & t_1 + t_2 \\ 1 & t_2t_3 & t_2 + t_3 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

$$= \frac{a^2}{2}\begin{vmatrix} 0 & (t_1 - t_3)t_2 & t_1 - t_3 \\ 0 & (t_2 - t_1)t_3 & t_2 - t_1 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

Expanding along C_1,

$$= \left|\frac{a^2}{2}(t_1 - t_3)(t_2 - t_1)(t_2 - t_3)\right| \qquad \text{(v)}$$

From Eqs. (iv) and (v), we get

$$\frac{\text{Ar}(\Delta ABC)}{\text{Ar}(\Delta ABC)} = \frac{a^2 \left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|}{\frac{a^2}{2}\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|}$$

$$= \frac{2}{1}$$

Therefore, the required ratio is 2:1.

8. Equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}$$

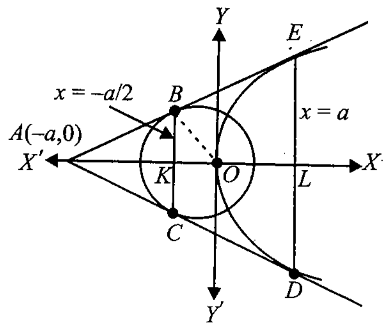


Fig. 3.141

This line will touch the circle $x^2 + y^2 = \frac{a^2}{2}$

if $$\frac{a}{m} = \pm \frac{a}{\sqrt{2}}\sqrt{m^2 + 1}$$

$$[c = \pm r\sqrt{1 + m^2}]$$

$$\Rightarrow \quad \frac{a^2}{m^2} = \frac{a^2}{2}(m^2 + 1)$$

$$\Rightarrow \quad 2 = m^4 + m^2$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = 1, -1$

Thus, the two tangents (common one) are $y = x + a$ and $y = -x - a$

These two intersect each other at $(-a, 0)$.

The chord of contact of circle w.r.t. $A(-a, 0)$ is

$$(-a)x + (0)y = \frac{a^2}{2}$$

or $\quad x = -\frac{a}{2}$

and the chord of contact of parabola w.r.t. $A(-a, 0)$ is

$$(0)y = 2a(x - a)$$

or $\quad x = a$

Note that DE is latus return of parabola $y^2 = 4ax$, therefore, its length is $4a$.

Chords of contact are clearly parallel to each other, so required quadrilateral is a trapezium.

$$\text{Ar(trap } BCDE) = \frac{1}{2}(BC + DE) \times KL$$

$$= \frac{1}{2}(a + 4a)\left(\frac{3a}{2}\right)$$

$$= \frac{15a^2}{4}$$

9. Point of intersection of tangents at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$R(h, k) \equiv (at_1t_2, a(t_1 + t_2))$$

$\Rightarrow \quad t_1 + t_2 = \frac{k}{a}$ and $t_1t_2 = \frac{h}{a}$

Now $\quad \tan 45° = \left|\dfrac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}\right|$

$\Rightarrow \quad 1 = \left|\dfrac{t_2 - t_1}{t_1t_2 + 1}\right|$

$$= \left|\frac{\sqrt{(t_1 + t_2)^2 - 4t_1t_2}}{t_1t_2 + 1}\right|$$

$\Rightarrow \quad 1 = \left|\dfrac{\sqrt{\frac{k^2}{a^2} - 4\frac{h}{a}}}{\frac{h}{a} + 1}\right|$

$\Rightarrow \quad k^2 - 4ah = (h + a)^2$

$\Rightarrow \quad x^2 - y^2 + 6ay + a^2 = 0$

which is parabola.

10. Given that $C_1 : x^2 = y - 1$, $C_2 : y^2 = x - 1$

Here C_1 and C_2 are symmetrical about the line $y = x$.

Let $P(x_1, x_1^2 + 1)$ on C_1 and $Q(y_2^2 + 1, y_2)$ on C_2

Then image of P in $y = x$ is $P_1(x_1^2 + 1, x_1)$ on C_2 and image of Q in $y = x$ is $Q_1(y_2, y_2^2 + 1)$ on C_1.

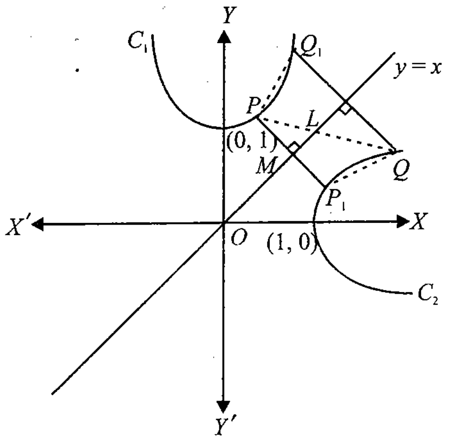


Fig. 3.142

Now PP_1 and QQ_1 both are perpendicular to mirror line $y = x$.

Also M is midpoint of PP_1

($\because P_1$ is mirror image of P in $y = x$)

$\therefore \quad PM = \frac{1}{2}PP_1$

In ΔPML, $\quad PL > PM$

$\Rightarrow \quad PL > \frac{1}{2}PP_1 \quad$ (i)

Similarly, $\quad LQ > \frac{1}{2}QQ_1 \quad$ (ii)

Adding Eqs. (i) and (ii), we get

$$PL + LQ > \frac{1}{2}(PP_1 + QQ_1)$$

$\Rightarrow \quad PQ > \frac{1}{2}(PP_1 + QQ_1)$

$\Rightarrow PQ$ is more then mean of PP_1 and QQ_1.

$\Rightarrow \quad PQ \geq \min(PP_1, QQ_1)$

Let $\min(PP_1, QQ_1) = PP_1$

Then

$$PQ^2 \geq PP_1^2 = \left(x_1^2 + 1 - x_1\right)^2 + \left(x_1^2 + 1 - x_1\right)^2$$

$$= 2\left(x_1^2 + 1 - x_1\right)^2$$

$$= f(x_1)$$

$\Rightarrow$ $$f'(x_1) = 4(x_1^2 + 1 - x_1)(2x_1 - 1)$$

$$= 4\left(\left(x_1 - \frac{1}{2}\right)^2 + \frac{3}{4}\right)(2x_1 - 1)$$

$\therefore$ $f'(x_1) = 0$ when $x_1 = \frac{1}{2}$

Also $f'(x_1) < 0$ if $x_1 < \frac{1}{2}$

and $f'(x_1) > 0$ if $x_1 > \frac{1}{2}$

$\Rightarrow f(x_1)$ is minimum when $x_1 = \frac{1}{2}$

Thus, at $x_1 = \frac{1}{2}$, point P is P_0 on C_1

$$P_0\left(\frac{1}{2}, \left(\frac{1}{2}\right)^2 + 1\right) \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$$

Similarly, Q_0 on C_2 will be image of P_0 with respect to $y = x$.

$\therefore$ $$Q_0 \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$$

11. Equation of normal to parabola $y^2 = 4x$ having slope m is

$$y = mx - 2m - m^3$$

It passes through the point $P(h, k)$

$\Rightarrow$ $$mh - k - 2m - m^3 = 0$$

$\Rightarrow$ $$m^3 + (2 - h)m + k = 0 \quad \text{(i)}$$

which is cubic in m and has three roots such that product of roots

$$m_1 m_2 m_3 = -k \quad \text{(from Eq. (i))}$$

But given that $m_1 m_2 = \alpha$

$\therefore$ $$m_3 = -\frac{k}{\alpha}$$

But m_3 must satisfy Eq. (i)

$\Rightarrow$ $$\frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$$

$\Rightarrow$ $$k^2 - 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$$

$\therefore$ Locus of $P(h, k)$ is $y^2 = \alpha^2 x + (\alpha^3 - 2\alpha^2)$

But given that, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get

$$\alpha^2 = 4 \text{ and } \alpha^3 - 2\alpha^2 = 0$$

$\Rightarrow$ $$\alpha = 2$$

12. Parabola is $(y - 1)^2 = 4(x - 1)$ whose directrix is y-axis or $x = 0$.

Any point on this parabola is $P(t^2 + 1, 2t + 1), t \in R$

$\therefore$ Equation of tangent at $P(t^2 + 1, 2t + 1)$ is

$$t(y - 1) = (x - 1) + t^2$$

or $$x - ty + (t^2 + t - 1) = 0 \quad \text{(i)}$$

$\Rightarrow$ Tangent meets directrix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Now Q is mid point of PR

$\therefore$ $$\frac{h + t^2 + 1}{2} = 0$$

and $$\frac{k + 2t + 1}{2} = \frac{t^2 + t - 1}{t}$$

$\Rightarrow$ $$t^2 = -1 - h$$

and $$kt + 2t^2 + t = 2t^2 + 2t - 2$$

$\Rightarrow$ $$t^2 = -1 - h$$

and $$2 = t(1 - k)$$

Eliminating t, we have

$$4 = (-1 - h)(1 - k)^2$$

or $$(x + 1)(y - 1)^2 + 4 = 0$$

Objective Type

Fill in the blanks

1. Tangents at the extremities of the focal chord intersect on directrix and tangents at the end of latus rectum intersect at foot of directrix $(-1, 0)$.

Multiple choice questions with one correct answer

1. a. The focus of parabola $y^2 = 2px$ is $\left(\frac{p}{2}, 0\right)$ and directrix $x = -\frac{p}{2}$.

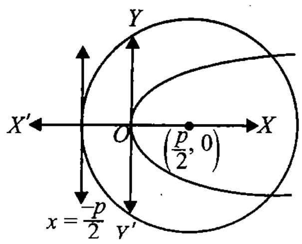


Fig. 3.143

$\therefore$ Centre of circle is $\left(\frac{p}{2}, 0\right)$ and radius $= \frac{p}{2} + \frac{p}{2} = p$

$\therefore$ Equation of circle is $\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$

or $$4x^2 + 4y^2 - 4px - 3p^2 = 0$$

Solving this circle with the given parabola, we have (eliminating y)

$$4x^2 + 8px - 4px - 3p^2 = 0$$

$\Rightarrow \quad 4x^2 + 4px - 3p^2 = 0$

$\Rightarrow \quad (2x + 3p)(2x - p) = 0$

$\Rightarrow \quad x = \frac{-3p}{2}, \frac{p}{2}$

$\Rightarrow \quad y^2 = -3p^2$ (not possible),

$\Rightarrow \quad y^2 = 2p.\frac{p}{2} \Rightarrow = \pm p$

Therefore, required points are $\left(\frac{p}{2}, p\right), \left(\frac{p}{2}, -p\right)$.

2. c. $\frac{x+y}{2} = t^2 + 1, \frac{x-y}{2} = t$

Eliminating t, $2(x + y) = (x - y)^2 + 4$

Since 2nd degree terms form a perfect square, it represents a parabola (also $\Delta \neq 0$).

3. b. $y = mx + c$ is normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$

Here $m = -1$ and $c = k$ and $a = 3$

$\therefore \quad c = k = -2(3)(-1) - 3(-1)^3$

$= 9$

4. c. $y^2 = kx - 8$

$\Rightarrow \quad y^2 = k\left(x - \frac{8}{k}\right)$

Directrix of parabola is $x = \frac{8}{k} - \frac{k}{4}$

Now $x = 1$ coincides with $x = \frac{8}{k} - \frac{k}{4}$

$\Rightarrow \quad \frac{8}{k} - \frac{k}{4} = 1$, we get $k = 4$

5. c. Equation of tangent to the given parabola having slope m is

$$y = mx + \frac{1}{m} \qquad \text{(i)}$$

Equation of tangent to the given circle having slope m is

$$y = m(x - 3) \pm 3\sqrt{1 + m^2} \qquad \text{(ii)}$$

Equations (i) and (ii) are identical,

$\Rightarrow \quad \frac{1}{m} = -3m \pm 3\sqrt{1 + m^2}$

$\Rightarrow \quad 1 + 3m^2 = \pm 3m\sqrt{1 + m^2}$

$\Rightarrow \quad 1 + 6m^2 + 9m^4 = 9(m^2 + m^4)$

$\Rightarrow \quad 3m^2 = 1$

$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$

Hence, equation of common tangent is $\sqrt{3}y = x + 3$ (as tangent is lying above x-axis).

6. d. $y^2 + 4y + 4x + 2 = 0$

$y^2 + 4y + 4 = -4x + 2$

$(y + 2)^2 = -4(x - 1/2)$

It is of the form $Y^2 = 4AX$ whose directrix is given by $X = A$

$\therefore$ Required equation is $x - \frac{1}{2} = 1$

$\Rightarrow \quad x = \frac{3}{2}$

7. c. If (h, k) is the midpoint of line joining focus $(a, 0)$ and $Q(at^2, 2at)$ on parabola then

$$h = \frac{a + at^2}{2} \quad k = at$$

Eliminating t, we get,

$2h = a + a\left(\frac{k^2}{a^2}\right)$

$\Rightarrow \quad k^2 = a(2h - a)$

$\Rightarrow \quad k^2 = 2a(h - \frac{a}{2})$

Therefore, locus of (h, k) is

$$y^2 = 2a(x - \frac{a}{2})$$

whose directrix is $(x - \frac{a}{2}) = -\frac{a}{2}$

$\Rightarrow \quad x = 0$

8. a. For parabola $y^2 = 16x$, focus $= (4, 0)$. Let m be the slope of focal chord, then equation is

$$y = m(x - 4) \qquad \text{(i)}$$

Given that above line is a tangent to the circle $(x - 6)^2 + y^2 = 2$ for which centre $C(6, 0)$ and radius $r = \sqrt{2}$

Therefore, length of perpendicular from $(6, 0)$ to Eq. (i) $= r$

$\Rightarrow \quad \frac{|6m - 4m|}{\sqrt{m^2 + 1}} = \sqrt{2}$

$\Rightarrow \quad 2m^2 = m^2 + 1$

$\Rightarrow \quad m^2 = 1$

$\Rightarrow \quad m = \pm 1$

9. c. $y = mx + \frac{1}{m}$

Above tangent passes through $(1, 4)$

$\Rightarrow \quad 4 = m + \frac{1}{m}$

$\Rightarrow \quad m^2 - 4m + 1 = 0$

Now angle between the lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \frac{\sqrt{16 - 4}}{1 + 1} = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Alternative Solution:

The combined equation of tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is

$$(y^2 - 4x)(4^2 - 4 \times 1) = [y \times 4 - 2(x + 1)]^2$$

[Using $SS_1 = T^2$]

$$\Rightarrow \quad 12(y^2 - 4x) = 4(2y - x - 1)^2$$

$$\Rightarrow \quad 3(y^2 - 4x) = 4y^2 + x^2 + 1 - 4xy + 2x - 4y$$

$\Rightarrow x^2 + y^2 - 4xy + 14x - 4y + 1 = 0$

Now we know angle between two lines, given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

∴ The required angle is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{2^2 - 1 \times 1}}{1 + 1}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \pi/3$$

10. d. The given curve is

$$y = x^2 + 6$$

Equation of tangent at (1, 7) is

$$\frac{1}{2}(y + 7) = x(1) + 6$$

$$\Rightarrow \quad 2x - y + 5 = 0 \quad \text{(i)}$$

According to question, this tangent Eq. (i) touches the circle $x^2 + y^2 + 16x + 12y + C = 0$ at Q (centre of circle (−8, − 6))

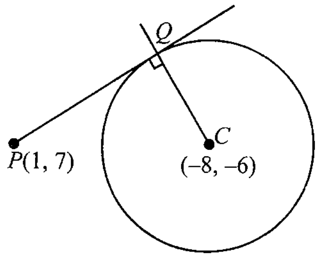


Fig. 3.144

Then equation of CQ which is perpendicular to Eq. (i) and passes through (−8, − 6) is

$$y + 6 = -\frac{1}{2}(x + 8)$$

$$\Rightarrow \quad x + 2y + 20 = 0 \quad \text{(ii)}$$

Now Q is point of intersection of Eqs. (i) and (ii),

i.e., $x = -6, y = -7$

Therefore, required point is (−6, − 7).

11. d. Vertex is (1, 1), focus (2, 2), directrix $x + y = 0$

∴ Equation of parabola is

$$(x - 2)^2 + (y - 2)^2 = \left(\frac{x + y}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 - 2xy = 8(x + y - 2)$$

$$\Rightarrow \quad (x - y)^2 = 8(x + y - 2)$$

12. b. Solving the curves, we have

$$x^2 + 4x - 6x + 1 = 0$$

or $(x - 1)^2 = 0$

or $x = 1.$

Hence, two curves touch each other. For $x = 1, y = \pm 2$.

The circle and the parabola touch each other at the point (1, 2) and (1, − 2) as shown in the figure.

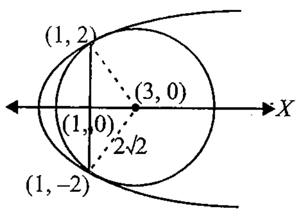


Fig. 3.145

13. c. $y^2 = x$

and Q will lie on it

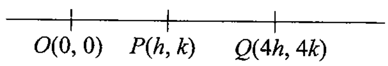


Fig. 3.146

$\Rightarrow (4k)^2 = 4 \times 4h$

$\Rightarrow y^2 = x$ (replacing h by x and k by y)

Multiple choice questions with one or more than one correct answer

1. a., b. If $y = mx + c$ is tangent to $y = x^2$, then $x^2 - mx - c = 0$ has equal roots.

$\Rightarrow \quad m^2 + 4c = 0$

$\Rightarrow \quad c = -\frac{m^2}{4}$

$\therefore \quad y = mx - \frac{m^2}{4}$

is tangent to $y = x^2$

$\because$ This is also tangent to $y = -(x-2)^2$

$\Rightarrow mx - \frac{m^2}{4} = -x^2 + 4x - 4$ has equal roots.

$\Rightarrow x^2 + (m-4)x + \left(4 - \frac{m^2}{4}\right) = 0$ has equal roots.

$\Rightarrow \quad (m-4)^2 - 4\left(4 - \frac{m^2}{4}\right) = 0$

$\Rightarrow \quad m^2 - 8m + 16 + m^2 - 16 = 0$

$\Rightarrow \quad m = 0, 4$

$\Rightarrow \quad y = 0$

or $\quad y = 4x - 4$ are the tangents.

2. b., c.

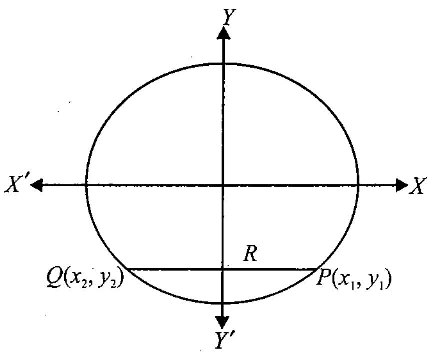


Fig 3.147

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

$\Rightarrow P\left(\sqrt{3}, -\frac{1}{2}\right)$ and $Q\left(-\sqrt{3}, -\frac{1}{2}\right)$ (given y_1 and y_2 less than 0).

Coordinates of midpoint of PQ are $R \equiv \left(0, -\frac{1}{2}\right)$

$$PQ = 2\sqrt{3}$$
$$= \text{length of latus rectum.}$$

$\Rightarrow$ Two parabola are possible whose vertices are $\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

Hence, the equations of the parabolas are

$$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$$

and $\quad x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. a., d.

Tangent at point $P(at^2, 2at)$ is

$$ty = x + at^2.$$

It meets the x-axis at $(-at^2, 0)$

Normal at point P is $y = -tx + 2at + at^3$

It meets the x-axis at $(2a + at^2, 0)$

Let centroid of triangle PNT is $G \equiv (h, k)$

$\Rightarrow \quad h = \frac{2a + at^2}{3}$ and $k = \frac{2at}{3}$

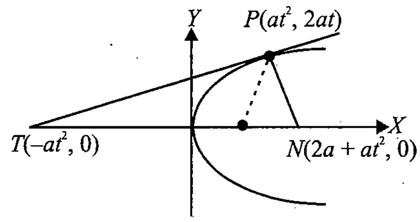


Fig. 3.148

$\Rightarrow \quad \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$

$\Rightarrow$ Required parabola is

$$\frac{9y^2}{4a^2} = \frac{(3x - 2a)}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$$

$\Rightarrow \quad y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

Vertex $\equiv \left(\frac{2a}{3}, 0\right)$; focus $\equiv (a, 0)$

4. c., d.

$$A = (t_1^2, 2t_1) B = (t_2^2, 2t_2)$$

Centre $= \left[\frac{t_1^2 + t_2^2}{2}, (t_1 + t_2)\right]$

$t_1 + t_2 = \pm r$

$$m = \frac{2(t_1 - t_2)}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm\frac{2}{r}.$$

5. a., b., d.

$y^2 = 4x$

Equation of normal is $y = mx - 2m - m^3$.

It passes through $(9, 6)$

$\Rightarrow m^3 - 7m + 6 = 0$

$\Rightarrow m = 1, 2, -3$

$\Rightarrow y - x + 3 = 0, y + 3x - 33 = 0, y - 2x + 12 = 0$

Match the following

1. i. → a.; ii → b.; iii → d.; iv → c.

Equation of normal to $y^2 = 4x$ is

$$y = mx - 2m - m^3$$

As it passes through (3, 0), we get $m = 0, 1, -1$

Then three points on parabola are given by $(m^2, -2m)$ for $m = 0, 1, -1$

$\therefore$ $P(0, 0)$, $Q(1, -2)$, $R(1, 2)$

$\therefore$ $$\text{Area of } \Delta PQR = \frac{1}{2} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2 \text{ sq. units}$$

$$R = \frac{abc}{4\Delta} = \frac{\sqrt{5} \times \sqrt{5} \times 4}{4 \times 2} = \frac{5}{2}$$

Centroid of $\Delta PQR = \left(\frac{2}{3}, 0\right)$, (where a, b, c are the sides of ΔPQR)

Circumcentre $= \left(\frac{5}{2}, 0\right)$

Comprehension based questions

Sol. Solving circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$

$$x^2 + 8x - 9 = 0$$

$\Rightarrow$ $x = 1, -9$

$\Rightarrow$ $x = 1$ ($x = -9$ not possible)

$\Rightarrow$ $y^2 = 8$

$\Rightarrow$ $y = \pm 2\sqrt{2}$

Hence, points of intersection are $P(1, 2\sqrt{2})$ and $Q(1, -2\sqrt{2})$.

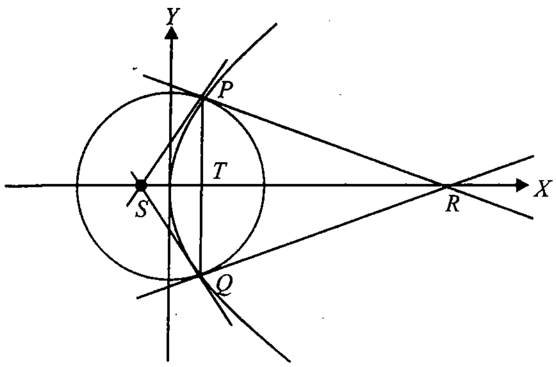


Fig. 3.149

Tangent to the parabola at point P is

$$2\sqrt{2}y = 4(x + 1)$$

It meets the x-axis at $S(-1, 0)$.

Tangent to the circle at point P is $(1)x + 2\sqrt{2}\,y = 9$

It meets the x-axis at $R(9, 0)$.

1. c. $$\frac{\text{Ar}(\Delta PQS)}{\text{Ar}(\Delta PQR)} = \frac{\frac{1}{2}PQ \times ST}{\frac{1}{2}PQ \times TR}$$

$$= \frac{ST}{TR} = \frac{2}{8} = \frac{1}{4}$$

2. b. For ΔPRS,

$$\text{Ar}(\Delta PRS) = \Delta = \frac{1}{2} \times SR \times PT$$

$$= \frac{1}{2} \times 10 \times 2\sqrt{2}$$

$\therefore$ $$\Delta = 10\sqrt{2},\ a = PS = 2\sqrt{3}$$

$$b = PR = 6\sqrt{2},\ c = SR = 10$$

$\therefore$ Radius of circumference $= R = \frac{abc}{2\Delta}$

$$= \frac{2\sqrt{3} \times 6\sqrt{2} \times 10}{4 \times 10\sqrt{2}} = 3\sqrt{3}$$

3. d. Radius of incircle of triangle PQR is

$$= \frac{\text{Area of } \Delta PQR}{\text{Semi-perimeter of } \Delta PQR}$$

$$= \frac{\Delta}{s}$$

We have $a = PR = 6\sqrt{2}$, $b = QP = PR = 6\sqrt{2}$ and $c = PQ = 4\sqrt{2}$

Also $$\Delta = \frac{1}{2} \times PQ \times TR = 16\sqrt{2}$$

$\therefore$ $$s = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$$

$\therefore$ $$r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Assertion and reasoning

1. a. The given curve is

$$y = -\frac{x^2}{2} + x + 1$$

or $$(x - 1)^2 = -2\left(y - \frac{3}{2}\right)$$

which is a parabola, so should be symmetric with respect to its axis $x - 1 = 0$.

$\therefore$ Both the statement are true and statement 2 is a correct explanation for statement 1.

Integer type

1. (2)

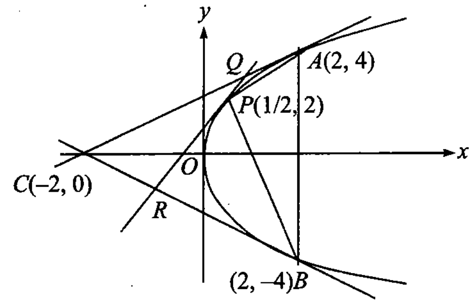


Fig. 3.150

$y^2 = 8x = 4 \cdot 2 \cdot x$

Tangents at end points of latus- rectum meet on directrix on x-axis $(-2, 0)$

Δ_1 = Area of ΔABC

$= \frac{1}{2}(8)\left(2 - \frac{1}{2}\right) = 6$

Δ_2 = Area of ΔCQR

Now tangent at point $A(2, 4)$ is $4y = 4(x + 2)$ or $y = x + 2$

Tangent at point $(2, -4)$ is $-4y = 4(x + 2)$ or $y = -x - 2$

Also tangent at point $P(1/2, 2)$ is $2y = 4(x + 1/2)$ or $y = 2x + 1$

Solving for Q and R we get $Q(1, 3)$ and $R(-1, -1)$

Hence area of ΔCQR is

$$\frac{1}{2}\left\|\begin{matrix} -1 & -1 & 1 \\ 1 & 3 & 1 \\ -2 & 0 & 1 \end{matrix}\right\| = \frac{1}{2}(-3+2+6+1) = 3$$

Hence the ratio of area is $6/3 = 2$

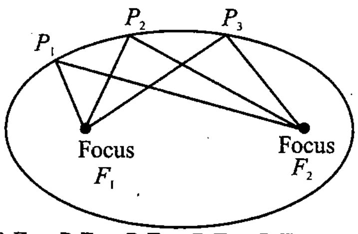
CHAPTER

4

Ellipse

- Ellipse: Definition 1
- Ellipse: Definition 2
- Auxiliary Circle and Eccentric Angle
- Intersection of a Line and an Ellipse
- Important Properties Related to Tangent
- Equation of Normal
- Chord of Contact
- Equation of Chord Joining Points $P(\alpha)$ and $Q(\beta)$
- Point of Intersection of Tangents at Points $P(\alpha)$ and $Q(\beta)$
- Equation of the Chord of the Ellipse whose Midpoint is (x_1, y_1)
- Concyclic Points

ELLIPSE: DEFINITION 1



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$

Fig. 4.1

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. The two fixed points are called the foci (plural of 'focus') of the ellipse. The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points. The midpoint of the line segment joining the foci is called the centre of the ellipse. The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis. The end points of the major axis are called the vertices of the ellipse.

Standard Equation of an Ellipse

Let the foci of an ellipse be $(\pm c, 0)$, then its centre is $(0, 0)$.

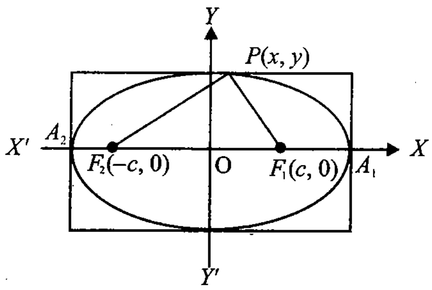


Fig. 4.2

According to the definition of an ellipse,

$$PF_1 + PF_2 = 2a \text{ (constant)}$$

$$\Rightarrow \quad \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \; (2a > 2c) \qquad \text{(i)}$$

$$\text{Now} [(x-c)^2 + y^2] - [(x+c)^2 + y^2] = -4cx \qquad \text{(ii)}$$

Dividing (ii) by (i), we get

$$\sqrt{(x^2-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = -\frac{2cx}{a} \qquad \text{(iii)}$$

$$\Rightarrow \quad \sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a}$$

Simplifying, we get $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$

Let $\quad a^2 - c^2 = b^2$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad$ (iv)

where $\quad a > b$

It is a 2nd degree equation with powers of x and y both even and hence is symmetric about both the x-and y-axis. The entire curve is confined within the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$.

Eccentricity

Degree of flatness of an ellipse is defined as

$$\text{Eccentricity } (e) = \frac{c}{a} = \frac{OF_1}{OA_1}$$

$$= \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

$$\therefore \quad e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} < 1$$

Now if $c \to 0$ (i.e. the two foci come closer and coalesce to form the centre), then,

$e \to 0 \Rightarrow b \to a$.

Hence, the ellipse gets thicker and $\to$ circle.

Again if $c \to a$ (i.e., the two foci tend to coincide with the vertex of the ellipse),

we have $e \to 1 \Rightarrow b \to 0$.

Hence, the ellipse gets thinner and thinner and tends to a line segment between the two foci.

Equation of an ellipse in terms of eccentricity becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

where $b^2 = a^2(1 - e^2)$ or $a^2e^2 = a^2 - b^2$ $(a > b)$.

Note:

1. ***Two ellipses are said to be similar if they have the same value of eccentricity.***
2. ***Distance of every focus from the extremity of minor axis is equal to a, as $b^2 + a^2e^2 = a^2$.***

Directrix

It is possible to define two lines $x = \pm \frac{a}{e}$ corresponding to each focus, which satisfy the focal directrix property of the ellipse, i.e., $PF_1 = ePM_1$ and $PF_2 = ePM_2$.

Hence, for any point P on the ellipse $\frac{PF_1}{PM_1} = e$ (constant)

Focal Distance

The sum of the focal radii of any point on the ellipse is equal to the length of the major axis.

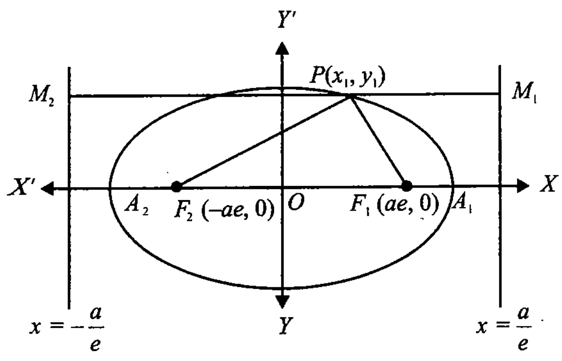


Fig. 4.3

We have

$$PF_1 = ePM_1 = e\left(\frac{a}{e} - x_1\right) = a - ex_1 \qquad \text{(i)}$$

$$PF_2 = ePM_2 = e\left(\frac{a}{e} + x_1\right) = a + ex_1 \qquad \text{(ii)}$$

(i) + (ii) gives $PF_1 + PF_2 = 2a$

Equation of an Ellipse Whose Axes are Parallel to Coordinate Axis and Centre is *(h, k)*

Equation of such an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, (a > b)$

Foci: $(h \pm ae, k)$

Directrix: $x = h \pm \frac{a}{e}$

Definition and Basic Terminology of Ellipse

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

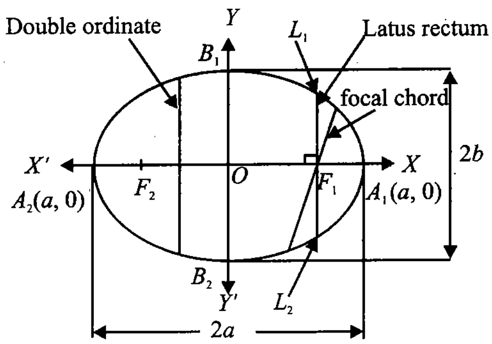


Fig. 4.4

- The line containing the two fixed points (called foci) is called the focal axis (major axis) and points of intersection of the curve with focal axis are called the vertices of the ellipse, i.e., A $(a, 0)$ and A_2 $(-a, 0)$. The distance between F_1 and F_2 is called the focal length. The distance between the two vertices, i.e., $2a$ is called the major axis.

  The distance $2b$, i.e., B_1B_2 is called the minor axes.

- Point of intersection of the major and minor axes is called the centre of the ellipse. Any chord of the ellipse passing through it gets bisected by it and is called the diameter. Major and minor axes together are known as principal axes of the ellipse.
- Any chord through focus is called a focal chord and any chord perpendicular to the focal axis is called double ordinate.
- A particular double ordinate through focus and perpendicular to focal axis is called its *latus rectum*.

Latus-rectum Length

The two foci are $(\pm ae, 0)$.

Putting $x = ae$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - e^2 = 1 - \left(1 - \frac{b^2}{a^2}\right) = \frac{b^2}{a^2};$$

$$\therefore \quad y = \pm\frac{b^2}{a}$$

$\therefore$ Coordinate of the extremities of the latus rectum $= \left(\pm ae, \pm\frac{b^2}{a}\right)$

Length of latus rectum $= \frac{2b^2}{a} = 2a(1 - e^2)$

$$= \frac{4b^2}{2a} = \frac{(\text{Minor axis})^2}{\text{Major axis}}$$

Also $\quad L_1L_2 = 2a(1 - e^2) = 2e\left(\frac{a}{e} - ae\right)$

$= 2e$ (distance between focus and corresponding foot of the directrix).

Tracing Out Ellipse

Method 1

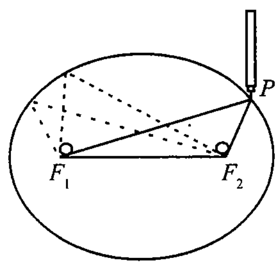


Fig. 4.5

We know that $\quad PF_1 + PF_2 = \text{constant}$

Hence $\quad PF_1 + PF_2 + F_1F_2 = \text{constant}$

Stick two drawing-pins into a board (though not pressed too far in) and slip the loop of thread over them. Insert a pencil point in the loop and position it so that the thread is tight. Move the pencil round the pins, always keeping the thread tight and thus trace out an ellipse.

Method 2

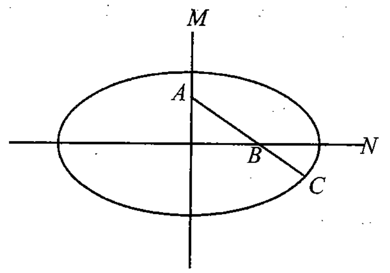


Fig 4.6

Draw two perpendicular lines M, N on the paper; these will be the major and minor axes of the ellipse. Mark three points A, B, C on the ruler. With one hand, move the ruler onto the paper, turning and sliding it so as to keep point A always on line M, and B on line N. With the other hand, keep the pencil's tip on the paper, following point C of the ruler. The tip will trace out an ellipse.

Position of a Point (h, k) with Respect to an Ellipse

Let an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $\qquad P \equiv (h, k)$

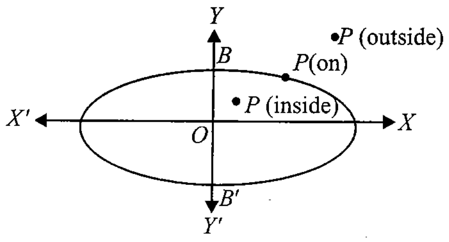


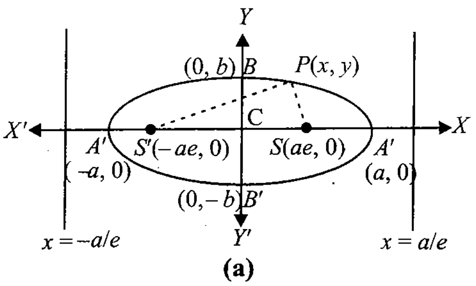
Fig. 4.7

Now P will lie outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to as $\frac{h^2}{a^2}+\frac{k^2}{b^2}-1>,=,<0$

Comparison of Standard Equation of an Ellipse when $a > b$ and $a < b$

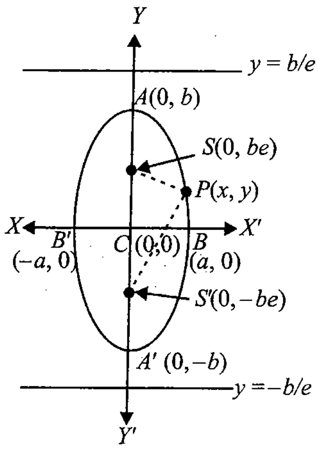
Equation of ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a > b$



(a)

$b > a$



(b)

Fig. 4.8

	$a > b$	$a < b$
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of the latus rectum	$2b^2/a$	$2a^2/b$
Ends of latus rectum	$(\pm ae, \pm b^2/a)$	$(\pm a^2/b, \pm be)$
Focal radii of any point $P(x_1, y_1)$ on ellipse	$PS = a - ex_1$ and $PS' = a + ex_1$	$PS = b - ey_1$ and $PS' = b + ey_1$
Sum of focal radii $SP + S'P$	$2a$	$2b$

Example 4.1 **A ladder 12 units long slides in a vertical plane with its ends in contact with a vertical wall and a horizontal floor along x-axis. Find the locus of a point on the ladder 4 units from its foot.**

Sol.

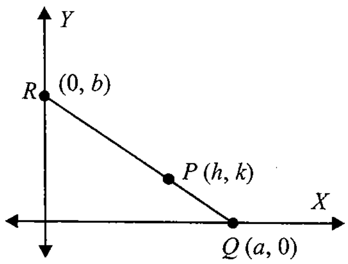


Fig. 4.9

Given $a^2 + b^2 = 12^2 = 144$ (i)

also $\frac{PR}{PQ} = \frac{8}{4} = \frac{2}{1}$

$\Rightarrow$ $h = \frac{2a}{3}$ and $k = \frac{b}{3}$

$\Rightarrow$ $a = \frac{3h}{2}$ and $b = 3k$

From (i), we get $\frac{9h^2}{4} + 9k^2 = 144$

$\Rightarrow$ $\frac{x^2}{64} + \frac{y^2}{16} = 1$

which is an ellipse.

Example 4.2 **Two circles are given such that one is completely lying inside other without touching. Prove that the locus of the centre of variable circle which touches smaller circle from outside and bigger circle from inside is an ellipse.**

Sol.

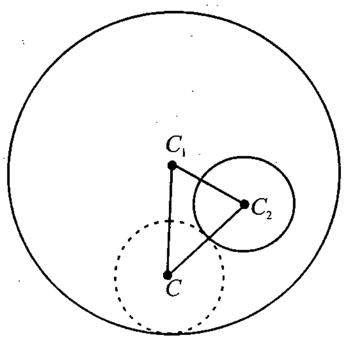


Fig. 4.10

In the figure, circles with hard line are given circles with centres C_1 and C_2 and radius r_1 and r_2 respectively. Let the circle with dotted line is a variable circle which touches given two circles as explained in the question which has centre C and radius r.

Now $CC_2 = r + r_2$ and $CC_1 = r_1 - r$

Hence, $CC_1 + CC_2 = r_1 + r_2 = \text{(constant)}$

Then locus of C is ellipse whose foci are C_1 and C_2.

Example 4.3 **Coordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0), respectively. The vertex A is moving in such a way that $4 \tan\frac{B}{2} \tan\frac{C}{2} = 1$. Then find the locus of A.**

Sol. $4 \tan\frac{B}{2} \tan\frac{C}{2} = 1$

$\Rightarrow \sqrt{\frac{(s-c)(s-a)}{s(s-b)} \times \frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{4}$

$\Rightarrow$ $\frac{s-a}{s} = \frac{1}{4}$

$\Rightarrow$ $\frac{2s-a}{a} = \frac{5}{3}$

$\Rightarrow$ $b + c = \frac{5}{3} \times 6 = 10$

$(\because a = BC = 6)$

Thus, sum of distance of variable point A from two given fixed points B and C is always 10, therefore equation of locus of A is an ellipse. Also centre is midpoint of BC, which is (5, 0).

$2ae = BC = 6$ and sum of focal distance for any point on the ellipse is 10.

Hence, $e = \frac{6}{10} = \frac{3}{5}$.

Length of semi-major axis = 5.

Length of semi-minor axis = 4.

Hence, equation of ellipse is $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$.

Example 4.4 **Find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose latus rectum is half of its major axis.**

Sol. Let $a > b$, then latus rectum of the ellipse is $\frac{2b^2}{a}$ and semi-major axis is a.

Given $\frac{2b^2}{a} = a$

$\Rightarrow$ $2b^2 = a^2$

Also for the ellipse $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $2a^2(1 - e^2) = a^2$

$\Rightarrow$ $e = \frac{1}{\sqrt{2}}$

Example 4.5 **Find the equation of the ellipse (referred to its axes as the axes of x and y, respectively) whose foci are (± 2, 0) and eccentricity $\frac{1}{2}$.**

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given $e = \frac{1}{2}$.

Also foci of the ellipse are $(\pm ae, 0) \equiv (\pm 2, 0)$

$\Rightarrow$ $ae = 2 \Rightarrow a = 4$.

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 12$

Thus, the required ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

Example 4.6 **If $P(x, y)$ be any point of the ellipse $16x^2 + 25y^2 = 400$ and $F_1 = (3, 0)$, $F_2 = (-3, 0)$, then find the value of $PF_1 + PF_2$.**

Sol. We have $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow \qquad a^2 = 25$ and $b^2 = 16$

Then, eccentricity is given by $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{16}{25} = \frac{9}{25}$

$\Rightarrow \qquad e = \frac{3}{5}$

So, the coordinates of the foci are $(\pm ae, 0)$ or $(\pm 3, 0)$.

Thus, F_1 and F_2 are the foci of the ellipse.

Since, the sum of the focal distances of a point on an ellipse is equal to its major axis, $PF_1 + PF_2 = 2a = 10$.

Example 4.7 **If the focal distance of an end of the minor axis of an ellipse (referred to its axes as the axes of x and y, respectively) is k and distance between its foci is $2h$, then find its equation.**

Sol. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It is given that distance between foci is $2h$

$\Rightarrow \qquad 2ae = 2h$

$\Rightarrow \qquad ae = h \qquad$ (i)

Focal distance of one end of minor axis is $a = k$

$\Rightarrow \qquad b^2 = a^2 - a^2e^2 = k^2 - h^2$

So, the equation of the ellipse is $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$.

Example 4.8 **If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then find the eccentricity of the ellipse.**

Sol. If two foci are S (5, 12) and S' (24, 7) and the ellipse passes through origin O.

Then $\qquad SO = \sqrt{25 + 144} = 13;$

$S'O = \sqrt{576 + 49} = 25$ and $SS' = \sqrt{386}$

If the conic is an ellipse, then $SO + S'O = 2a$ and

$SS' = 2ae$

$\therefore \qquad e = \frac{SS'}{S'O + SO} = \frac{\sqrt{386}}{38}$

Example 4.9 **Find the centre, foci, the length of the axes and the eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.**

Sol. The given equation can be rewritten as

$$2(x^2 - 2x) + 3(y^2 - 4y) + 13 = 0$$

$$\Rightarrow \quad 2(x - 1)^2 + 3(y - 2)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{(1/\sqrt{2})^2} + \frac{(y-2)^2}{(1/\sqrt{3})^2} = 1,$$

$$\Rightarrow \quad \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$\Rightarrow$ Centre $\qquad X = 0, Y = 0$, i.e. (1, 2)

Length of major axis $= 2a = \sqrt{2}$,

Length of minor axis $= 2b = \frac{2}{\sqrt{3}}$

and $\qquad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{3}}$

$ae = \frac{1}{\sqrt{6}}$

Then foci are $\left(1 \pm \frac{1}{\sqrt{6}}, 2\right)$.

Concept Application Exercise 4.1

1. P is a variable on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with AA' as the major axis. Find the maximum area of the triangle APA'.
2. Prove that the curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$ $t \in R$ is an ellipse.
3. An arc of a bridge is semi-elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal, then prove that the best approximation of the height of the arch 2 m from the centre of the base is $\frac{8}{3}$ m.
4. An ellipse has OB as a semi-minor axis, F, F' as its foci and the angle $\angle FBF'$ is a right angle. Then, find the eccentricity of the ellipse
5. Find the equation of an ellipse whose axes are x- and y-axis and whose one focus is at (4, 0) and eccentricity is $\frac{4}{5}$.
6. If $P(\alpha, \beta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S and S' and eccentricity e, then prove that area be of $\Delta SPS'$ is be $\sqrt{a^2 - \alpha^2}$.
7. An ellipse is drawn with major and minor axes of lengths 10 and 8, respectively. Using one focus as centre a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Then find the radius of the circle.
8. Find the foci of the ellipse $25(x + 1)^2 + 9(y + 2)^2 = 225$.
9. Find the sum of the focal distances of any point on the ellipse $9x^2 + 16y^2 = 144$.
10. If C is the centre of the ellipse $9x^2 + 16y^2 = 144$ and S is a focus. Then find the ratio of CS to semi-major axis.
11. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of eccentricity e. If A, A' are the vertices and S, S' are the foci of the ellipse, then find the ratio area $\Delta PSS'$: area $\Delta APA'$.
12. If the foci of an ellipse are $(0, \pm 1)$ and minor axis is of unit length. Then find the equation of the ellipse. Axis of ellipse are coordinate axes.

ELLIPSE: DEFINITION 2

From the discussions in the previous sections we can also define an ellipse with respect to one fixed point and fixed line. An ellipse is the locus of a point which moves in a plane such that the ratio of its distances from a fixed point (i.e., focus) and the fixed line (i.e., directrix) is constant and less than 1. This ratio is called eccentricity and is denoted by e. For an ellipse $e < 1$.

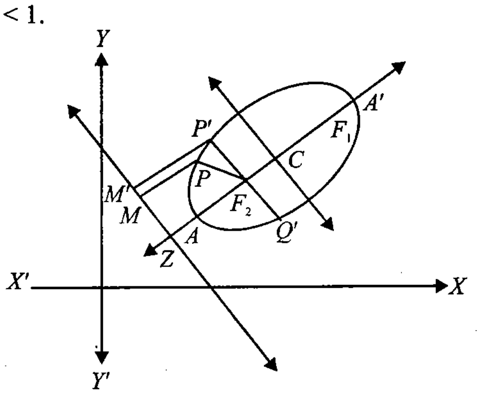


Fig. 4.11

From the diagram, for any point P on the curve, we have by definition,

$\frac{F_2P}{PM} = e$, or $F_2P = ePM$ (focal length or focal radius of point P)

Also A and A' divide F_2Z in the ratio e: 1 internally and externally, respectively.

If the focus F_2 has coordinates (α, β) and equation of directrix ZM is $lx + my + n = 0$, then equation of the ellipse is

$$\sqrt{(x-\alpha)+(y-\beta)^2} = e\frac{|lx+my+n|}{\sqrt{l^2+m^2}}$$

which is of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$,

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and $h^2 < ab$.

From the diagram, length of latus rectum.

$= P'Q'$

$= 2F_2P'$

$= 2(eP'M')$

$= 2(eF_2Z)$

$= 2(e)$(distance of focus from corresponding directrix)

Example 4.10 **Find the equation of the ellipse whose focus is $S(-1, 1)$, the corresponding directrix is $x - y + 3 = 0$ and the eccentricity is $\frac{1}{2}$. Also find its centre, the second focus, the equation of the second directrix and the length of latus rectum.**

Sol.

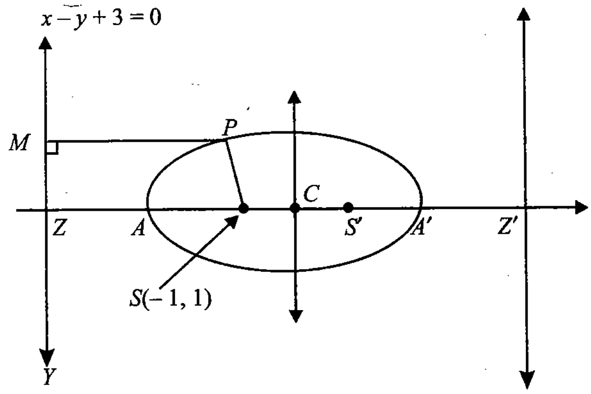


Fig. 4.12

Let $P(x, y)$ be any point on the ellipse and PM be perpendicular to the directrix, then $PS = ePM$ gives

$$(x+1)^2 + (y-1)^2 = \frac{1}{4}[(x-y+3)/\sqrt{2}]^2$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 2xy + 10x - 10y - 7 = 0 \qquad \text{(i)}$$

The major axis passes through $S(-1, 1)$ and is perpendicular to the directrix.

So the equation of the major axis is $x + y = 0$ (ii)

Axis meets the directrix at Z, then Z is $(-\frac{3}{2}, \frac{3}{2})$

A and A′ divide ZS in the ratio 1: e, i.e., 1: $\frac{1}{2}$

or in 2: 1 internally and externally, respectively.

Therefore, we have

$$A = \left(-\frac{7}{6}, \frac{7}{6}\right) \text{ and } A' = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Thus, the centre C (midpoint of AA') $\equiv \left(-\frac{5}{6}, \frac{5}{6}\right)$ (iii)

Let other focus S' be (h, k).

Then $\quad \frac{1}{2}(h-1) = -\frac{5}{6}$ and $\frac{1}{2}(k+1) = \frac{5}{6}$

so $\quad h = -\frac{2}{3}$, and $k = \frac{2}{3}$

$\therefore$ S' is $\left(-\frac{2}{3}, \frac{2}{3}\right)$ (iv)

If major axis meets the other directrix at $Z'(\alpha, \beta)$ then since C is the midpoint of ZZ', we have

$$\frac{1}{2}(\alpha - \frac{3}{2}) = -\frac{5}{6}, \frac{1}{2}(\beta + \frac{3}{2}) = \frac{5}{6} \Rightarrow Z'(-\frac{1}{6}, \frac{1}{6})$$

The second directrix is the line perpendicular to the axis passing through $Z'(-\frac{1}{6}, \frac{1}{6})$

$\therefore$ Equation of other directrix is $x - y + \frac{1}{3} = 0$ ...(v)

Also length of latus rectum = $2(e)$(distance of $(-1, 1)$ from the line $x - y + 3 = 0$)

$$= 2 \times \frac{1}{2}\frac{|-1-1+3|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Example 4.11 **If the equation $(5x - 1)^2 + (5y - 2)^2 = (\lambda^2 - 2\lambda + 1)(3x + 4y - 1)^2$ represents an ellipse, then find values of λ.**

Sol. Given equation of an ellipse is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=(\lambda-1)^2\left[\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right]^2$$

or $\sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=|\lambda-1|\frac{|3x+4y-1|}{\sqrt{3^2+4^2}}$

For ellipse $SP = ePM$, where $0 < e < 1$

$\Rightarrow$ $0 < |\lambda - 1| < 1$

$\Rightarrow$ $\lambda \in (0, 2) - \{1\}$

Equation of an Ellipse Referred to Two Perpendicular Lines

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in the figure.

Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.

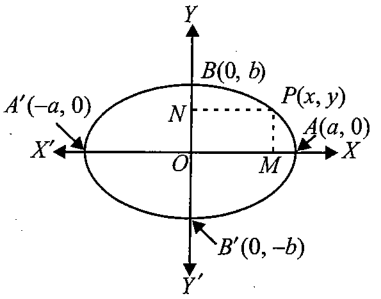


Fig. 4.13

$$\Rightarrow \quad \frac{PN^2}{a^2}+\frac{PM^2}{b^2}=1$$

It follows from this, that if perpendicular distances p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight line $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv b_1x - a_1y + c_2 = 0$, respectively, satisfy the equation

$$\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}=1,$$

i.e. $$\frac{\left(\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right)^2}{a^2}+\frac{\left(\frac{b_1x-a_1y+c_2}{\sqrt{b_1^2+a_1^2}}\right)^2}{b^2}=1$$

then locus of the point P is an ellipse in the plane of the given lines such that

i. The centre of the ellipse is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$

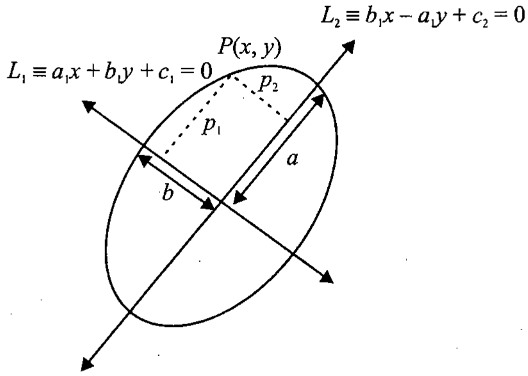


Fig. 4.14

ii. The major axis lies along $L_2 = 0$ and the minor axis lies along $L_1 = 0$, if $a > b$

Example 4.12 **Find the equation of the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.**

Sol. Equation of the ellipse is $\left(\frac{\frac{x-3y+3}{\sqrt{1+9}}}{\sqrt{6}}\right)^2+\left(\frac{\frac{3x+y-1}{\sqrt{9+1}}}{3}\right)^2=1$

$$\Rightarrow \quad \frac{(x-3y+3)^2}{60}+\frac{(3x+y-1)^2}{90}=1$$

$$\Rightarrow \quad 3(x-3y+3)^2+2(3x+y-1)^2=180$$

$$\Rightarrow \quad 21x^2-6xy+29y^2+6x-58y-151=0$$

Concept Application Exercise 4.2

1. Find the eccentricity, one of the foci, directrix and length of the latus rectum for the conic $(3x-12)^2+(3y+15)^2=\frac{(3x-4y+5)^2}{25}$.
2. Find the length of major axis, minor axis, eccentricity of the ellipse $\frac{(3x-4y+2)^2}{16}+\frac{(4x+3y-5)^2}{9}=1$.

AUXILIARY CIRCLE AND ECCENTRIC ANGLE

Definition

A circle described on the major axis as diameter is called the auxiliary circle. For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$ it has equation

$$x^2+y^2=a^2 \quad \text{(i)}$$

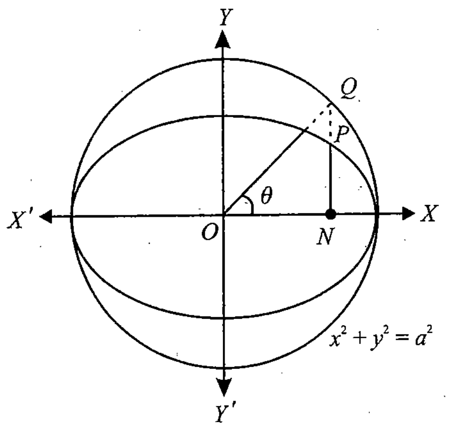


Fig. 4.15

$$P \equiv (a \cos \theta, b \sin \theta)$$

$$Q \equiv (a \cos \theta, a \sin \theta) \quad 0 \le \theta < 2\pi.$$

Here θ is called eccentric angle of point P. P and Q are corresponding points and θ is called the eccentric angle of the point P.

We have $\quad \dfrac{PN}{PQ} = \dfrac{b \sin \theta}{a \sin \theta - b \sin \theta} = \dfrac{b}{a - b}$

= constant

Hence, if from each point on a circle, perpendiculars are drawn on a fixed diameter then the locus of a point P dividing these perpendiculars in a constant ratio is an ellipse whose auxiliary circle is the original circle.

Example 4.13 **Find the equation of the curve whose parametric equations are $x = 1 + 4 \cos \theta$, $y = 2 + 3 \sin \theta$, $\theta \in R$.**

Sol. We have $x = 1 + 4 \cos \theta$, $y = 2 + 3 \sin \theta$

$$\therefore \quad \left(\frac{x-1}{4}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

which is an ellipse.

Example 4.14 **Prove that any point on the ellipse whose foci are (–1, 0) and (7, 0) and eccentricity 1/2 is $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$, $\theta \in R$.**

Sol. Foci are (–1, 0) and (7, 0).

Distance between foci is $2ae = 8$.

$\Rightarrow \quad ae = 4$ and since $e = \frac{1}{2}$, $a = 8$.

Now, $\quad b^2 = a^2(1 - e^2) \Rightarrow b^2 = 48$

$\Rightarrow \quad b = 4\sqrt{3}$.

The centre of the ellipse is the midpoint of the line joining two foci, therefore the coordinates of the centre are (3, 0).

Hence its equation is $\dfrac{(x-3)^2}{8^2} + \dfrac{(y-0)^2}{(4\sqrt{3})^2} = 1 \quad$ (i)

Thus, the parametric coordinates of a point on (i) are $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$.

Example 4.15 **Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre of the ellipse is $\sqrt{5}$.**

Sol. Any point on the ellipse is $P(\sqrt{6} \cos \theta, \sqrt{2} \sin \theta)$

Now $CP = \sqrt{6 \cos^2 \theta + 2 \sin^2 \theta} = \sqrt{5}$ where C is a centre

$\Rightarrow \quad 6(1 - \sin^2 \theta) + 2 \sin^2 \theta = 5$

$\Rightarrow \quad 4 \sin^2 \theta = 1$

$\Rightarrow \quad \sin \theta = \pm \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Example 4.16 **Find the area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol. Let $PQRS$ be a rectangle, where P is $(a \cos \theta, b \sin \theta)$

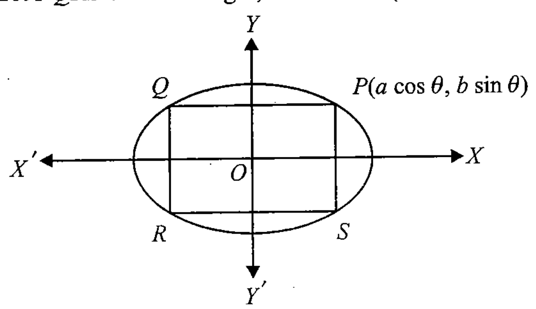


Fig. 4.16

$\therefore \quad$ Area of the rectangle $= 4(a \cos \theta)(b \sin \theta)$
$= 2ab \sin 2\theta$

This is max. when $\sin 2\theta = 1$

Hence, max. area $= 2ab(1) = 2ab$

Example 4.17 **If the line $lx + my + n = 0$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points whose eccentric angles differ by $\frac{\pi}{2}$, then find the value of $\frac{a^2l^2 + b^2m^2}{n^2}$.**

Sol. Let the points of intersection of the line and the ellipse be $(a \cos \theta, b \sin \theta)$ and $\left(a \cos\left(\frac{\pi}{2} + \theta\right), b \sin\left(\frac{\pi}{2} + \theta\right)\right)$. Since they lie on the given line $lx + my + n = 0$

$la \cos \theta + mb \sin \theta + n = 0$

$\Rightarrow \quad la \cos \theta + mb \sin \theta = -n$

and $\; -la \sin \theta + mb \cos \theta + n = 0$

$\Rightarrow \quad la \sin \theta - mb \cos \theta = n$

Squaring and adding, we get

$$a^2l^2 + b^2m^2 = 2n^2$$

$$\Rightarrow \quad \frac{a^2l^2 + b^2m^2}{n^2} = 2$$

Example 4.18 **Find the area of the greatest isosceles triangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having its vertex coincident with one extremity of major axis.**

Sol. Let APQ be the isosceles Δ inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with one vertex A at the extremity $(a, 0)$ of the major axis. If coordinates of P are $(a \cos \theta, b \sin \theta)$, then those of Q will be $(a \cos \theta, -b \sin \theta)$.

So area of $\Delta\, APQ$ is given by

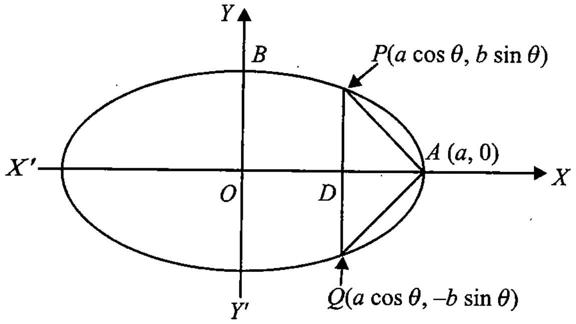


Fig. 4.17

$$\Delta = \frac{1}{2} PQ.AD$$
$$= PD.AD$$
$$= PD\,(OA - OD)$$
$$= b \sin \theta\,(a - a \cos \theta)$$
$$= \frac{1}{2} ab\,(2 \sin \theta - \sin 2\theta),$$

$$0 < \theta < \pi$$

$\therefore \quad \frac{d\Delta}{d\theta} = ab\,(\cos \theta - \cos 2\theta),$

and $\quad \frac{d^2\Delta}{d\theta^2} = ab\,(-\sin \theta + 2 \sin 2\theta)$

For max. or min. of A, $\frac{d\Delta}{d\theta} = 0$

$\Rightarrow \quad \cos \theta - \cos 2\theta = 0$

$\Rightarrow \quad 2 \cos^2 \theta - \cos \theta - 1 = 0$

$\Rightarrow (2 \cos \theta + 1)(\cos \theta - 1) = 0$

$\Rightarrow \quad \cos \theta = -\frac{1}{2},$

$(\because \cos \theta \neq 1 \text{ as } \theta \neq 0)$

$\Rightarrow \quad \theta = 2\pi/3,$

when $\quad \theta = 2\pi/3,$

$$\frac{d^2\Delta}{d\theta^2} = -\frac{1}{2}(3\sqrt{3})\,ab, \ (-\text{ ve})$$

$\therefore \quad \Delta$ has max. when $\theta = 2\pi/3$

$\therefore$ Max. area, $\quad \Delta = \frac{1}{2} ab\,[2 \sin (2\pi/3) - \sin (4\pi/3)]$

$$= \frac{1}{4}(3\sqrt{3})\,ab$$

Concept Application Exercise 4.3

1. Find the eccentric angles of the extremities of the latus recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then prove that the eccentricity of the ellipse is $\frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)}$.
3. If the chord joining points $P(\alpha)$ and $Q(\beta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtends a right angle at the vertex $A(a, 0)$, then prove that $\tan (\alpha/2) \tan (\beta/2) = -\frac{b^2}{a^2}$.

Some Important Properties of Ellipse

1. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab

Proof :

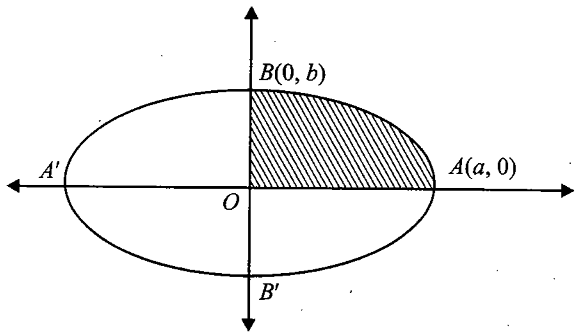


Fig. 4.18

$$\text{Area} = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[4\int_0^a \sqrt{a^2 - x^2}\,dx\right]$$

$$= \frac{b}{a}\,[\text{area of circle having radius } a]$$

$$= \frac{b}{a}\pi a^2$$

$$= \pi ab$$

2. Ratio of area of any triangle PQR inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and that of triangle formed by corresponding points on the auxiliary circle is b/a.

Proof :

Let the three point on the ellipse be $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\gamma, b\sin\gamma)$.

Then corresponding points on the auxiliary circle are $A(a\cos\alpha, a\sin\alpha)$, $B(a\cos\beta, a\sin\beta)$ and $C(a\cos\gamma, a\sin\gamma)$.

$$\text{Now } \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}{a^2\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{b}{a}$$

3. Semi latus rectum is harmonic mean of segments of focal chord or $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$ $(a > b)$ (where PQ is focal chord through focus S)

4. Circle described on focal length as diameter always touches auxiliary circle.

Proof : Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

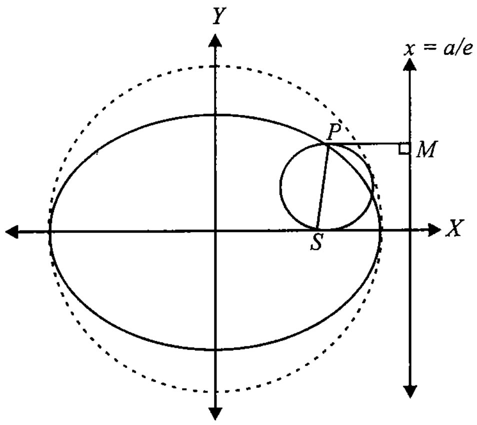


Fig. 4.19

Let P be $(a\cos\theta, b\sin\theta)$

$$SP = ePM = e\left(\frac{a}{e} - a\cos\theta\right)$$
$$= a(1 - e\cos\theta)$$

Auxiliary circle is $x^2 + y^2 = a^2$ having centre $C_1(0, 0)$ and $r_1 = a$.

Circle having SP as a diameter has centre

$C_2\left(\frac{ae + a\cos\theta}{2}, \frac{b\sin\theta}{2}\right)$ and radius

$$r_2 = \frac{SP}{2} = \frac{a(1 - e\cos\theta)}{2}$$

$$\text{Now} \quad r_1 - r_2 = \frac{a(1 + e\cos\theta)}{2}$$

$$\text{and} \quad C_1C_2 = \sqrt{\frac{a^2(e + \cos\theta)^2}{4} + \frac{b^2\sin^2\theta}{4}}$$

$$= \frac{a}{2}\sqrt{e^2 + 2e\cos\theta + \cos^2\theta + \frac{b^2}{a^2}\sin^2\theta}$$

$$= \frac{a}{2}\sqrt{e^2 + 2e\cos\theta + \cos^2\theta + (1 - e^2)\sin^2\theta}$$

$$= \frac{a}{2}\sqrt{e^2\cos^2\theta + 2e\cos\theta + 1}$$

$$= \frac{a}{2}(e\cos\theta + 1)$$

Hence, circle on SP as a diameter touches the auxiliary circle internally.

Example 4.19 **If PSQ is a focal chord of the ellipse $16x^2 + 25y^2 = 400$ such that $SP = 8$, then find length of SQ.**

Sol. Given ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

then $\frac{1}{8} + \frac{1}{SQ} = \frac{2(5)}{16} = \frac{5}{8} \Rightarrow SQ = 2$

Example 4.20 **The ratio of area of triangle inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to that of triangle formed by the corresponding points on the auxiliary circle is 0.5, then find the eccentricity of ellipse.**

Sol. The given ratio is $\frac{b}{a} = \frac{1}{2}$

Now $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$.

Example 4.21 **If S and S' are two foci of the ellipse $16x^2 + 25y^2 = 400$ and PSQ is a focal chord such that $SP = 16$, then find $S'Q$.**

Sol. We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

Then for given ellipse $\frac{1}{16} + \frac{1}{SQ} = 2 \times \frac{5}{16} = \frac{5}{8} \Rightarrow \frac{1}{SQ} = \frac{9}{16}$

$$\text{Now} \quad SQ + SQ' = 2a = 10 \Rightarrow SQ'$$
$$= 10 - \frac{16}{9} = \frac{74}{9}$$

Example 4.22 ***AOB* is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in which $OA = a$, $OB = b$. Then find the area between the arc AB and the chord AB of the ellipse.**

Sol.

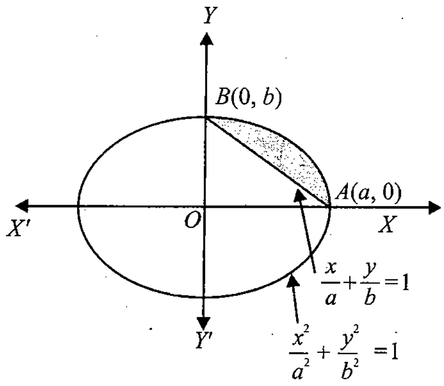


Fig. 4.20

Area of ellipse is πab. Then area of ellipse in first quadrant is $\frac{1}{4}\pi ab$ sq. units.

Now area of triangle $OAB = \frac{1}{2}ab$ sq. units

Hence, the required area is $\frac{1}{4}\pi ab - \frac{1}{2}ab = \frac{ab}{4}(\pi - 2)$ sq. units

Example 4.23 **Prove that area bounded by the circle $x^2 + y^2 = a^2$ and ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is equal to the area of another ellipse having semi-axis $a - b$ and b $(a > b)$.**

Sol. Area bounded by the given circle and ellipse

= area of circle – area of ellipse

$= \pi a^2 - \pi ab$

$= \pi a(a - b)$

= area of ellipse having semi-axis $a - b$ and b

INTERSECTION OF A LINE AND AN ELLIPSE

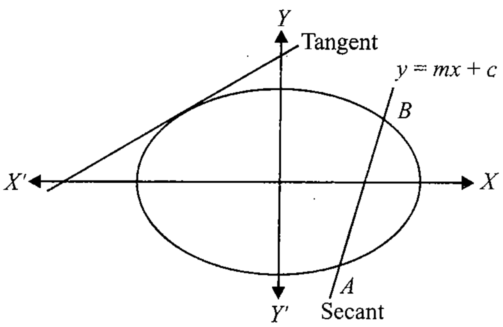


Fig. 4.21

Line $\qquad y = mx + c \qquad$ (i)

Ellipse: $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad$ (ii)

Solving Eqs. (i) and (ii), we get

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

i.e., $(a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$

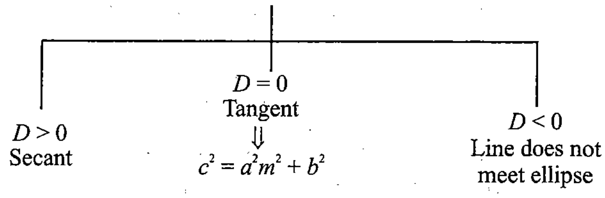


Fig. 4.22

Hence, $y = mx \pm \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all $m \in R$.

Note there are two parallel tangents for a given m.

If it passes through (h, k), then $k = mh \pm \sqrt{a^2m^2 + b^2}$

or $\qquad (k - mh)^2 = a^2m^2 + b^2$

$\Rightarrow (h^2 - a^2)m^2 - 2khm + k^2 - b^2 = 0 \qquad$ (iii)

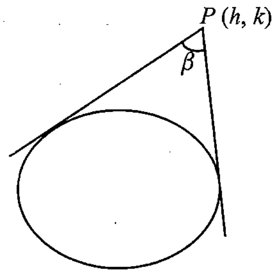


Fig. 4.23

Hence, passing through a given point there can be a maximum of two tangents.

Equation (iii) can be used to determine the locus of the point of intersection of two tangents enclosing an angle β.

If $\qquad \beta = 90°$, then $m_1m_2 = -1$

$\Rightarrow \qquad k^2 - b^2 = a^2 - h^2$

$\Rightarrow \qquad x^2 + y^2 = a^2 + b^2$

which is known as *director circle* of the ellipse.

Hence, *director circle* of an ellipse is a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axes.

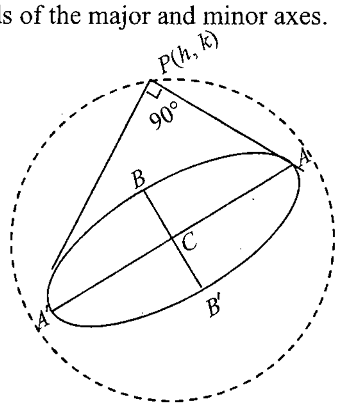


Fig. 4.24

Equation of Tangent to the Ellipse at Point (x_1, y_1)

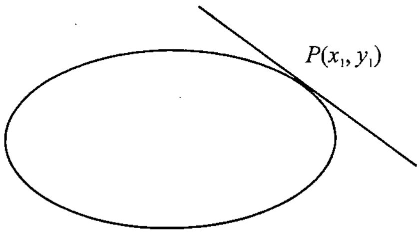


Fig. 4.25

Differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t. x, we have

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2x_1}{a^2y_1}$$

Hence, equation of the tangent is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

or $\quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

But (x_1, y_1) lies on the ellipse $\Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

Hence, equation of the tangent is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{(i)}$$

or $\quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$ or $T = 0$

where $\quad T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

Equation of Tangent at Point $(a \cos\theta, b \sin\theta)$

Putting $x_1 = a\cos\theta$ and $y_1 = b\sin\theta$ in (1), we get

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \text{(ii)}$$

Point of Contact where Line $y = mx + c$ Touches the Ellipse

Line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $c = \pm\sqrt{a^2m^2 + b^2}$

Comparing lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ with (i), we get

$$\frac{\frac{x_1}{a^2}}{m} = \frac{\frac{y_1}{b^2}}{-1} = \frac{1}{\mp\sqrt{a^2m^2 + b^2}}$$

$$\Rightarrow (x_1, y_1) \equiv \left(\pm\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

or $\left(\pm\frac{a^2m}{c}, \mp\frac{b^2}{c}\right)$, where $c = \sqrt{a^2m^2 + b^2}$

Example 4.24 **Find the equations of the tangents drawn from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.**

Sol. Let the equation of the tangent is $y = mx \pm \sqrt{16m^2 + 9}$

It passes through the point (2, 3)

$$\Rightarrow \quad 3 = 2m + \sqrt{16m^2 + 9}$$

$$\Rightarrow \quad (3 - 2m)^2 = 16m^2 + 9$$

$$\Rightarrow \quad 12m^2 + 12m = 0$$

$$\Rightarrow \quad m = 0, -1$$

$\Rightarrow$ Tangents are $\quad y = 3$ or $y = -x + 5$

Draw the diagram and verify that both tangents have + ve y-intercept.

Example 4.25 **If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then find its eccentric angle θ of point of contact.**

Sol. Let θ be the eccentric angle of the point of contact, then tangent at this point is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Also $\quad \frac{x}{a\sqrt{2}} + \frac{y}{b\sqrt{2}} = 1$ is the tangent

$$\therefore \quad \frac{\cos\theta}{\frac{1}{\sqrt{2}}} + \frac{\sin\theta}{\frac{1}{\sqrt{2}}} = 1$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}};\ \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = 45°$$

Example 4.26 **Find the locus of the middle point of the portion of a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ included between the axes.**

Sol.

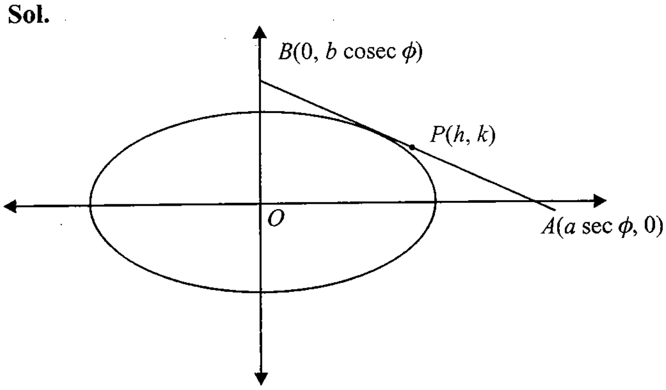


Fig. 4.26

The equation of the tangent at any point ϕ of the ellipse is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$.

It meets the axes at the point $A(a \sec \phi, 0)$ and $B(0, b \operatorname{cosec} \phi)$

Let $P(h, k)$ be the midpoint of AB,

then $$h = \frac{a \sec\phi}{2}, k = \frac{b \operatorname{cosec} \phi}{2}$$

$\therefore$ $$\cos \phi = \frac{a}{2h} \text{ and } \sin \phi = \frac{b}{2k}$$

$\Rightarrow$ $$\frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1 \text{ (eliminating } \phi)$$

Thus, the locus of (h, k) is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$

Example 4.27 **If the line $3x + 4y = \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$, then find the point of contact.**

Sol. Let the given line touches the ellipse at point $P(\theta)$.

The equation of the tangent at P is

$\sqrt{3}\, x \cos \theta + 2y \sin \theta = 1$ (i)

Comparing (i) with the given equation of the line $3x + 4y = \sqrt{7}$, we get

$$\frac{\cos \theta}{\sqrt{3}} = \frac{\sin \theta}{2} = \frac{1}{\sqrt{7}}$$

The point of the contact $(a \cos \theta, b \sin \theta)$ is $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$.

Example 4.28 **Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangent at each point makes equal angles with the axes.**

Sol. Let P be $(a \cos \theta, b \sin \theta)$, then the equation of tangent at P is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

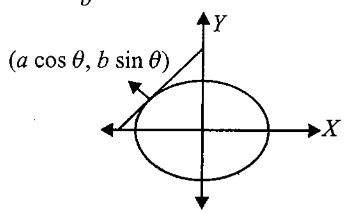


Fig. 4.27

$$\text{Slope of tangent} = \frac{-\cos \theta}{\sin \theta}\frac{b}{a} = \pm \tan 45° = \pm 1$$

$\Rightarrow$ $$\cot \theta = \pm \frac{a}{b}$$

$\Rightarrow$ $$\cos \theta = \pm \frac{a}{\sqrt{a^2 + b^2}} \text{ and}$$

$$\sin \theta = \pm \frac{b}{\sqrt{a^2 + b^2}}$$

$\therefore$ Coordinates of the required points are

$$\left[\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}}\right]$$

Example 4.29 **An ellipse passes through the point (4, –1) and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with coordinate axes.**

Sol. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (i)

It passes through (4, –1) so $a^2 + 16b^2 = a^2b^2$ (ii)

Also $x + 4y - 10 = 0$ touches the ellipse

$\Rightarrow$ $$y = (-1/4)x + (5/2)$$

$\Rightarrow$ $$\frac{25}{4} = \frac{a^2}{16} + b^2$$

$\Rightarrow$ $$a^2 + 16b^2 = 100 \quad \text{(iii)}$$

From (ii) and (iii), we get

$$a^2b^2 = 100 \text{ or } ab = 10$$

Solving (ii) and (iii), we have (iv)

$(a = 4\sqrt{5}, b = \sqrt{5}/2)$ or $(a = 2\sqrt{5}, b = \sqrt{5})$

Hence, there are two ellipses satisfying the given conditions, i.e.,

$$\frac{x^2}{80} + \frac{4y^2}{5} = 1 \text{ and } \frac{x^2}{20} + \frac{y^2}{5} = 1$$

Example 4.30 **An ellipse slides between two perpendicular straight lines. Then identify the locus of its centre.**

Sol. Clearly, P is the point of intersection of perpendicular tangents.

So, P lies on the director circle of the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (say).

This means that the centre of the ellipse will always remain at a constant distance $\sqrt{a^2 + b^2}$ from P.

Hence, the locus of C is a circle.

Example 4.31 **Find the locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol. Let the foot of perpendicular is $P(h, k)$

Then the slope of OP is $\frac{k}{h}$. (O is centre)

$\Rightarrow$ Slope of the tangent is $-\frac{h}{k}$

$\Rightarrow$ Equation of the tangent is $y - k = -\frac{h}{k}(x - h)$

or $$y = -\frac{h}{k}x + \frac{h^2 + k^2}{k}$$

This touches the ellipse, then $\left(\frac{h^2 + k^2}{k}\right)^2 = a^2\left(\frac{h}{k}\right)^2 + b^2$

$\Rightarrow$ $$a^2x^2 + b^2y^2 = (x^2 + y^2)^2$$

Equation of Pair of Tangents from Point (h, k)

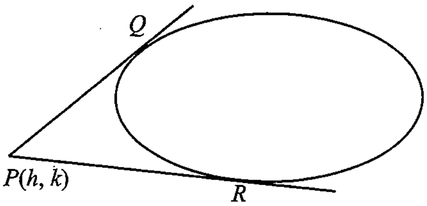


Fig. 4.28

Combined equation pair of tangents PQ and PR is given by $T^2 = SS_1$

where $\quad T = \frac{hx}{a^2} + \frac{ky}{b^2} - 1, S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

and $\quad S_1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

Example 4.32 **Find the angle between the pair of tangents from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$.**

Sol. The combined equation of the pair of tangents drawn from (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is

$(3x^2 + 2y^2 - 5)(3 + 8 - 5) = (3x + 4y - 5)^2$

$\Rightarrow 9x^2 - 24xy - 4y^2 + ... = 0$ [Using $SS' = T^2$]

If the angle between these lines is θ, then

$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$, where $a = 9, h = -12, b = -4$

$\Rightarrow \quad \tan\theta = \frac{12}{\sqrt{5}}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$

IMPORTANT PROPERTIES RELATED TO TANGENT

1. Locus of feet of perpendiculars from foci upon any tangent is an auxiliary circle.

Proof: For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the equation of tangent at any point θ, i.e., at $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Equation of perpendiculars from foci $(\pm ae, 0)$ to tangent (i) is $\frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \pm\frac{ae\sin\theta}{b}$ (ii)

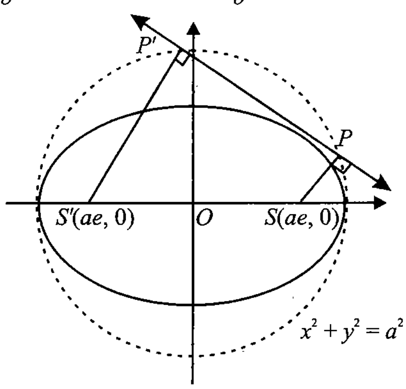


Fig. 4.29

Locus of the feet of perpendicular, i.e., of points of intersetion of (i) and (ii) is obtained by eliminating θ.

Squaring (i) and (ii) and adding, we get

$$x^2\left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right] + y^2\left[\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right]$$

$$= 1 + \frac{a^2e^2\sin^2\theta}{b^2}$$

$$= a^2\left[\frac{b^2 + a^2\sin^2\theta - b^2\sin^2\theta}{a^2b^2}\right]$$

$$= a^2\left[\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}\right]$$

$[\because a^2e^2 = a^2 - b^2]$

$$= a^2\left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right]$$

Hence, cancelling $[\cos^2\theta/a^2 + \sin^2\theta/b^2]$,

the locus is $x^2 + y^2 = a^2$, which is an auxiliary circle.

2. Product of perpendiculars from foci upon any tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is b^2.

Proof: Equation of tangent having slope m is

$$mx - y + \sqrt{a^2m^2 + b^2} = 0$$

$$p_1p_2 = \left[\frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}} \cdot \frac{-mea + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right]$$

$$= \left|\frac{(a^2m^2 + b^2 - m^2a^2e^2}{1+m^2}\right|$$

$$= \left|\frac{a^2m^2 + b^2 - m^2(a^2 - b^2)}{1+m^2}\right|$$

$$= \left|\frac{b^2(1+m^2)}{1+m^2}\right| = b^2$$

3. Tangents at the extremities of latus rectum pass through the corresponding foot of directrix on major axis.
4. Length of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.

Example 4.33 **If F_1 and F_2 are the feet of the perpendiculars from the foci S_1 and S_2 of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ on the tangent at any point P on the ellipse, then prove that $S_1F_1 + S_2F_2 \geq 8$.**

Sol. We known that the product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of the semi-minor axis.

Then $\quad (S_1F_1)(S_2F_2) = 16$

Now A.M. ≥ G.M.

$\Rightarrow \quad \frac{S_1F_1 + S_2F_2}{2} \geq \sqrt{S_1F_1 \cdot S_2F_2}$

$\Rightarrow \quad S_1F_1 + S_2F_2 \geq 8$

Concept Application Exercise 4.4

1. If the line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.
2. Find the slope of a common tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a concentric circle of radius r.
3. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then find l.
4. If the tangent to the ellipse $x^2 + 2y^2 = 1$ at point $P\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ meets the auxiliary circle at point R and Q. Then find point of intersection of tangents to circle at Q and R.
5. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32}$ intersects the major and minor axes in points A and B, respectively. If C is the centre of the ellipse, then find the area of the triangle ABC.
6. Find the two perpendicular tangents drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ intersect on the curve.
7. If the tangent at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact, then show that the eccentricity of the ellipse is given by $e = \frac{\cos \beta}{\cos \alpha}$.

EQUATION OF NORMAL

Equation of Normal to the Ellipse at Point (x_1, y_1)

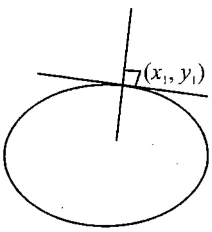


Fig. 4.30

Slope of normal at point (x_1, y_1) is $\frac{a^2 y_1}{b^2 x_1}$. (i)

Hence, equation of normal is $(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

or $\quad \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$ (iii)

Normal at Point $(a \cos \theta, b \sin \theta)$

Putting $x_1 = a \cos \theta$ and $y_1 = b \sin \theta$ in (iii), we get

$\Rightarrow \quad \frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$ (iv)

Note:

1. ***Normal other than major axis never passes through the focus.***
2. ***Normal at point P bisects the angle SPS′***

$\boldsymbol{SP = a - ex,}$

$\boldsymbol{S'P = a + ex,}$

Normal NP is $\boldsymbol{\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2}$

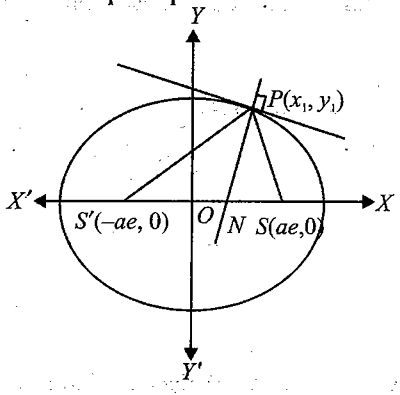


Fig. 4.31

For point N, putting y = 0, we get

$\Rightarrow \quad \boldsymbol{x = x_1\left(\frac{a^2 - b^2}{a^2}\right) = x_1\left(1 - \frac{b^2}{a^2}\right)}$

$\boldsymbol{= x_1 e^2}$

$\Rightarrow \quad \boldsymbol{SN = ae - x_1 e^2}$

$\boldsymbol{= e(a - ex_1) = eSP}$

and $\quad \boldsymbol{S'N = ae + x_1 e^2}$

$\boldsymbol{= e(a + ex_1) = eS'P}$

$\Rightarrow \quad \boldsymbol{\frac{SP}{S'P} = \frac{SN}{S'N}}$

$\Rightarrow$ ***PN bisect the*** $\angle SPS'$

Thus, incident ray from focus S after reflection by ellipse at point P passes through other focus S′.

Co-normal Points

From any point in the plane maximum four normals can be drawn to ellipse.

Four feet of normals on the ellipse are called co-normal points. The condition for their eccentric angles is $\alpha + \beta + \gamma + \delta = (2n+1)\pi, n \in Z$.

Proof:

Normal at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

If it passes through point (h, k), then

$$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow \frac{ah\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}} - \frac{bk\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}} = a^2 - b^2$$

$$\Rightarrow \frac{(1+t^2)\,ah}{(1-t^2)} - \frac{(1+t^2)\,bk}{2t} = a^2 - b^2 \text{ where } t = \tan\frac{\theta}{2}$$

$$\Rightarrow 2t(1+t^2)\,ah - (1-t^4)\,bk = 2t(1-t^2)(a^2-b^2)$$

$$\Rightarrow bkt^4 + [2ah - 2(a^2-b^2)]\,t^3 + [2ah - 2(a^2-b^2)]t - bk = 0$$

This equation has four roots, i.e., $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$ $\tan\frac{\gamma}{2}$, $\tan\frac{\delta}{2}$, where $\alpha, \beta, \gamma, \delta$ are eccentric angles of feet of normals on the ellipse.

Now $s_1 = \Sigma \tan\frac{\alpha}{2}$

$$s_2 = \Sigma \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 0$$

$$s_3 = \Sigma \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$s_4 = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = -1$$

Now $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4}$

But $1 - s_2 + s_4 = 0$

$$\Rightarrow \left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right) = (2n+1)\frac{\pi}{2}, n \in Z$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = (2n+1)\pi, n \in Z$$

Example 4.34 **Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{9}$, on which the normals are parallel to the line $2x - y = 1$**

Sol. Normal at $P(2\cos\theta, 3\sin\theta)$ is $\frac{2x}{\cos\theta} - \frac{3y}{\sin\theta} = -5$

Now this normal is parallel to $2x - y = 1$, then $\frac{\frac{2}{\cos\theta}}{\frac{3}{\sin\theta}} = 2$

$$\Rightarrow \tan\theta = \frac{3}{1}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{10}} \text{ and } \sin\theta = \pm\frac{3}{\sqrt{10}}$$

Hence, one of the points is $\left(\pm\frac{2}{\sqrt{10}}, \pm\frac{9}{\sqrt{10}}\right)$.

Example 4.35 **If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in G and g, respectively, then find the ratio $PG{:}Pg$.**

Sol. Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the equation of the normal at P is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

It meets the axes at $G\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$ and $g\left(0, -\frac{a^2-b^2}{a}\sin\theta\right)$

$$\therefore PG^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

$$= \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

and $$Pg^2 = \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$\therefore PG{:}Pg = b^2{:}a^2$$

Example 4.36 **P is the point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9}$ and Q is the corresponding point on the auxiliary circle of the ellipse. If the line joining centre C to Q meets the normal at P with respect to the given ellipse at K, then find the value of CK.**

Sol.

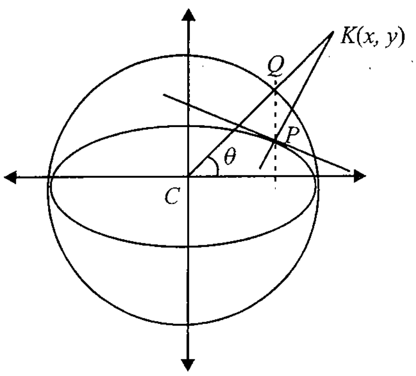


Fig. 4.32

Equation of normal at P is $\frac{4x}{\cos\theta} - \frac{3y}{\sin\theta} = 7$

It passes through K

$$\Rightarrow 4CK - 3CK = 7$$

$$\Rightarrow CK = 7$$

Example 4.37 **If normal at $P\left(2, \frac{3\sqrt{3}}{2}\right)$ meets the major axis of ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at Q and S and S' are foci of given ellipse, then find the ratio $SQ:S'Q$.**

Sol. Equation of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

$$9 = 16(1-e^2)$$

$$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$$

Hence, foci are $S(\sqrt{7}, 0)$ and $S'(-\sqrt{7}, 0)$

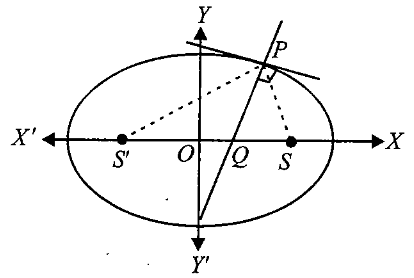


Fig. 4.33

Normal at P is a bisector of angle between $S'P$ and SP

Hence, $$\frac{SQ}{S'Q} = \frac{SP}{S'P} = \frac{8-\sqrt{7}}{8+\sqrt{7}}$$

Concept Application Exercise 4.5

1. The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.
2. Find the equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the positive end of the latus rectum.
3. If the normal at one end of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the one end of the mirror axis, then prove that eccentricity is constant.
4. If the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are concurrent, then prove that

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0.$$

CHORD OF CONTACT

Let PQ and PR be tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from any external point $P(h, k)$.

Then QR is called chord of contact of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $Q(x_1, y_1)$ and $R(x_2, y_2)$

$\therefore$ Equations of tangent PQ and PR are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{(i)}$$

and $$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \quad \text{(ii)}$$

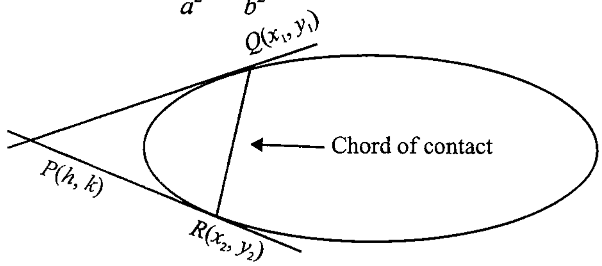


Fig. 4.34

Since (i) and (ii) pass through $P(h, k)$ then

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=1 \quad \text{(iii)}$$

and $$\frac{hx_2}{a^2}+\frac{ky_2}{b^2}=1 \quad \text{(iv)}$$

Hence, it is clear that $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

or $T = 0$ which is chord of contact QR.

where $T = \frac{hx}{a^2}+\frac{ky}{b^2}-1$

Example 4.38 **If from a point P tangents PQ and PR are drawn to the ellipse $\frac{x^2}{2}+y^2=1$ so that equation of QR is $x + 3y = 1$, then find the coordinates of P.**

Sol. Let coordinates of P be (h, k), then equation of QR is

$$\frac{hx}{2}+ky=1 \quad \text{(i)}$$

but is given as $x + 3y = 1$ (ii)

$\therefore$ (i) and (ii) are identical

$$\therefore \quad \frac{h}{2}=\frac{k}{3}=1 \Rightarrow h=2 \text{ and } k=3$$

$\therefore$ Coordinates of P are (2, 3).

Example 4.39 **Tangents are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$. Then all the chords of contact pass through a fixed point, find its coordinates.**

Sol. Any point on the line $x - y - 5 = 0$ will be of the form $(t, t-5)$ where $t \in R$.

Chord of contact of this point with respect to curve $x^2 + 4y^2 = 4$ is

$$tx + 4(t-5)y - 4 = 0$$

or $(-20y - 4) + t(x + 4y) = 0$ which is a family of straight lines, each member of this family passes through the point of intersection of straight lines $-20y - 4 = 0$ and $x + 4y = 0$ which is $(4/5, -1/5)$.

Example 4.40 **Find the locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms a triangle of constant area with the coordinate axes.**

Sol. The chord of contact of tangents from (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

It meets the axes at the points $\left(\frac{a^2}{x_1}, 0\right)$ and $\left(0, \frac{b^2}{y_1}\right)$.

$$\text{Area of the triangle} = \frac{1}{2}\frac{a^2}{x_1}\frac{b^2}{y_1} = k \text{ (constant)}$$

$$\Rightarrow \quad x_1y_1 = \frac{a^2b^2}{2k} = c^2 \text{ (c is constant)}$$

$$\Rightarrow \quad xy = c^2, \text{ which is a hyperbola}$$

Example 4.41 **Prove that the chord of contact of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to any point on the directrix is a focal chord.**

Sol. Let any point on the directrix is $(a/e, k)$.

Chord of contact with respect to this point is $\frac{(a/e)x}{a^2} + \frac{ky}{b^2} = 1$

Chearly focus $S(ae, 0)$ satisfies the above line.

Hence, proved.

EQUATION OF CHORD JOINING POINTS $P(\alpha)$ AND $Q(\beta)$

Equation of chord passing through the points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$

$$\text{Then its equation is } \begin{vmatrix} x & y & 1 \\ a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \end{vmatrix} = 0$$

$$\Rightarrow bx(\sin\alpha - \sin\beta) - ay(\cos\alpha - \cos\beta) + ab\sin(\beta - \alpha) = 0$$

$$\Rightarrow \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Example 4.42 **Find the equation of a chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.**

Sol. Equation of chord is

$$\frac{x}{5}\cos\left(\frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2}\right) + \frac{y}{4}\sin\left(\frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2}\right) = \cos\left(\frac{\frac{\pi}{4}-\frac{5\pi}{4}}{2}\right)$$

$$\Rightarrow \frac{x}{5}\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}\sin\left(\frac{3\pi}{4}\right) = 0$$

$$\Rightarrow \quad -\frac{x}{5} + \frac{y}{5} = 0$$

$$\Rightarrow \quad y = x$$

POINT OF INTERSECTION OF TANGENTS AT POINTS $P(\alpha)$ AND $Q(\beta)$

Point of intersection of the tangents at points α and β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $Q(\alpha)$ and $R(\beta)$ with chord of contact of the pair of tangents from (x_1, y_1) on the ellipse.

Proof:

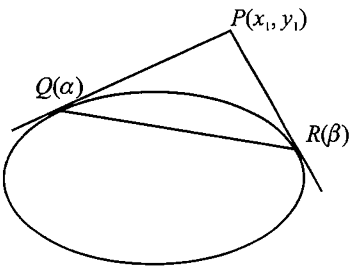


Fig. 4.35

Equation of chord of contact QR with respect to point P is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{(i)}$$

Also equation of chord PQ is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \qquad \text{(ii)}$$

Comparing Eqs. (i) and (ii),

$$\Rightarrow \quad \frac{\frac{x_1}{a^2}}{\frac{a}{\cos\frac{\alpha+\beta}{2}}} = \frac{\frac{y_1}{b^2}}{\frac{b}{\sin\frac{\alpha+\beta}{2}}} = \frac{1}{\cos\frac{\alpha-\beta}{2}}$$

$$\therefore \quad x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}};$$

$$\text{and} \quad y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

Example 4.43 **Find the locus of the point of intersection of tangents to the ellipse if the difference of the eccentric angle of the points is $\frac{2\pi}{3}$.**

Sol. $|\alpha - \beta| = \frac{2\pi}{3}$

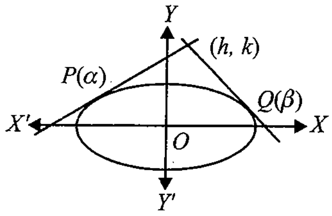


Fig. 4.36

$$h = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = 2a\cos\frac{\alpha+\beta}{2}$$

and $$k = 2b\sin\frac{\alpha+\beta}{2}$$

$$\Rightarrow \quad \frac{h^2}{4a^2} + \frac{k^2}{4b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$$

EQUATION OF THE CHORD OF THE ELLIPSE WHOSE MIDPOINT IS (x_1, y_1)

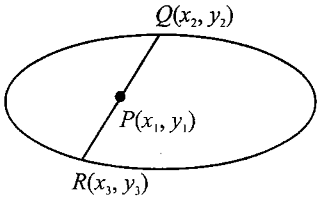


Fig. 4.37

Let the slope of the chord be $\tan\theta$, then any point on the chord at distance r from the point (x_1, y_1) is $(x_1 + r\cos\theta, y_1 + r\sin\theta)$

If this point lies on the ellipse, then

$$\frac{(x_1 + r\cos\theta)^2}{a^2} + \frac{(y_1 + r\sin\theta)^2}{b^2} = 1 \quad \text{(i)}$$

Since line cuts the ellipse in two point Q and R, this is quadratic in r, whose roots are $r_1 = PQ$ and $r_2 = -QR$

Hence, sum of roots, $r_1 + r_2 = 0$ (as $PQ = QR$)

Then from (i), coefficient of $r = 0$

$$\Rightarrow \quad \frac{2x_1\cos\theta}{a^2} + \frac{2y_1\sin\theta}{b^2} = 0$$

$$\Rightarrow \quad \tan\theta = -\frac{b^2x_1}{a^2y_1}$$

Hence, equation of chord is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

or $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

or $$T = S_1, \text{ where } S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

Example 4.44 **Tangents are drawn from the point (3, 2) to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the middle point of this chord of contact.**

Sol. $x^2 + 4y^2 = 9$

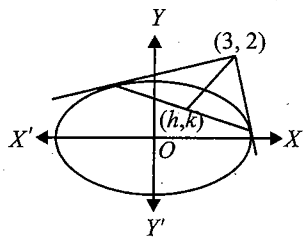


Fig. 4.38

Equation of the chord of contact of the pair of tangents from (3, 2)

$$3x + 8y = 9 \quad \text{(i)}$$

This must be the same as chord whose middle point is (h, k).

$$T = S_1$$

$$\frac{hx}{9} + \frac{ky}{9/4} = \frac{h^2}{9} + \frac{k^2}{9/4}$$

$$\Rightarrow \quad hx + 4ky = h^2 + 4k^2 \quad \text{(ii)}$$

Equations (i) and (ii) represent same straight lines.

Comparing coefficient of (i) and (ii), we get

$$\frac{h}{3} = \frac{4k}{8} = \frac{h^2 + 4k^2}{9}$$

$$\Rightarrow \quad 2h = 3k \text{ and } 3h = h^2 + 4k^2$$

$$\Rightarrow \quad 3h = h^2 + 4 \times \frac{4h^2}{9}$$

$$\Rightarrow \quad \frac{25h^2}{9} = 3h$$

$$\Rightarrow \quad h = \frac{27}{25} \text{ and } k = \frac{18}{25}$$

Example 4.45 **Find the locus of the midpoints of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol. Let the midpoint of the focal chord of the given ellipse be (h, k).

Then its equation is $\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$ [Using $T=S_1$]

Since this passes though $(ae, 0)$

$$\therefore \quad \frac{hae}{a^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$$\Rightarrow \quad \frac{he}{a}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$\therefore$ Locus of (h, k) is $\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Example 4.46 **Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose middle point is (1/2, 2/5).**

Sol. Equation of the chord having (1/2, 2/5) as midpoint is

$$\frac{1/4}{25}+\frac{4/25}{16}-1=\frac{(1/2)x}{25}+\frac{(2/5)y}{16}-1 \quad [T=S_1]$$

$$\Rightarrow \quad 4x+5y=4$$

$$\Rightarrow \quad 5y=4(1-x) \quad \text{(i)}$$

Solving with ellipse, we get

$$16x^2+16(1-x)^2=400$$

$$\Rightarrow \quad x^2-x-12=0$$

$$\Rightarrow \quad x=4,-3$$

$$\text{for } x=4,\ y=-\frac{12}{5},$$

$$\text{For } x=-3,\ y=\frac{16}{5}$$

Therefore, length of the chord $=\sqrt{\left\{7^2+\left(\frac{28}{5}\right)^2\right\}}=7\sqrt{\frac{41}{25}}$

Concept Application Exercise 4.6

1. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find the value of $\frac{x_1x_2}{y_1y_2}$.
2. From the point $A(4, 3)$, tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ to touch the ellipse at B and C. EF is a tangent to the ellipse parallel to the line BC and towards the point A. Then find the distance of A from EF.
3. Find the locus of the middle points of all chords of $\frac{x^2}{4}+\frac{y^2}{9}=1$, which are at a distance of 2 units from the vertex of parabola $y^2=-8ax$.
4. Tangents PQ and PR are drawn at the extremities of the chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, which get bisected at point $P(1, 1)$, then find the point of intersection of tangents.
5. Chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are drawn through the positive end of the minor axis. Then prove that their midpoint lies on the ellipse.

CONCYCLIC POINTS ON ELLIPSE

Let the circle $x^2+y^2+2gx+2fy+c=0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in four points P, Q, R, S.

Solving circle and ellipse $(x=a\cos\theta,\ y=b\sin\theta)$, we have

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2bf\sin\theta+c=0$$

$$\Rightarrow a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ag\left(\frac{1-t^2}{1+t^2}\right)+2bf\left(\frac{2t}{1+t^2}\right)+c=0, \text{ where } t=\tan\frac{\theta}{2}$$

$$\Rightarrow a^2(1-t^2)^2+4b^2t^2+2ag(1-t^2)(1+t^2)+4bft(1+t^2)+c(1+t^2)^2=0$$

$$\Rightarrow (a^2-2ag+c)t^4+4bft^3+(-2a^2+4b^2+2c)t^2+4bft+(a^2+2ag+c)=0$$

Roots of the equation are $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}, \tan\frac{\gamma}{2}\tan\frac{\delta}{2}$, where $\alpha, \beta, \gamma, \delta$ are eccentric angles of P, Q, R, S, respectively.

Also $\quad s_1=s_3$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}=0$$

$$\Rightarrow \quad \frac{\alpha+\beta+\gamma+\delta}{2}=n\pi,\ n\in Z$$

$$\Rightarrow \quad \alpha+\beta+\gamma+\delta=2n\pi,\ n\in Z$$

EXERCISES

Subjective Type

Solutions on page 4.34

1. A circle which is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and passes through the foci F_1 and F_2 of the ellipse. The two curves intersect at four points. Let P be any point of intersection. If the major axis of the ellipse is 15 and area of the triangle PF_1F_2 is 26, then find the value of $4a^2-4b^2$.

2. Find the values of α for which three distinct chords drawn from $(\alpha, 0)$ to the ellipse $x^2 + 2y^2 = 1$ are bisected by the parabola $y^2 = 4x$.
3. Prove that if any tangent to the ellipse is cut by the tangents at the ends points of the major axis in T and T', then the circle whose diameter is TT' will pass through the foci of the ellipse.
4. Let P be a point on an ellipse with eccentricity $\frac{1}{2}$, such that $\angle PS_1S_2 = \alpha$, $\angle PS_2S_1 = \beta$ and $\angle S_1PS_2 = \gamma$ where S_1 and S_2 are foci of the ellipse. Then prove that $\cot\frac{\alpha}{2}$, $\cot\frac{\gamma}{2}$ and $\cot\frac{\beta}{2}$ are in A.P.
5. Find the range of eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ such that the line segment joining the foci does not subtend a right angle at any point on the ellipse.
6. From any point on the line $y = x + 4$, tangent are drawn to the auxiliary circle of the ellipse $x^2 + 4y^2 = 4$. If P, Q are the points of contact and A, B are the corresponding points of P and Q on the ellipse respectively, then find the locus of the midpoint of AB.
7. If a triangle is inscribed is an ellipse and two of its sides are parallel to the given straight lines, then prove that third side touches the fixed ellipse.
8. The tangent at a point P $(a\cos\phi, b\sin\phi)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre. Find the eccentricity of the ellipse.
9. Tangents are drawn to the ellipse from the point $\left(\frac{a^2}{\sqrt{a^2-b^2}}, \sqrt{a^2+b^2}\right)$.

   Prove that the tangents intercept on the ordinate through the nearer focus a distance equal to the major axis.
10. From any point on any directrix of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ a pair of tangents is drawn to the auxiliary circle. Show that chord of contact will pass through the corresponding focus of the ellipse.
11. A tangent is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to cut the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ at the points P and Q. If tangents at P and Q to the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ intersect at right angle, then prove that $\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$.
12. Origin O is the centre of two concentric circles whose radii are a and b, respectively, $a < b$. A line OPQ is drawn to cut the inner circle at P and the outer circle at Q. PR is drawn parallel to the y-axis and QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the foci of this ellipse lie on the inner circle, find the ratio of inner: outer radii and also find the eccentricity of the ellipse.
13. The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.
14. Find the locus of point P such that tangents drawn from it to the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the coordinate axes in concyclic points.
15. Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum.

Objective Type

Solutions on page 4.38

Each question has four choices a, b, c, d, out of which *only one* answer is correct. Find the correct answer.

1. P and Q are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and B is an end of the minor axis. If PBQ is an equilateral triangle, then eccentricity of the ellipse is

   a. $\frac{1}{\sqrt{2}}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{\sqrt{3}}{2}$
2. An ellipse having foci at (3, 3) and (– 4, 4) and passing through the origin has eccentricity equal to

   a. $\frac{3}{7}$ **b.** $\frac{2}{7}$ **c.** $\frac{5}{7}$ **d.** $\frac{3}{5}$
3. If the eccentricity of the ellipse $\frac{x^2}{a^2+1} + \frac{y^2}{a^2+2} = 1$ is $\frac{1}{\sqrt{6}}$, then latus rectum of ellipse is

   a. $\frac{5}{\sqrt{6}}$ **b.** $\frac{10}{\sqrt{6}}$ **c.** $\frac{8}{\sqrt{6}}$ **d.** none of these
4. If PQR is an equilateral triangle inscribed in the auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ and $P'Q'R'$ is corresponding triangle inscribed within the ellipse then centroid of the triangle $P'Q'R'$ lies at

   a. centre of ellipse
   b. focus of ellipse
   c. between focus and centre on major axis
   d. none of these
5. S_1, S_2 are foci of an ellipse of major axis of length 10 units and P is any point on the ellipse such that perimeter of triangle PS_1S_2 is 15. Then eccentricity of the ellipse is

   a. 0.5 **b.** 0.25 **c.** 0.28 **d.** 0.75
6. If the ellipse $\frac{x^2}{4} + y^2 = 1$ meets the ellipse $x^2 + \frac{y^2}{a^2} = 1$ in four distinct points and $a = b^2 - 5b + 7$, then b does not lie in

   a. [4, 5] **b.** $(-\infty, 2) \cup (3, \infty)$
   c. $(-\infty, 0)$ **d.** [2, 3]

7. With a given point and line as focus and directrix, a series of ellipses are described, the locus of the extremities of their minor axis is

   a. ellipse b. parabola
   c. hyperbola d. none of these

8. A line of fixed length $a + b$ moves so that its ends are always on two fixed perpendicular straight lines, then the locus of a point, which divides this line into portions of lengths a and b is a/an

   a. ellipse b. parabola
   c. straight line d. none of these

9. The length of the major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x - 4y + 7)^2}{4}$ is

   a. 10 b. $\frac{20}{3}$ c. $\frac{20}{7}$ d. 4

10. Angle subtended by common tangents of two ellipses $4(x - 4)^2 + 25y^2 = 100$ and $4(x + 1)^2 + y^2 = 4$ at origin is

    a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{2}$

11. A circle has the same centre as an ellipse and passes through the focii F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of the triangle PF_1F_2 is 30, then the distance between the focii is

    a. 13 b. 10 c. 11 d. None of these

12. The line $x = t^2$ meets the ellipse $x^2 + \frac{y^2}{9} = 1$ in the real and distinct points if and only if

    a. $|t| < 2$ b. $|t| < 1$
    c. $|t| > 1$ d. None of these

13. The eccentric angle of a point on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at a distance of $\frac{5}{4}$ units from the focus on the positive x-axis, is

    a. $\cos^{-1}\left(\frac{3}{4}\right)$ b. $\pi - \cos^{-1}\left(\frac{3}{4}\right)$
    c. $\pi + \cos^{-1}\left(\frac{3}{4}\right)$ d. None of these

14. If S and S' are the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, and P is any point on it then range of values of $SP \cdot S'P$ is

    a. $9 \le f(\theta) \le 16$ b. $9 \le f(\theta) \le 25$
    c. $16 \le f(\theta) \le 25$ d. $1 \le f(\theta) \le 16$.

15. Let d_1 and d_2 be the lengths of the perpendiculars drawn from foci S and S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at any point P on the ellipse. Then, $SP{:}S'P =$

    a. $d_1{:}d_2$ b. $d_2{:}d_1$ c. $d_1^2 : d_2^2$ d. $\sqrt{d_1}/\sqrt{d_2}$

16. The auxiliary circle of a family of ellipse passes through origin and makes intercept of 8 and 6 units on the x-axis and the y-axis, respectively. If eccentricity of all such family of ellipse is $\frac{1}{2}$, then locus of the focus will be

    a. $\frac{x^2}{16} + \frac{y^2}{9} = 25$ b. $4x^2 + 4y^2 - 32x - 24y + 75 = 0$
    c. $\frac{x^2}{16} + \frac{y^2}{9} = 25$ d. none of these

17. A man running round a race course notes that the sum of the distances of two flagposts from him is always 10 m and the distance between the flag posts is 8 m. Then the area of the path he encloses in square metres is

    a. 15π b. 20π c. 27π d. 30π

18. There are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose distance from its centre is the same and is equal to $\frac{\sqrt{a^2 + 2b^2}}{2}$. Then the eccentricity of the ellipse is

    a. $\frac{1}{2}$ b. $\frac{1}{\sqrt{2}}$ c. $\frac{1}{3}$ d. $\frac{1}{3\sqrt{2}}$

19. The eccentricity of locus of point $(3h + 2, k)$ where (h, k) lies on the circle $x^2 + y^2 = 1$ is

    a. $\frac{1}{3}$ b. $\frac{\sqrt{2}}{3}$ c. $\frac{2\sqrt{2}}{3}$ d. $\frac{1}{\sqrt{3}}$

20. Let S and S' be two foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If a circle described on SS' as diameter intersects the ellipse in real and distinct points, then the eccentricity e of the ellipse satisfies

    a. $e = 1\sqrt{2}$ b. $e \in (1/\sqrt{2}, 1)$
    c. $e \in (0, 1/\sqrt{2})$ d. none of these

21. If the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{a^2} + y^2 = 1$ for suitable value of a cut on four concyclic points, the equation of the circle passing through these four points is

    a. $x^2 + y^2 = 2$ b. $x^2 + y^2 = 1$
    c. $x^2 + y^2 = 4$ d. none of these

22. From any point P lying in first quadrant on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, PN is drawn perpendicular to the major axis and produced at Q so that NQ equals to PS, where S is a focus. Then the locus of Q is

    a. $5y - 3x - 25 = 0$
    b. $3x + 5y + 25 = 0$
    c. $3x - 5y - 25 = 0$
    d. none of these

23. Locus of the point which divides double ordinates of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the ratio 1:2 internally is

a. $\frac{x^2}{a^2}+\frac{9y^2}{b^2}=1$ **b.** $\frac{x^2}{a^2}+\frac{9y^2}{b^2}=\frac{1}{9}$

c. $\frac{9x^2}{a^2}+\frac{9y^2}{b^2}=1$ **d.** none of these

24. The slopes of the common tangents of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ and the circle $x^2+y^2=3$ are

a. ± 1 **b.** $\pm\sqrt{2}$

c. $\pm\sqrt{3}$ **d.** none of these

25. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ and the circle $x^2+y^2=a^2$ at the points where a common ordinate cuts them (on the same side of the x-axis). Then the greatest acute angle between these tangents is given by

a. $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ **b.** $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$

c. $\tan^{-1}\left(\frac{2ab}{\sqrt{a-b}}\right)$ **d.** $\tan^{-1}\left(\frac{2ab}{\sqrt{a+b}}\right)$

26. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxiliary circle meet on the line

a. $x=a/e$ **b.** $x=0$

c. $y=0$ **d.** none of these

27. If the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles α and β with the major axis such that $\tan\alpha+\tan\beta=\lambda$, then the locus of their point of intersection is

a. $x^2+y^2=a^2$ **b.** $x^2+y^2=b^2$

c. $x^2-a^2=2\lambda xy$ **d.** $\lambda(x^2-a^2)=2xy$

28. If $\alpha-\beta=$ constant, then the locus of the point of intersection of tangents at $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

a. a circle **b.** a straight line

c. an ellipse **d.** a parabola

29. The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant, is a/an

a. parabola **b.** circle

c. ellipse **d.** straight line

30. The sum of the squares of the perpendiculars on any tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from two points on the minor axis each at a distance ae from the centre is

a. $2a^2$ **b.** $2b^2$ **c.** a^2+b^2 **d.** a^2-b^2

31. A tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ at any point P meets the line $x=0$ at a point Q. Let R be the image of Q in the line $y=x$, then the circle whose extremities of a diameter are Q and R passes through a fixed point. The fixed point is

a. (3, 0) **b.** (5, 0) **c.** (0, 0) **d.** (4, 0)

32. For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangent drawn at the point P in the first quadrant meets the y-axis at Q and the chord $A'P$ meets the y-axis at M. If O is the origin, then OQ^2-MQ^2 equals to

a. 9 **b.** 13 **c.** 4 **d.** 5

33. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes at points A and B respectively. If C is the centre of the ellipses, then the area of the triangle ABC is

a. 12 sq. units **b.** 24 sq. units

c. 36 sq. units **d.** 48 sq. units

34. Let P be any point on a directrix of an ellipse of eccentricity e. S be the corresponding focus and C the centre of the ellipse. The line PC meets the ellipse at A. The angle between PS and tangent at A is α, then α is equal to

a. $\tan^{-1} e$ **b.** $\frac{\pi}{2}$

c. $\tan^{-1}(1-e^2)$ **d.** none of these

35. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is normal to the circle $x^2+y^2+4x+1=0$, then the maximum value of ab is

a. 4 **b.** 2

c. 1 **d.** none of these

36. If $(\sqrt{3})\,bx+ay=2ab$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the eccentric angle of the point of contact is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

37. If the ellipse $\frac{x^2}{a^2-7}+\frac{y^2}{13-5a}=1$ is inscribed in a square of side length $\sqrt{2}a$, then a is equal to

a. $\frac{6}{5}$

b. $(-\infty,-\sqrt{7})\cup(\sqrt{7}, 13/5)$

c. $(-\infty,-\sqrt{7})\cup(13/5,\sqrt{7})$

d. no such a exists

38. Locus of the point of intersection of the tangent at the end points of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(b<a)$ is a/an

a. circle **b.** ellipse
c. hyperbola **d.** pair of straight lines

39. The normal at a variable point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse at Q and R, then the locus of the midpoint of QR is a conic with an eccentricity e' such that

a. e' is independent of e
b. $e'=1$
c. $e'=e$
d. $e'=1/e$

40. Any ordinate MP of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ meets the auxiliary circle at Q, then locus of the point of intersection of normals at P and Q to the respective curves is

a. $x^2+y^2=8$ **b.** $x^2+y^2=34$
c. $x^2+y^2=64$ **d.** $x^2+y^2=15$

41. Number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ from the point $P(0, 6)$ is

a. one **b.** two **c.** three **d.** four

42. An ellipse has the points $(1,-1)$ and $(2,-1)$ as its foci and $x+y-5=0$ as one of its tangents. Then the point where this line touches the ellipse from origin is

a. $\left(\frac{32}{9},\frac{22}{9}\right)$ **b.** $\left(\frac{23}{9},\frac{2}{9}\right)$
c. $\left(\frac{34}{9},\frac{11}{9}\right)$ **d.** none of these

43. If tangents PQ and PR are drawn from a point on the circle $x^2+y^2=25$ to the ellipse $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, $(b<4)$, so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of triangle PQR, then eccentricity of the ellipse is

a. $\frac{\sqrt{5}}{4}$ **b.** $\frac{\sqrt{7}}{3}$
c. $\frac{\sqrt{7}}{4}$ **d.** $\frac{\sqrt{5}}{3}$

44. If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is inscribed in a rectangle whose length to breadth ratio is 2:1 then the area of the rectangle is

a. $4\frac{a^2+b^2}{7}$ **b.** $4\frac{a^2+b^2}{3}$
c. $12\frac{a^2+b^2}{5}$ **d.** $8\frac{a^2+b^2}{5}$

45. If the normals at $P(\theta)$ and $Q(\pi/2+\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the major axis at G and g, respectively, then $PG^2+Qg^2=$

a. $b^2(1-e^2)(2-e^2)$
b. $a^2(e^4-e^2+2)$
c. $a^2(1+e^2)(2+e^2)$
d. $b^2(1+e^2)(2+e^2)$

46. The line $y=mx-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m belongs to

a. $(0, 1)$ **b.** $(0, \infty)$
c. R **d.** none of these

47. The length of the sides of square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{7}+\frac{2y^2}{11}=1$ is

a. 10 units **b.** 8 units **c.** 6 units **d.** 5 units

48. From point $P(8, 27)$, tangent PQ and PR are drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. Then the angle subtended by QR at origin is

a. $\tan^{-1}\frac{\sqrt{6}}{65}$ **b.** $\tan^{-1}\frac{4\sqrt{6}}{65}$
c. $\tan^{-1}\frac{8\sqrt{2}}{65}$ **d.** $\tan^{-1}\frac{48\sqrt{6}}{455}$

49. Let P be any point on any directrix of an ellipse. Then chords of contact of point P with respect to the ellipse and its auxiliary circle intersect at

a. some point on the major axis depending upon the position of point P
b. midpoint of the line segment joining the centre to the corresponding focus
c. corresponding focus
d. none of these

50. The equation of the line passing through the centre and bisecting the chord $7x+y-1=0$ of the ellipse $\frac{x^2}{1}+\frac{y^2}{7}=1$ is

a. $x=y$ **b.** $2x=y$ **c.** $x=2y$ **d.** $x+y=0$

51. Equation of the chord of contact of pair of tangents drawn to the ellipse $4x^2+9y^2=36$ from the point (m, n) where $m.n=m+n$, m, n being non-zero positive integers is

a. $2x+9y=18$ **b.** $2x+2y=1$
c. $4x+9y=18$ **d.** none of these

52. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms a triangle of constant area with the coordinate axes is

a. a straight line **b.** a hyperbola
c. an ellipse **d.** a circle

53. An ellipse is sliding along the co-ordinate axes. If the foci of the ellipse are (1, 1) and (3, 3), then area of the director circle of the ellipse (in sq. units) is

a. 2π **b.** 4π **c.** 6π **d.** 8π

54. The equation of the ellipse whose axes are coincident with the co-ordinates axes and which touches the straight lines $3x-2y-20=0$ and $x+6y-20=0$ is

a. $\frac{x^2}{40}+\frac{y^2}{10}=1$ **b.** $\frac{x^2}{5}+\frac{y^2}{8}=1$
c. $\frac{x^2}{10}+\frac{y^2}{40}=1$ **d.** $\frac{x^2}{40}+\frac{y^2}{30}=1$

55. An ellipse with major and minor axes length as $2a$ and $2b$ touches coordinate axes in first quadrant and having foci (x_1, y_1) and (x_2, y_2) then the value of x_1x_2 and y_1y_2 is

a. a^2 **b.** b^2 **c.** a^2b^2 **d.** a^2+b^2

56. Let P_i and P_i' be the feet of the perpendiculars drawn from foci S, S' on a tangent T_i to an ellipse whose length of semi-major axis is 20, if $\sum_{i=1}^{10}(SP_i)(S'P_i')=2560$, then the value of eccentricity is

a. $\frac{1}{5}$ **b.** $\frac{2}{5}$ **c.** $\frac{3}{5}$ **d.** $\frac{4}{5}$

57. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

a. 0 **b.** 2 **c.** 1 **d.** 4

58. A parabola is drawn with focus is at one of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b$) and directrix passing through the other focus and perpendicular to the major axes of the ellipse. If latus rectum of the ellipse and the parabola are same, then the eccentricity of the ellipse is

a. $1-\frac{1}{\sqrt{2}}$ **b.** $2\sqrt{2}-2$
c. $\sqrt{2}-1$ **d.** None of these

59. If maximum distance of any point on the ellipse $x^2+2y^2+2xy=1$ from its centre be r, then r is equal to

a. $3+\sqrt{3}$ **b.** $2+\sqrt{2}$
c. $\frac{\sqrt{2}}{\sqrt{3-\sqrt{5}}}$ **d.** $\sqrt{2-\sqrt{2}}$

60. The set of values of m for which it is possible to draw the chord $y=\sqrt{m}\,x+1$ to the curve $x^2+2xy+(2+\sin^2\alpha)y^2=1$, which subtends a right angle at the origin for some value of α is

a. [2, 3] **b.** [0, 1] **c.** [1, 3] **d.** none of these

Multiple Correct Answers Type

Solutions on page 4.48

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. $\frac{x^2}{r^2-r-6}+\frac{y^2}{r^2-6r+5}=1$ will represents the ellipse, if r lies in the interval

a. $(-\infty,-2)$ **b.** $(3,\infty)$ **c.** $(5,\infty)$ **d.** $(1,\infty)$

2. On the x-y plane, the eccentricity of an ellipse is fixed (in size and position) by

a. both foci
b. both directrices
c. one focus and corresponding directrix
d. length of major axis

3. The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2. Then the eccentric angle of the point is

a. $\frac{\pi}{4}$ **b.** $\frac{3\pi}{4}$ **c.** $\frac{5\pi}{6}$ **d.** $\pi/6$

4. If the equation of the ellipse is $3x^2+2y^2+6x-8y+5=0$, then which of the following is/are true?

a. $e=\frac{1}{\sqrt{3}}$
b. centre is $(-1, 2)$
c. foci are $(-1, 1)$ and $(-1, 3)$
d. directrices are $y=2\pm\sqrt{3}$

5. If the tangent at the point $P(\theta)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x=15$, then $\theta=$

a. $\frac{2\pi}{3}$ **b.** $\frac{4\pi}{3}$ **c.** $\frac{5\pi}{3}$ **d.** $\frac{\pi}{3}$

6. The co-ordinates (2, 3) and (1, 5) are the foci of an ellipse which passes through the origin, then the equation of

a. tangent at the origin is $(3\sqrt{2}-5)x+(1-2\sqrt{2})y=0$
b. tangent at the origin is $(3\sqrt{2}+5)x+(1+2\sqrt{2}y)=0$
c. normal at the origin is $(3\sqrt{2}+5)x-(2\sqrt{2}+1)y=0$
d. normal at the origin is $x(3\sqrt{2}-5)-y(1-2\sqrt{2})=0$

7. If the chord through the points whose eccentric angles are θ and ϕ on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ passes through a focus, then the value of $\tan(\theta/2)\tan(\phi/2)$ is

a. $\frac{1}{9}$ **b.** -9 **c.** $-\frac{1}{9}$ **d.** 9

8. The equation $3x^2 + 4y^2 - 18x + 16y + 43 = k$

a. represents empty set, if $k < 0$

b. represents an ellipse, if $k > 0$

c. a point, if $k = 0$

d. cannot represent a real pair of straight lines for any value of k

9. If a pair of variable straight lines $x^2 + 4y^2 + \alpha xy = 0$ (where α is a real parameter) cut the ellipse $x^2 + 4y^2 = 4$ at two points A and B, then the locus of the point of intersection of tangents at A and B is

a. $x - 2y = 0$ **b.** $2x - y = 0$

c. $x + 2y = 0$ **d.** $2x + y = 0$

10. Which of the following is/are true?

a. There are infinite positive integral values of a for which $(13x - 1)^2 + (13y - 2)^2 = \left(\frac{5x + 12y - 1}{a}\right)^2$ represents an ellipse

b. The minimum distance of a point (1, 2) from the ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ is 1

c. If from a point $P(0, \alpha)$ two normals other than axes are drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $|\alpha| < \frac{9}{4}$

d. If the length of latus rectum of an ellipse is one-third of its major axis, then its eccentricity is equal to $\frac{1}{\sqrt{3}}$

11. Which of the following is/are true about the ellipse $x^2 + 4y^2 - 2x - 16y + 13 = 0$?

a. The latus rectum of the ellipse is 1

b. Distance between foci of the ellipse is $4\sqrt{3}$

c. Sum of the focal distances of a point $P(x, y)$ on the ellipse is 4

d. $y = 3$ meets the tangents drawn at the vertices of the ellipse at points P and Q then PQ subtends a right angle at any of its foci

12. A point on the ellipse $x^2 + 3y^2 = 37$ where the normal is parallel to the line $6x - 5y = 2$ is

a. $(5, -2)$ **b.** $(5, 2)$ **c.** $(-5, 2)$ **d.** $(-5, -2)$

13. The locus of the image of the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ $(a > b)$ with respect to any of the tangents to the ellipse is

a. $(x + 4)^2 + y^2 = 100$ **b.** $(x + 2)^2 + y^2 = 50$

c. $(x - 4)^2 + y^2 = 100$ **d.** $(x - 2)^2 + y^2 = 50$

14. Let E_1 and E_2 be two ellipses $\frac{x^2}{a^2} + y^2 = 1$ and $x^2 + \frac{y^2}{a^2} = 1$ (where a is a parameter). Then the locus of the points of intersection of the ellipses E_1 and E_2 is a set of curves comprising

a. two straight lines **b.** one straight line

c. one circle **d.** one parabola

15. Consider the ellipse $\frac{x^2}{f(k^2 + 2k + 5)} + \frac{y^2}{f(k + 11)} = 1$ and $f(x)$ is a positive decreasing function, then

a. the set of values of k, for which the major axis is x-axis is $(-3, 2)$

b. the set of values of k, for which the major axis is y-axis is $(-\infty, 2)$

c. the set of values of k, for which the major axis is y-axis is $(-\infty, -3) \cup (2, \infty)$

d. the set of values of k, for which the major axis is y-axis is $(-3, \infty)$

16. If two concentric ellipses are such that the foci of one are on the other and their major axes are equal. Let e and e' be their eccentricities, then

a. the quadrilateral formed by joining the foci of the two ellipses is a parallelogram

b. the angle θ between their axes is given by $\theta = \cos^{-1}\sqrt{\frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2 e'^2}}$

c. if $e^2 + e'^2 = 1$, then the angle between the axes of the two ellipses is 90°

d. none of these

17. If the tangent drawn at point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Then

a. $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ **b.** $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

c. $t = -\frac{2}{\sqrt{5}}$ **d.** $t = -\frac{1}{\sqrt{5}}$

Reasoning Type

Solutions on page 4.52

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.

b. Both the statements are True but Statement 2 is NOT the correct explanation of Statement 1.

c. Statement 1 is True and Statement 2 is False.

d. Statement 1 is False and Statement 2 is True.

1. Statement 1: The locus of a moving point (x, y) satisfying $\sqrt{(x-2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4$ is ellipse.

Statement 2: Distance between $(-2, 0)$ and $(2, 0)$ is 4.

2. Statement 1: In a triangle ABC, if base BC is fixed and perimeter of the triangle is constant, then vertex A moves on an ellipse.

Statement 2: If the sum of distances of a point P from two fixed points is constant, then locus of P is a real ellipse.

3. **Statement 1:** In an ellipse the sum of the distances between foci is always less than the sum of focal distances of any point on it.

Statement 2: The eccentricity of any ellipse is less than 1.

4. **Statement 1:** The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$ is $y = 0, y = 6$.

Statement 2: The equation of the tangent drawn at the ends of major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is always parallel to y-axis.

5. **Statement 1:** There can be maximum two points on the line $px + qy + r = 0$, from which perpendicular tangents can be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Statement 2: Circle $x^2 + y^2 = a^2 + b^2$ and the given line can intersect in maximum two distinct points.

6. **Statement 1:** Circle $x^2 + y^2 = 9$, and the circle $(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$ touches each other internally.

Statement 2: Circle described on the focal distance as diameter of the ellipse $4x^2 + 9y^2 = 36$ touches the auxiliary circle $x^2 + y^2 = 9$ internally.

7. **Statement 1:** Locus of the centre of a variable circle touching two circles $(x - 1)^2 + (y - 2)^2 = 25$ and $(x - 2)^2 + (y - 1)^2 = 16$ is an ellipse.

Statement 2: If a circle $S_2 = 0$ lies completely inside the circle $S_1 = 0$, then locus of the centre of a variable circle $S = 0$ that touches both the circles is an ellipse.

8. **Statement 1:** For the ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ the product of the perpendiculars drawn from foci on any tangent is 3.

Statement 2: For ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$, the foot of the perpendiculars drawn from foci on any tangent lies on the circle $x^2 + y^2 = 5$ which is an auxiliary circle of the ellipse.

9. **Statement 1:** If there is exactly one point on the line $3x + 4y + 5\sqrt{5} = 0$, from which perpendicular tangents can be drawn to the ellipse $\frac{x^2}{a^2} + y^2 = 1$ $(a > 1)$, then the eccentricity of the ellipse is $\frac{1}{3}$.

Statement 2: For the condition given in statement 1, given line must touch the circle $x^2 + y^2 = a^2 + 1$.

10. **Statement 1:** Any chord of the conic $x^2 + y^2 + xy = 1$ through $(0, 0)$ is bisected at $(0, 0)$.

Statement 2: The centre of a conic is a point through which every chord is bisected.

11. **Statement 1:** The area of the ellipse $2x^2 + 3y^2 = 6$ is more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

Statement 2: the length of semi-major axes of an ellipse is more than the radius of the circle.

12. **Statement 1:** If line $x + y = 3$ is a tangent to an ellipse with foci $(4, 3)$ and $(6, y)$ at the point $(1, 2)$, then $y = 17$.

Statement 2: Tangent and normal to the ellipse at any point bisects the angle subtended by foci at that point.

13. **Statement 1:** Diagonals of any parallelogram inscribed in an ellipse always intersect at the centre of the ellipse

Statement 2: Centre of the ellipse is the point at which chord passing through the centre of the ellipse gets bisected at the centre.

14. **Statement 1:** A triangle ABC right angled at A moves so that its perpendicular sides touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ all the time. Then loci of the points A, B and C are circle.

Statement 2: Locus of point of intersection of two perpendicular tangents to the curve is a director circle.

15. **Statement 1:** Tangents are drawn to the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at the points, where it is intersected by the line $2x + 3y = 1$. Point of intersection of these tangents is $(8, 6)$.

Statement 2: Equation of the chord of contact to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from an external point is given by $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.

16. **Statement 1:** If tangent at point P (in first quadrant) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$, meets corresponding directrix $x = a/e$ at point Q, then circle with minimum radius having PQ as chord passes through the corresponding focus.

Statement 2: PQ subtends right angle at corresponding focus.

17. **Statement 1:** If a, b are real numbers and $c > 0$, then the locus represented by the equation $|ay - bx| = c\sqrt{(x - a)^2 + (y - b)^2}$ is an ellipse.

Statement 2: An ellipse is the locus of a point which moves in a plane such that ratio of its distances from a fixed point (i.e., focus) to the fixed line (i.e., directrix) is constant and less than 1.

Linked Comprehension Type

Solutions on page 4.53

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

An ellipse (E) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centred at point O have AB and CD as its major and minor axes, respectively, let the S_1 be one of the foci of the ellipse, radius of incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units. Then

1. perimeter of ΔOCS_1 is

   a. 20 units **b.** 10 units **c.** 15 units **d.** 25 units

2. Equation of director the director circle of ellipse (E) is

   a. $x^2 + y^2 = (48.5)$ **b.** $x^2 + y^2 = \sqrt{97}$

   c. $x^2 + y^2 = 97$ **d.** $x^2 + y^2 = \sqrt{48.5}$

3. Area of ellipse (E) is

   a. $\frac{65\pi}{4}$ **b.** $\frac{64\pi}{5}$ **c.** 64π **d.** 65π

For Problems 4–6

Consider the ellipse whose major and minor axes are x-axis and y-axis, respectively. If ϕ is the angle between the CP and the normal at point P on the ellipse, and the greatest value tan ϕ is $\frac{3}{2}$ (where C is the centre of the ellipse). Also semi-major axis is 10 units.

4. The eccentricity of the ellipse is

   a. $\frac{1}{2}$ **b.** $\frac{1}{3}$

   c. $\frac{\sqrt{3}}{2}$ **d.** none of these

5. A rectangle is inscribed in the ellipse whose sides are parallel to the co-ordinates axes, then maximum area of rectangle is

   a. 50 units **b.** 100 units

   c. 25 units **d.** none of these

6. Locus of the point of intersection of perpendicular tangents to the ellipse is

   a. $x^2 + y^2 = 125$ **b.** $x^2 + y^2 = 150$

   c. $x^2 + y^2 = 200$ **d.** none of these

For Problems 7–9

A curve is represented by $C = 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

7. Eccentricity of curve is

   a. 1/3 **b.** $1/\sqrt{3}$ **c.** 2/3 **d.** $2/\sqrt{5}$

8. The lengths of axes

   a. $6, 2\sqrt{6}$ **b.** $5, 2\sqrt{5}$

   c. $4, 4\sqrt{5}$ **d.** none of these

9. The centre of the conic C is

   a. (1, 0) **b.** (0,0)

   c. (0, 1) **d.** none of these

For Problems 10–12

For all real p, the line $2px + y\sqrt{1-p^2} = 1$ touches a fixed ellipse whose axes are coordinate axes.

10. The eccentricity of the ellipse is

    a. $\frac{2}{3}$ **b.** $\frac{\sqrt{3}}{2}$ **c.** $\frac{1}{\sqrt{3}}$ **d.** $\frac{1}{2}$

11. The foci of ellipse are

    a. $(0, \pm\sqrt{3})$ **b.** $(0, \pm 2/3)$

    c. $(\pm\sqrt{3}/2, 0)$ **d.** none of these

12. The locus of point of intersection of perpendicular tangents is

    a. $x^2 + y^2 = \frac{5}{4}$ **b.** $x^2 + y^2 = \frac{3}{2}$

    c. $x^2 + y^2 = 2$ **d.** none of these

For Problems 13–15

Let S, S' be the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentricity is e. P is a variable point on the ellipse. Consider the locus of the incentre of the $\Delta PSS'$.

13. The locus of incentre is

    a. ellipse **b.** hyperbola

    c. parabola **d.** circle

14. The eccentricity of locus of P is

    a. $\sqrt{\frac{2e}{1-e}}$ **b.** $\sqrt{\frac{2e}{1+e}}$

    c. 1 **d.** none of these

15. Maximum area of rectangle inscribed in the locus is

    a. $\frac{2abe^2}{1+e}$ **b.** $\frac{2abe}{1-e}$

    c. $\frac{abe}{1+e}$ **d.** none of these

For Problems 16–18

$C_1: x^2 + y^2 = r^2$ and $C_2: \frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four distinct points A, B, C and D. Their common tangents form a parallelogram $A'B'C'D'$.

16. If $ABCD$ is a square then r is equal to

    a. $\frac{12}{5}\sqrt{2}$ **b.** $\frac{12}{5}$

    c. $\frac{12}{5\sqrt{5}}$ **d.** none of these

17. If A', B', C', D' is a square then r is equal to

    a. $\sqrt{20}$ **b.** $\sqrt{12}$

    c. $\sqrt{15}$ **d.** none of these

18. If $A'B'C'D'$ is a square, then the ratio of area of the circle C_1 to the area of the circumcircle of $\Delta A'B'C'$ is

    a. $\frac{9}{16}$ **b.** $\frac{3}{4}$

    c. $\frac{1}{2}$ **d.** none of these

For Problems 19–21

A coplanar beam of light emerging from a point source has the equation $\lambda x - y + 2(1 + \lambda) = 0$, $\lambda \in R$, the rays of the beam strike an elliptical surface and get reflected. The reflected rays form another convergent beam having equation $\mu x - y + 2(1 - \mu) = 0$, $\mu \in R$. Further it is found that the foot of the perpendicular from the point (2, 2) upon any tangent to the ellipse lies on the circle $x^2 + y^2 - 4y - 5 = 0$.

19. The eccentricity of the ellipse is equal to

a. $\frac{1}{3}$ **b.** $\frac{1}{\sqrt{3}}$ **c.** $\frac{2}{3}$ **d.** $\frac{1}{2}$

20. The area of the largest triangle that an incident ray and the corresponding reflected ray can enclose with axis of the ellipse is equal to

a. $4\sqrt{5}$ **b.** $2\sqrt{5}$
c. $\sqrt{5}$ **d.** none of these

21. Total distance travelled by an incident ray and the corresponding reflected ray is the least if the point of incidence coincides with

a. an end of the minor with
b. an end of the major axis
c. an end of this latus rectum
d. none of these

For Problems 22–24

The tangent at any point P of the circle $x^2 + y^2 = 16$ meets the tangent at a fixed point A at T, and T is joined to B, the other end of the diameter through A.

22. The locus of the intersection of AP and BT is conic whose eccentricity is

a. $\frac{1}{2}$ **b.** $\frac{1}{\sqrt{2}}$ **c.** $\frac{1}{3}$ **d.** $\frac{1}{\sqrt{3}}$

23. Sum of focal distances of any point on the curve is

a. 12 **b.** 16 **c.** 20 **d.** 8

24. Which of the following does not change by changing the radius of the circle?

a. coordinates of focii
b. length of major axis
c. eccentricity
d. length of minor axis

For Problems 25–27

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that it has the least area but contains the circle $(x - 1)^2 + y^2 = 1$.

25. The eccentricity of the ellipse is

a. $\sqrt{\frac{2}{3}}$ **b.** $\frac{1}{\sqrt{3}}$
c. $\frac{1}{2}$ **d.** none of these

26. Equation of auxiliary circle of ellipse is

a. $x^2 + y^4 = 6.5$ **b.** $x^2 + y^4 = 5$
c. $x^2 + y^4 = 45$ **d.** none of these

27. Length of latus-rectum of the ellipse is

a. 2 units **b.** 1 unit **c.** 3 unit **d.** 2.5 unit

Matrix-Match Type

Solutions on page 4.57

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, , b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

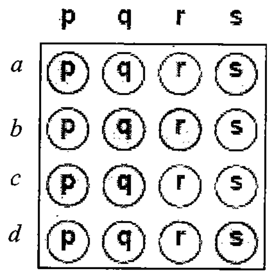


Fig. 4.35

1.

Column I	Column II
a. Distance between the points on the curve $4x^2 + 9y^2 = 1$, where tangent is parallel to the line $8x = 9y$, is less than	*p.* 1
b. Sum of distance between the foci of the curve $25(x + 1)^2 + 9(y + 2)^2 = 225$ from (–1, 0) is more than	*q.* 4
c. Sum of distances from the x-axis of the points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4}$, where the normal is parallel to the line $2x + y = 1$, is less than	*r.* 7
d. Tangents are drawn from points on the line $x - y + 2 = 0$ to the ellipse $x^2 + 2y^2 = 2$, then all the chords of contact pass through the point whose distance from (2, 1/2) is more than	*s.* 5

2. The tangents drawn from a point P to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make angle α and β with the major axis.

Column I	Column II
a. If $\alpha+\beta=\frac{c\pi}{2}$ $(c \in N)$	*p.* circle
b. If $\tan\alpha\tan\beta=c$ {where $c \in R$}, then locus of P can be	*q.* ellipse
c. If $\tan\alpha+\tan\beta=c$ {where $c \in R$}, then locus of P can be	*r.* hyperbola
d. If $\cot\alpha+\cot\beta=c$ {where $c \in R$}, then locus of P can be	*s.* pair of straight lines

3.

Column I	Column II
a. If the tangent to the ellipse $x^2+4y^2=16$ at the point $P(\phi)$ is a normal to the circle $x^2+y^2-8x-4y=0$, then $\frac{\phi}{2}$ may be	*p.* 0
b. The eccentric angle(s) of a point on the ellipse $x^2+3y^2=6$ at a distance 2 units from the centre of the ellipse is/are	*q.* $\cos^{-1}\left(-\frac{2}{3}\right)$
c. The eccentric angle of intersection of the ellipse $x^2+4y^2=4$ and the parabola $x^2+1=y$ is	*r.* $\frac{\pi}{4}$
d. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then θ is	*s.* $\frac{5\pi}{4}$

4.

Column I	Column II
a. A stick of length 10 m slides on co-ordinate axes, then locus of a point dividing this stick from x-axis in the ratio 6 : 4 is a curve whose eccentrictly is e, then $3e$ is equal to	*p.* $\sqrt{6}$
b. AA' is a major axis of an ellipse $3x^2+2y^2+6x-4y-1=0$ and P is a variable point on it, then greatest area of triangle APA' is	*q.* $2\sqrt{7}$
c. Distance between foci of the curve represented by the equation $x=1+4\cos\theta$, $y=2+3\sin\theta$ is	*r.* $\frac{128}{3}$
d. Tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1$ at end points of the latus rectum. The area of equadrilateral so formed is	*s.* $\sqrt{5}$

5.

Column I	Column II
a. An ellipse passing through the origin has its foci (3, 4) and (6, 8), then length of its minor axis is	*p.* 8
b. If PQ is focal chord of ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ which passes through $S \equiv (3, 0)$ and $PS = 2$, then length of chord PQ is	*q.* $10\sqrt{2}$
c. If the line $y=x+K$ touches the ellipse $9x^2+16y^2=144$, then the difference of values of K is	*r.* 10
d. Sum of distances of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ from the foci	*s.* 12

6.

Column I	Column II
a. If vertices of a rectangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are extremities of latus rectum. Then eccentricity of ellipse is	*p.* $\frac{2}{\sqrt{5}}$
b. If extremities of diameter of the circle $x^2+y^2=16$ are foci of a ellipse, then eccentricity of the ellipse, if its size is just sufficient to contain the circle, is	*q.* $\frac{1}{\sqrt{2}}$
c. If normal at point (6, 2) to the ellipse passes through its nearest focus (5, 2), having centre at (4, 2) then its eccentricity is	*r.* $\frac{1}{3}$
d. If extremities of latus rectum of the parabola $y^2=24x$ are foci of ellipse and if ellipse passes through the vertex of the parabola, then its eccentricity is	*s.* $\frac{1}{2}$

Integer type

Solutions on page 4.60

1. If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$, then the difference between the largest and smallest value of the expression $\frac{x^2}{4}+\frac{y^2}{9}$ is
2. The value of a for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$, if the extremities of the latus rectum of the ellipse having positive ordinate lies on the parabola $x^2=-2(y-2)$, is

3. If the variable line $y = kx + 2h$ is tangent to an ellipse $2x^2 + 3y^2 = 6$, then locus of $P(h, k)$ is a conic C whose eccentricity is e then the value of $3e^2$ is

4. Tangents drawn from the point $P(2, 3)$ to the circle $x^2 + y^2 - 8x + 6y + 1 = 0$ touch the circle at the points A and B. The circumcircle of the ΔPAB cuts the director circle of ellipse $\frac{(x+5)^2}{9} + \frac{(y-3)^2}{b^2} = 1$ orthogonally. Then the value of $b^2/6$ is

5. If from a point $P(0, \alpha)$ two normals other than axes are drawn to ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, such that $|\alpha| < k$, then the value of $4k$ is

6. An ellipse passing through the origin has its foci (3, 4) and (6, 8) and length of its semi-minor axis is b, then the value of $b/\sqrt{2}$ is

7. If the mid point of a chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is (0, 3), and length of the chord is $\frac{4k}{5}$, then k is

8. Let the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$. If the length of the minor axis is k, then $\sqrt{3}\,k/2$ is

9. Consider an ellipse (E) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centered at point '$O$' and having AB and CD as its major and minor axes respectively if S_1 be one of the foci of the ellipse, radius of incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units, then the value of $(a - b)/2$ is

10. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

11. Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$, then the maximum value of $(4x - 9y)/2$ is

12. Rectangle $ABCD$ has area 200. An ellipse with area 200π passes through A and C and has foci at B and D. If the perimeter of the rectangle is P, then the value of $P/20$ is

Archives

Solutions on page 4.62

Subjective Type

1. Let 'd' be the perpendicular distance from the centre of the ellipse to any tangent to ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$. **(IIT-JEE, 1995)**

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **(IIT-JEE, 1997)**

3. Consider the family of circle $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the co-ordiante axes at A and B, then find the equation of the locus of the midpoint of AB. **(IIT-JEE, 1999)**

4. Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the triangle PON is maximum, where O denotes the origin and N be the foot of the perpendicular from O to the tangent at P. **(IIT-JEE, 1999)**

5. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendicular from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ meets the ellipse, respectively, at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C, respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. **(IIT-JEE, 2000)**

6. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR:RQ = r:s$ as P varies over the ellipse. **(IIT-JEE, 2001)**

7. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **(IIT-JEE, 2002)**

8. From a point, common tangents are drawn to the curve $x^2 + y^2 = 16$ and $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Find the slope of common tangent in 1st quadrant and also find the length of intercept between coordinate axes. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. An ellipse has OB as a semi-minor axis, F, F' as its foci and the angle $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is.

Multiple choice questions with one correct answer

1. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}$, and having its centre (0, 3) is

   a. 4 b. 3 c. $\sqrt{12}$ d. $\frac{7}{2}$

   (IIT-JEE, 1995)

2. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4}+\frac{y^2}{1}=1$ is

   a. 0 b. 1 c. 2 d. infinite

   (IIT-JEE, 1998)

3. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

   a. 8 b. 6 c. 10 d. 12

   (IIT-JEE, 1998)

4. The area of the quadrilateral formed by the tangents at the end point of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is

   a. 27/4 sq. units b. 9 sq. units
   c. 27/2 sq. units d. 27 sq. units

   (IIT-JEE, 2003)

5. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the midpoint of the intercept made by the tangents between the coordinate axes is

   a. $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ b. $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

   c. $\frac{x^2}{2}+\frac{y^2}{4}=1$ d. $\frac{x^2}{4}+\frac{y^2}{2}=1$

   (IIT-JEE, 2004)

6. The minimum area of a triangle formed by the tangent to the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes is

   a. ab sq. units b. $\frac{a^2+b^2}{2}$ sq. units

   c. $\frac{(a+b)^2}{2}$ sq. units d. $\frac{a^2+ab+b^2}{3}$ sq. units

   (IIT-JEE, 2004)

7. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

   a. $\frac{31}{10}$ b. $\frac{29}{10}$ c. $\frac{21}{10}$ d. $\frac{27}{10}$

   (IIT-JEE, 2009)

8. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

   a. $(\pm (3\sqrt{5})/2, \pm 2/7)$ b. $(\pm (3\sqrt{5})/2, \pm \sqrt{19}/7)$

   c. $(\pm 2\sqrt{3}, \pm 1/7)$ d. $(\pm 2\sqrt{3}, \pm 4\sqrt{3}/7)$

   (IIT-JEE, 2009)

9. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then

   a. Q lies inside C but outside E
   b. Q lies outside both C and E
   c. P lies inside both C and E
   d. P lies inside C but outside E

   (IIT-JEE, 1994)

Multiple choice questions with one or more than one correct answer

1. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

   (IIT-JEE, 1999)

   a. $\left(\frac{2}{5},\frac{1}{5}\right)$ b. $\left(-\frac{2}{5},\frac{1}{5}\right)$

   c. $\left(-\frac{2}{5},-\frac{1}{5}\right)$ d. $-\left(\frac{2}{5},-\frac{1}{5}\right)$

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are

   a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
   b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
   d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

   (IIT-JEE, 2008)

3. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then **(IIT-JEE, 2009)**

   a. $b + c = 4a$
   b. $b + c = 2a$
   c. locus of point A is an ellipse
   d. locus of point A is a pair of straight line

Comprehension type

Tangents are drawn from the point $P(3,4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

1. The coordinate of A and B are

   a. (3,0) and (0,2)

b. $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

c. $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0,2)

d. (3,0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$ **(IIT-JEE, 2010)**

2. The orthocenter of the triangle PAB is

a. $\left(5, \frac{8}{7}\right)$ **b.** $\left(\frac{7}{5}, \frac{25}{8}\right)$ **c.** $\left(\frac{11}{5}, \frac{8}{5}\right)$ **d.** $\left(\frac{8}{25}, \frac{7}{5}\right)$

(IIT-JEE, 2010)

3. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

a. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

b. $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

c. $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

d. $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1.

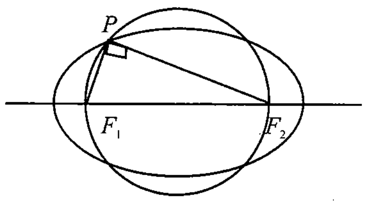


Fig. 4.39

$PF_1 + PF_2 = 15$

$PF_1 \times PF_2 = 52$

$(F_1F_2)^2 = (PF_1 + PF_2)^2 - 2PF_1 \times PF_2$

$= 225 - 104 = 121$

$\Rightarrow \quad F_1F_2 = 11$

$\Rightarrow \quad 2ae = 11$

$\Rightarrow \quad ae = \frac{11}{2}$

Also $\quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2$

$\Rightarrow \quad a^2 - b^2 = (ae)^2$

$\Rightarrow \quad 4(a^2 - b^2) = 4\left(\frac{11}{2}\right)^2 = 121$

2. Let the middle point of chord be $(t^2, 2t)$

Midpoint of chord must lie inside the ellipse

$\Rightarrow \quad t^4 + 8t^2 - 1 < 0$

$\Rightarrow \quad t^2 \in (0, -4 + \sqrt{17})$ (i)

Also equation of the chord is $T = S_1$

$\Rightarrow \quad t^2x + 4ty = t^4 + 8t^2$

Let this passes through $(\alpha, 0)$

$\Rightarrow \quad \alpha t^2 = t^4 + 8t^2$

$\Rightarrow \quad t^4 + (8 - \alpha)t^2 = 0$

$\Rightarrow \quad t^2 = 0$ or $t^2 = \alpha - 8$

$\Rightarrow \quad \alpha = t^2 + 8$

$\Rightarrow \quad \alpha \in (8, 4 + \sqrt{17})$

3. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Any tangent to the ellipse be

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Tangents at the vertices are $x = a$ and $x = -a$.

Solving with (i), we get the points T and T' as

$T\left[a, \frac{b(1 - \cos\theta)}{\sin\theta}\right], T'\left[-a, \frac{b(1 + \cos\theta)}{\sin\theta}\right]$

i.e., $T\left[a, b\tan\frac{\theta}{2}\right], T'\left[-a, b\cot\frac{\theta}{2}\right]$

circle on TT' as diameter is

$(x - a)(x + a) + (y - b\tan\frac{\theta}{2})(y - b\cot\frac{\theta}{2}) = 0$

$\Rightarrow \quad x^2 - a^2 + y^2 + b^2 - b\left[\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right]y = 0$

It will pass through the foci (± ae, 0)

If $\quad a^2e^2 - a^2 + b^2 = 0$

$\Rightarrow \quad b^2 = a^2(1 - e^2)$, which is true.

4.

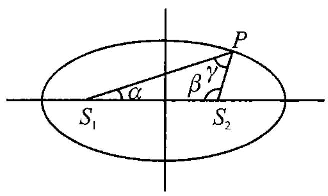


Fig. 4.40

We know that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}$$

In triangle ABC, we know that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$\Rightarrow \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=3\cot\frac{\gamma}{2}$$

$$\Rightarrow \quad 2\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}$$

$\Rightarrow$ $\cot\frac{\alpha}{2}, \cot\frac{\gamma}{2}, \cot\frac{\beta}{2}$ in A.P.

5.

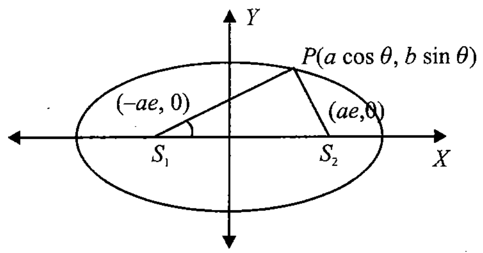


Fig. 4.41

Let any point P on the ellipse be $P(a\cos\theta, b\sin\theta)$.

If $\angle S_1PS_2=\frac{\pi}{2}$, then P lies on the circle having S_1S_2 as its diameter.

$\Rightarrow$ Equation of the circle drawn on S_1S_2 as diameter is

$$x^2+y^2=a^2e^2=a^2-b^2$$

Since the point P should not lie on the ellipse.

$\Rightarrow$ There should not be any point on intersection of

$$x^2+y^2=a^2-b^2 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \quad a^2-b^2<b^2$$

$$\Rightarrow \quad 2b^2>a^2$$

$$\Rightarrow \quad \frac{b^2}{a^2}>\frac{1}{2}$$

$$\Rightarrow \quad 1-e^2>\frac{1}{2}$$

$$\Rightarrow \quad e^2<\frac{1}{2}$$

$$\Rightarrow \quad e\in\left(0, \frac{1}{\sqrt{2}}\right)$$

6. Let $M(h, k)$ be the midpoint of AB.

Let $R(t, t+4)$, $t\in R$ be the point on the line $y=x+4$ and points be $P(x_1, y_1)$, $Q(x_2, y_2)$.

$A(x_1', y_1')$, $B(x_2', y_2')$

Now, $\quad x_1=x_1'$ and $x_2=x_2'$

and $\quad y_1=\frac{a}{b}y_1', y_2=\frac{a}{b}y_2'$.

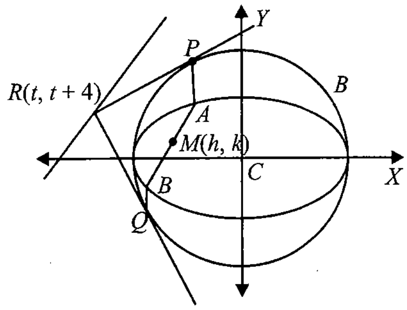


Fig. 4.42

Also $\quad h=\frac{x_1+x_2}{2}$ and $k=\frac{y_1'+y_2'}{2}$

Let N be the midpoint of PQ. So coordinates corresponding to N are

$$\frac{x_1+x_2}{2}=h, \frac{y_1+y_2}{2}=\frac{2(y_1'+y_2')}{2}$$

$$=2k \quad (\text{since } a/b=2)$$

So equation of PQ with $N(h, 2k)$ as midpoint is

$$xh+2yk=h^2+4k^2 \tag{i}$$

Also PQ is chord of contact with respect to R; So equation of PQ is

$$xt+y(t+4)=4 \tag{ii}$$

Comparing the coefficients of (i) and (ii), we get

$$\frac{h}{t}=\frac{2k}{t+4}=\frac{h^2+4k^2}{4}$$

$$\Rightarrow \quad t=\frac{4h}{h^2+4k^2} \text{ and}$$

$$8k=(h^2+4k^2)(t+4)$$

Eliminating t, we have

$$8k=(h^2+4k^2)\left(\frac{4h}{h^2+4k^2}+4\right)$$

$$\Rightarrow \quad 8k=4h+4h^2+16k^2.$$

$$\Rightarrow \quad 4k^2+h^2+h-2k=0$$

Hence, locus is

$$4y^2+x^2+x-2y=0.$$

7. Let the eccentric angles of the vertices P, Q, R of ΔPQR be $\theta_1, \theta_2, \theta_3$.

Then the equations of PQ and PR are

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}$$

and $\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2}+\frac{y}{b}\sin\frac{\theta_2+\theta_3}{2}=\cos\frac{\theta_2-\theta_3}{2}$, respectively

If PQ and PR are parallel to given straight lines, then we have

$$\theta_1+\theta_2=\text{constant}=2\alpha \text{ (say)}$$

and $\quad \theta_1+\theta_3=\text{constant}=2\beta$

Hence, $\theta_2 - \theta_3 = 2(\alpha - \beta)$ (i)

Now, the equation of QR is

$$\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2} + \frac{y}{b}\sin\frac{\theta_2+\theta_3}{2} = \cos\frac{\theta_2-\theta_3}{2} \quad \text{(ii)}$$

or $$\frac{x}{a}\cos\frac{\theta_2+\theta_3}{2} + \frac{y}{b}\sin\frac{\theta_2+\theta_3}{2} = \cos(\alpha-\beta) \quad \text{(iii)}$$

which shows that the line (ii), for different values of $\theta_2 + \theta_3$, is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\alpha - \beta)$$

8. Equation of the auxiliary circle is

$$x^2 + y^2 = a^2 \quad \text{(i)}$$

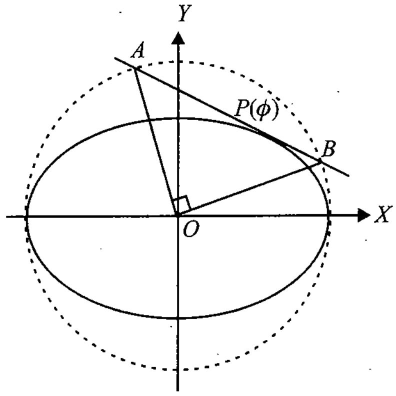


Fig. 4.43

Therefore, equation of tangent at a point $p(a\cos\phi, b\sin\phi)$ is

$$\left(\frac{x}{a}\right)\cos\phi + \left(\frac{y}{b}\right)\sin\phi = 1 \quad \text{(ii)}$$

which meets the auxiliary circle at points A and B.

Therefore, equation of the pair of lines OA and OB is obtained by making Eq. (i) homogeneous with the help of Eq. (ii).

$$\Rightarrow \quad x^2 + y^2 = a^2\left(\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi\right)^2$$

But $\angle AOB = 90°$

$\therefore$ Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow 1 - \cos^2\phi + 1 - \frac{a^2}{b^2}\sin^2\phi = 0$$

$$\Rightarrow \quad \sin^2\phi\left(1 - \frac{a^2}{b^2}\right) + 1 = 0$$

$$\Rightarrow \quad (a^2 - b^2)\sin^2\phi = b^2$$

$$\Rightarrow \quad a^2e^2\sin^2\phi = a^2(1 - e^2)$$

$$\Rightarrow \quad (1 + \sin^2\phi)e^2 = 1$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{1 + \sin^2\theta}}$$

9. Equation of pair of tangents from point P is $SS_1 = T^2$

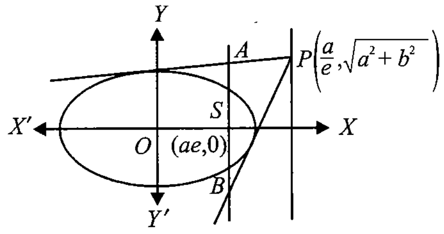


Fig. 4.44

$$\Rightarrow \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{1}{e^2} + \frac{a^2+b^2}{b^2} - 1\right) = \left(\frac{xa}{ea^2} + \frac{y\sqrt{a^2+b^2}}{b^2} - 1\right)^2$$

Solving this with $x = ae$, we get

$$\left(e^2 + \frac{y^2}{b^2} - 1\right)\left(\frac{1}{e^2} + \frac{a^2}{b^2}\right) = \frac{y^2(a^2+b^2)}{b^2\,b^2}$$

$$\Rightarrow \quad \left(\frac{y^2}{b^2} - \frac{b^2}{a^2}\right)\left(\frac{b^2 + a^2 - b^2}{b^2 e^2}\right) = \frac{y^2(a^2+b^2)}{b^2\,b^2}$$

(using $e^2 = 1 - \frac{b^2}{a^2}$)

$$\Rightarrow \quad \left(\frac{a^2y^2 - b^4}{a^2\,b^2}\right)\left(\frac{a^2}{b^2\,e^2}\right) = \frac{y^2(a^2+b^2)}{b^4}$$

$$\Rightarrow \quad a^2(a^2y^2 - b^4) = y^2(a^4 - b^4)$$

[using $(a^2e^2 = a^2 - b^2)$]

$$\Rightarrow \quad a^2b^4 = y^2b^4$$

$$\Rightarrow \quad y^2 = a^2$$

$$\Rightarrow \quad y = \pm a$$

$$\Rightarrow \quad AB = 2a$$

10.

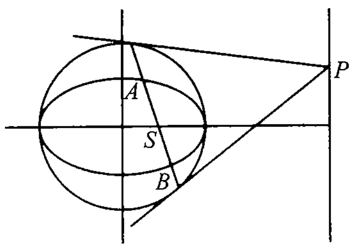


Fig. 4.45

Let the point on directrix be $\left(\frac{a}{e}, k\right)$.

Then equation of AB will be,

$$x\left(\frac{a}{e}\right) + yk = a^2.$$

Now it will clearly pass through focus of the ellipse as

$$(ae)\left(\frac{a}{e}\right) + k.0 = a^2,$$

so AB is a focal chord.

11. Equation of any tangent PQ to the ellipse

$$\frac{x^2}{y^2}+\frac{y^2}{b^2}=1 \text{ be}$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad \text{(i)}$$

This tangent cuts the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=a$ at the points P and Q.

Let tangents at P and Q intersect the point $R(h, k)$.

Then PQ becomes chord of contact with respect to the point R for the ellpse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$, i.e., equation of PQ

$$\frac{hx}{c^2}+\frac{ky}{d^2}=1 \quad \text{(ii)}$$

Equations (i) and (ii) represent same straight lines.

$$\Rightarrow \quad \frac{\frac{\cos\theta}{a}}{\frac{h}{c^2}}=\frac{\frac{\sin\theta}{a}}{\frac{k}{d^2}}=1$$

$$\Rightarrow \quad \cos\theta=\frac{ah}{c^2}\ \sin\theta=\frac{bk}{d^2}$$

Squaring and adding, we get

$$\frac{a^2h^2}{c^4}+\frac{b^2k^2}{d^4}=1$$

$$\Rightarrow \quad \frac{a^2x^2}{c^4}+\frac{b^2y^2}{d^4}=1 \quad \text{(iii)}$$

which is the locus of the point $R(h, k)$.

If $R(h, k)$ is the point of intersection of two perpendicular tangents, then locus of R should be the director circle of the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$,

i.e,. $\quad x^2+y^2=c^2+d^2,$

i.e,. $\quad \frac{x^2}{c^2+d^2}+\frac{y^2}{c^2+d^2}=1 \quad$ (iv)

Equations (iii) and (iv) represent the same locus

$$\Rightarrow \quad \frac{a^2}{c^4}=\frac{1}{c^2+d^2},$$

and $\quad \frac{b^2}{d^4}=\frac{1}{c^2+d^2}$

$$\Rightarrow \quad \frac{a^2}{c^2}+\frac{b^2}{d^2}=1$$

12. Let line OPQ makes an angle θ with x-axis so

$P\equiv(a\cos\theta, a\sin\theta)$,
$Q\,(b\cos\theta, b\sin\theta)$

AQ and let $R\,(x, y)$

So $\quad x=a\cos\theta, y=b\sin\theta$

Eliminating θ, we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, locus of R is an ellipse.

Also $a<b$ so vertices are $(0, b)$ and $(0, -b)$ and extremities of minor axis are $(\pm a, 0)$.

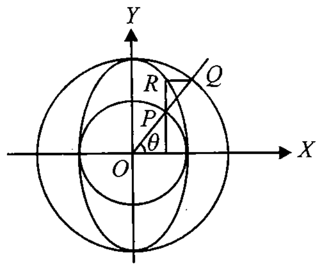


Fig. 4.46

So the ellipse touches both inner circle and outer circle if foci are $(0, \pm a)$

$\Rightarrow \quad a=be$, i.e., $e=\frac{a}{b}$

Also $\quad e=\sqrt{1-e^2}$

$\Rightarrow \quad e^2=1-e^2$

$\Rightarrow \quad e=\frac{1}{\sqrt{2}}$

and ratio of radii is $\quad \frac{a}{b}=e=\frac{1}{\sqrt{2}}$

13.

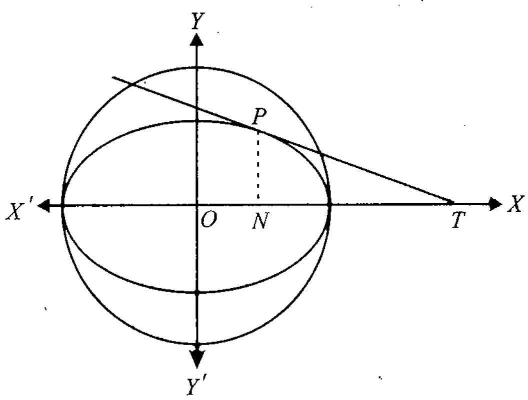


Fig. 4.47

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $(a\cos\theta, b\sin\theta)$ be a point on the ellipse.

The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

It meets the major axis at $T(a \sec\theta, 0)$.

The coordinates of N are $(a\cos\theta, 0)$.

The equation of the circle with NT as its diameter is

$$(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$$

$\Rightarrow$ It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally as $2g\cdot 0 + 2f\cdot 0 = a^2 - a^2 = 0$, which is true.

14. Let $P \equiv (h, k)$. The combined equation of tangents from (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2=0$$

Any curve that can be drawn through the points of intersection of these tangents with coordinate axes is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2+\lambda xy=0$$

or $x^2\left(\frac{k^2}{a^2b^2}-\frac{1}{a^2}\right)+y^2\left(\frac{h^2}{a^2b^2}-\frac{1}{b^2}\right)+xy\left(\lambda-\frac{2hk}{a^2b^2}\right)+\frac{2xh}{a^2}+\frac{2yk}{b^2}-\frac{h^2}{a^2}-\frac{k^2}{b^2}=0$

It should represent a circle

$$\Rightarrow \quad \lambda=\frac{2hk}{a^2b^2}, \frac{k^2}{a^2b^2}-\frac{1}{a^2}=\frac{h^2}{a^2b^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \lambda=\frac{2hk}{a^2b^2}, h^2-k^2=a^2-b^2$$

Hence, locus of P is

$$x^2 - y^2 = a^2 - b^2.$$

15. Equation of the ellipse is $\frac{x^2}{3}+\frac{y^2}{4}=1$.

Let point P be $(\sqrt{3}\cos\theta, 2\sin\theta)$, $\theta \in (0, \frac{\pi}{2})$

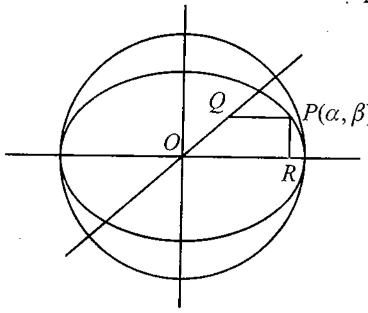


Fig. 4.48

Clearly line PQ is $y = 2\sin\theta$, line PR is $x = \sqrt{3}\cos\theta$ and OQ is $y = x$ and Q is $(2\sin\theta, 2\sin\theta)$.

$Z =$ area of the region PQORP (trapezium)

$$= \frac{1}{2}(OR + PQ)PR$$

$$= \frac{1}{2}(\sqrt{3}\cos\theta + (\sqrt{3}\cos\theta - 2\sin\theta))2\sin\theta$$

$$= \frac{1}{2}(2\sqrt{3}\cos\theta\sin\theta - 2\sin^2\theta)$$

$$= \frac{1}{2}(\sqrt{3}\sin 2\theta + \cos 2\theta - 1)$$

$$= \cos\left(2\theta - \frac{\pi}{3}\right) - \frac{1}{2}$$

which is maximum when $\cos\left(\theta - \frac{\pi}{3}\right)$ is maximum

or $\quad 2\theta - \frac{\pi}{3} = 0$

or $\quad \theta = \frac{\pi}{6}$

Hence, point P is $(\frac{3}{2}, 1)$.

Objective Type

1. c. We have $PQ = BP$

$$\Rightarrow \quad 2ae = \sqrt{a^2e^2 + b^2} = \sqrt{a^2} = a$$

$$\Rightarrow \quad e = \frac{1}{2}$$

2. c. Ellipse passing through $O(0, 0)$ and having foci $P(3, 3)$ and $Q(-4, 4)$,

then $\quad e = \frac{PQ}{OP + OQ}$

$$= \frac{\sqrt{50}}{3\sqrt{2} + 4\sqrt{2}}$$

$$= \frac{5}{7}$$

3. b. Here $a^2 + 2 > a^2 + 1$

$$\Rightarrow \quad a^2 + 1 = (a^2 + 2)(1 - e^2)$$

$$\Rightarrow \quad a^2 + 1 = (a^2 + 2)\frac{5}{6}$$

$$\Rightarrow \quad 6a^2 + 6 = 5a^2 + 10$$

$$\Rightarrow \quad a^2 = 10 - 6 = 4$$

$$\Rightarrow \quad a = \pm 2$$

Latus rectum $= \frac{2(a^2+1)}{\sqrt{a^2+2}} = \frac{2\times 5}{\sqrt{6}} = \frac{10}{\sqrt{6}}$

4. a. Let $P(\theta)$, $Q\left(\theta + \frac{2\pi}{3}\right)$, $R\left(\theta + \frac{4\pi}{3}\right)$

then $\quad P' \equiv (a\cos\theta, b\sin\theta)$,

$$Q' \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$R' \equiv \left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let centroid of $\Delta P'Q'R' \equiv (x', y')$

$$x' = a\left[\frac{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3}\right]$$

$$= \frac{a}{3}\left[\cos\theta + 2\cos(\theta+\pi)\cos\frac{\pi}{3}\right] = 0$$

$$y' = \frac{a}{3}\left[\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right]$$

$$= \frac{a}{3}\left[\sin\theta + 2\sin(\theta+\pi)\sin\frac{\pi}{3}\right] = 0$$

$$= 0$$

5. a. $2a(1 + e) = 15$

$$1 + e = \frac{3}{2}$$

$$e = 0.5$$

6. d. For the two ellipses to intersect at four distinct points, $a > 1$

$\Rightarrow \quad b^2 - 5b + 7 > 1$

$\Rightarrow \quad b^2 - 5b + 6 > 0$

$\Rightarrow \quad b \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ b does not lie in [2, 3]

7. b. Let S be the given focus and ZM be the given line.

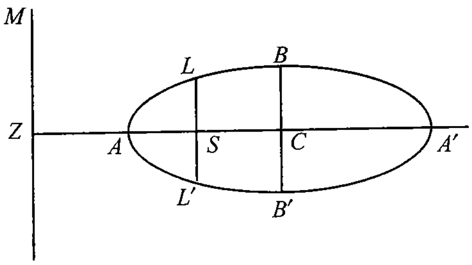


Fig. 4.49

Then $\quad SZ = \frac{a}{e} - ae$

$$= \frac{a}{e}(1 - e^2)$$

$$= \frac{b^2}{ae} = k \text{ (say)}$$

as $\quad b^2 = a^2(1 - e^2)$

Now take SC as x-axis and LSL' as y-axis. Let (x, y) be the coordinates of B with respect to these axes, then $x = SC = ae$, $y = CB = b$

Hence, $\quad \frac{y^2}{x} = \frac{b^2}{ae} = SZ$, which is constant.

$\therefore \quad y^2 = kx$ is the required locus which is a parabola.

8. a. Let AB be the line

Let $\quad AP = a, PB = b,$

so that $\quad AB = a + b$

If AB makes an angle θ with x-axis and coordinates of P are (x, y),

then in ΔAPL, $\quad x = a\cos\theta$

in ΔPBQ, $\quad y = b\sin\theta$

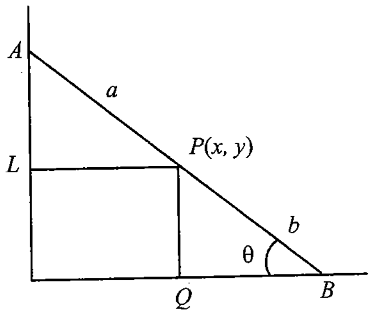


Fig. 4.50

$\therefore$ Locus of $P(x, y)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse

9. b. $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x - 4y + 7)^2}{4}$

$\Rightarrow \quad (x-2)^2 + (y+3)^2 = \left(\frac{1}{2}\,\frac{3x - 4y - 7}{5}\right)^2$

$\Rightarrow \quad \sqrt{(x-2)^2 + (y+3)^2} = \frac{1}{2}\frac{|3x - 4y - 7|}{5}$ is an ellipse, whose focus is $(2, -3)$, directrix $3x - 4y + 7 = 0$ and eccentricity is $\frac{1}{2}$.

Length of $\perp$ from focus to directrix is

$$\frac{|3 \times 2 - 4(-3) + 7|}{5} = 5$$

$\Rightarrow \quad \frac{a}{e} - ae = 5$

$\Rightarrow \quad 2a - \frac{a}{2} = 5$

$\Rightarrow \quad a = \frac{10}{3}$

So length of major axis is $\frac{20}{3}$.

10. d.

$$\frac{(x-4)^2}{25} + \frac{y^2}{4} = 1 \text{ and } (x-1)^2 + \frac{y^2}{4} = 1$$

Clearly $\quad m_{OP} \cdot m_{OQ} = -1$

$\Rightarrow$ OP and OQ are perpendicular to each other.

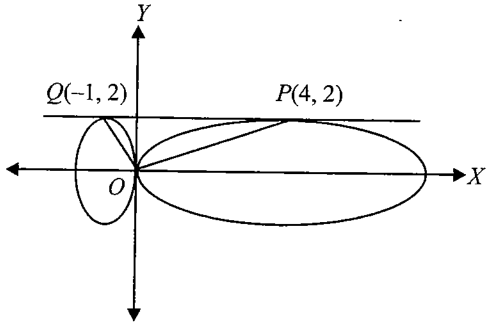


Fig. 4.51

11. a. Let ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=a^2e^2$

Radius of circle = ae

Point of intersection of circle and ellipse is $\left[\frac{a}{e}\sqrt{2e^2-1}, \frac{a}{e}(1-e^2)\right]$.

Now area of ΔPF_1F_2

$$=\frac{1}{2}\begin{vmatrix}\frac{a}{e}\sqrt{2e^2-1} & \frac{a}{e}(1-e^2) & 1\\ ae & 0 & 1\\ -ae & 0 & 1\end{vmatrix}=\frac{1}{2}\left|\frac{a}{e}(1-e^2)(2ae)\right|=30$$

$\Rightarrow$ $a^2(1-e^2)=30$ (given)

$\Rightarrow$ $a^2e^2=a^2-30=\left(\frac{17}{2}\right)^2-30=\frac{169}{4}$

$\Rightarrow$ $2ae=13$

12. b. Solving the given line and the ellipse, we get

$$t^2+\frac{y^2}{9}=1$$

$\Rightarrow$ $y^2=9(1-t^2)$,

which gives real and distinct values of y, if $1-t^2>0$

$\Rightarrow$ $t\in(-1, 1)$.

13. a. Any point on the ellipse is $(2\cos\theta, \sqrt{3}\sin\theta)$.

The focus on the positive x-axis is (1, 0).

Given that

$$(2\cos\theta-1)^2+3\sin^2\theta=\frac{25}{16}$$

$\Rightarrow$ $\cos\theta=\frac{3}{4}$.

14. c. Let $P(5\cos\theta, 4\sin\theta)$ be any point on the ellipse

Then $SP=5+5e\cos\theta$

$S'P=5-5e\cos\theta$

$SP\cdot S'P=25-25e^2\cos^2\theta$

$=25\sin^2\theta+16\cos^2\theta$

$=16+9\sin^2\theta=f(\theta)$ (say)

$\Rightarrow$ $16\le f(\theta)\le 25$.

15. a. Tangent at P($a\cos\alpha$, $b\sin\alpha$) is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1 \quad \text{(i)}$$

Distance of focus $S(ae, 0)$ from this tangent is

$$d_1=\frac{|e\cos\alpha-1|}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

$$=\frac{1-e\cos\alpha}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

Distance of focus $S'(-ae, 0)$ from this line

$$d_2=\frac{1+e\cos\alpha}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

$\Rightarrow$ $\frac{d_1}{d_2}=\frac{1-e\cos\alpha}{1+e\cos\alpha}$

Now $SP=a-ae\cos\alpha$ and $S'P=a+ae\cos\alpha$

$\Rightarrow$ $\frac{SP}{S'P}=\frac{1-e\cos\alpha}{1+e\cos\alpha}$

$\Rightarrow$ $\frac{SP}{S'P}=\frac{d_1}{d_2}$

16. b.

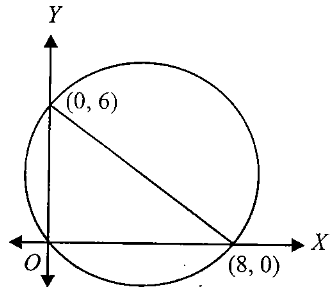


Fig. 4.52

Centre of family of ellipse is (4, 3) and distance of focus from centre = $ae=\frac{5}{2}$.

Hence, locus $(x-4)^2+(y-3)^2=\frac{25}{4}$

17. a. Let $P(x, y)$ be the position of the man at any time.

Let S (4, 0) and S' (– 4, 0) be the fixed flag, post, with C as the origin.

Since $SP+S'P=10$ m i.e., a constant, the locus of P is an ellipse with S and S' as foci

$\Rightarrow$ $ae=4$, and $2a=10$

$\Rightarrow$ $e=\frac{4}{5}$

Now $b^2=a^2(1-e^2)$

$\Rightarrow \quad b^2 = 25\left(1 - \frac{16}{25}\right) = 9$

$\Rightarrow \quad b = 3$

Hence, the area of the ellipse $= \pi ab = \pi \times 5 \times 3 = 15\pi$

18. b. Since there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from centre is same, the points would be either end points of the major axis or of the minor axis.

But $\sqrt{\frac{a^2 + 2b^2}{2}} > b$, so the points are the vertices of major axis.

Hence, $\quad a = \sqrt{\frac{a^2 + 2b^2}{2}}$

$\Rightarrow \quad a^2 = 2b^2$

$\Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$

19. c. Let $p = 3h + 2$ and $q = k$

$\Rightarrow \quad h = \frac{p-2}{3}$ and $k = q$

Since (h, k) lies on $x^2 + y^2 = 1$

$\Rightarrow \quad h^2 + k^2 = 1$

$\Rightarrow \quad \left(\frac{p-2}{3}\right)^2 + q^2 = 1$

Locus is $\quad \left(\frac{x-2}{3}\right)^2 + y^2 = 1$

which has eccentricity $e = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$

20. b.

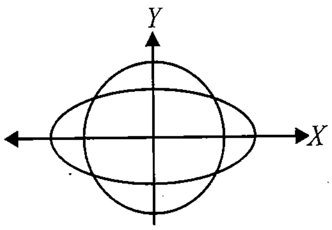


Fig. 4.53

Radius of the circle having SS' as diameter is $r = ae$

If it cuts an ellipse, then $r > b$

$\Rightarrow \quad ae > b$

$\Rightarrow \quad e^2 > \frac{b^2}{a^2}$

$\Rightarrow \quad e^2 > 1 - e^2$

$\Rightarrow \quad e^2 > \frac{1}{2}$

$\Rightarrow \quad e > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad e \in \left(\frac{1}{\sqrt{2}}, 1\right)$

21. b. Equation of conic through point of intersection of given two ellipse is

$$\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$$

$\Rightarrow \quad x^2\left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1 + \lambda) = 1 + \lambda$

$\Rightarrow \quad x^2\left(\frac{a^2 + 4\lambda}{4a^2(1+\lambda)}\right) + y^2 = 1$

This equation is a circle if $\frac{a^2 + 4\lambda}{4a^2(1+\lambda)} = 1$

$\Rightarrow$ Circle is $x^2 + y^2 = 1$

22. b.

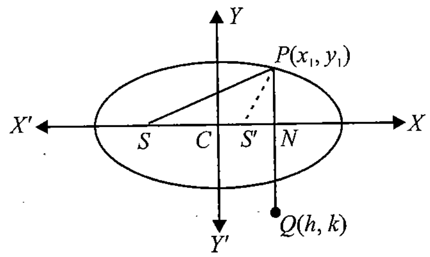


Fig. 4.54

$a^2 = 25$

and $\quad b^2 = 16$

$\Rightarrow \quad e = \sqrt{1 - \frac{16}{25}}$

$= \frac{3}{5}$

Let point Q be (h, k), where $k < 0$

Given that $k = SP = a + ex_1$, where $P(x_1, y_1)$ lies on the ellipse

$\Rightarrow \quad |k| = a + eh$ (as $x_1 = h$)

$\Rightarrow \quad -y = a + ex$

$\Rightarrow \quad 3x + 5y + 25 = 0$

23. a.

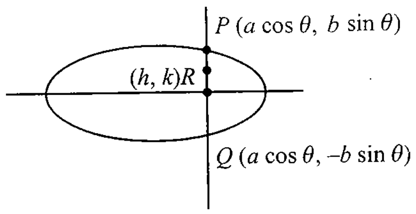


Fig. 4.55

Let $P(a\cos\theta, b\sin\theta)$, $Q(a\cos\theta, -b\sin\theta)$

$PR : RQ = 1:2$

Therefore, $\quad h = a\cos\theta$

$\Rightarrow \qquad \cos\theta = \frac{h}{a}$ (i)

and $\qquad k = \frac{b}{3}\sin\theta$

$\Rightarrow \qquad \sin\theta = \frac{3k}{b}$ (ii)

On squaring and adding Eqs. (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$$

24. b. Let m be the slope of the common tangent, then

$$\pm\sqrt{3}\sqrt{1+m^2} = \pm\sqrt{4m^2+1}$$

$\Rightarrow \qquad 3 + 3m^2 = 4m^2 + 1$

$\Rightarrow \qquad m^2 = 2$

$\Rightarrow \qquad m = \pm\sqrt{2}$

25. a.

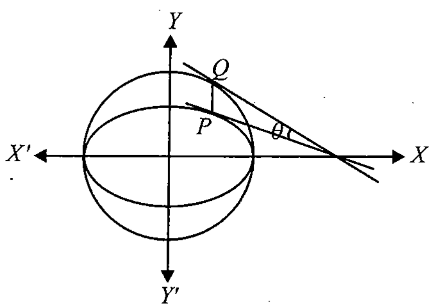


Fig. 4.56

Tangent to the ellipse at $P(a\cos\alpha, b\sin\alpha)$ is

$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ (i)

Tangent to the circle at $Q(a\cos\alpha, a\sin\alpha)$ is

$\cos\alpha x + \sin\alpha y = a$ (ii)

Now angle between tangents is θ,

then $\qquad \tan\theta = \left|\frac{-\frac{b}{a}\cot\alpha - (-\cot\alpha)}{1+\left(-\frac{b}{a}\cot\alpha\right)(-\cot\alpha)}\right|$

$$= \left|\frac{\cot\alpha\left(1-\frac{b}{a}\right)}{1+\frac{b}{a}\cot^2\alpha}\right|$$

$$= \left|\frac{a-b}{a\tan\alpha + b\cot\alpha}\right|$$

$$= \left|\frac{a-b}{(\sqrt{a}\tan\alpha - \sqrt{b}\cot\alpha)^2 + 2\sqrt{ab}}\right|$$

Now the greatest value of the above expression is $\left|\frac{a-b}{2\sqrt{ab}}\right|$

when $\sqrt{a}\tan\alpha = \sqrt{b}\cot\alpha$

$\Rightarrow \qquad \theta_{\text{maximum}} = \tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

26. c.

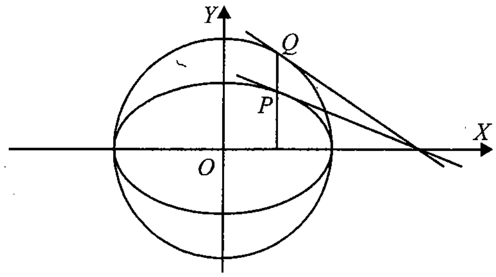


Fig. 4.57

Tangent to the ellipse at point $P(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (i)

Tangent to the circle at point $Q(a\cos\theta, a\sin\theta)$ is

$x\cos\theta + y\sin\theta = a$ (ii)

Equation (i) and (ii) intersect at $(\frac{a}{\cos\theta}, 0)$ which lies on $y = 0$

27. d. Tangent to the ellipse having slope m is

$$y = mx + \sqrt{a^2m^2+b^2}$$

If it passes through the point $P(h, k)$, then

$$k = mh + \sqrt{a^2m^2+b^2}$$

or $\quad (a^2-h^2)m^2 + 2hkm + b^2 - k^2 = 0$

Now given $\tan\alpha + \tan\beta = \lambda$

$\Rightarrow \qquad m_1 + m_2 = \lambda$

$\Rightarrow \qquad \frac{-2hk}{a^2-h^2} = \lambda$

$\Rightarrow \qquad$ locus is $\lambda(x^2 - a^2) = 2xy$

28. c. Tangent to the ellipse at P and Q are

$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ (i)

and $\quad \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1$ (ii)

Solving Eqs. (i) and (ii), we get

$$\frac{x}{\begin{vmatrix}\frac{\sin\alpha}{b} & 1\\ \frac{\sin\beta}{b} & 1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}\frac{\cos\alpha}{a} & 1\\ \frac{\cos\beta}{a} & 1\end{vmatrix}} = \frac{1}{\begin{vmatrix}\frac{\cos\alpha}{a} & \frac{\sin\alpha}{b}\\ \frac{\cos\beta}{a} & \frac{\sin\beta}{b}\end{vmatrix}}$$

$\Rightarrow \qquad x = \frac{a(\sin\alpha - \sin\beta)}{\sin(\beta-\alpha)},$

$$y = \frac{-b(\cos\alpha - \cos\beta)}{\sin(\beta - \alpha)}$$

$$\Rightarrow \quad \frac{x\sin(\beta-\alpha)}{a} = \sin\alpha - \sin\beta,$$

$$\frac{y\sin(\beta-\alpha)}{b} = -(\cos\alpha - \cos\beta)$$

Squaring and adding, we get

$$\sin^2(\beta-\alpha)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = 2$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{2}{\sin^2 c}$$

(where $\beta - \alpha = c$ (constant) given)

which is an ellipse.

29. d. As in above question point of intersection is

$$(h, k) \equiv \left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

It is given that $\alpha + \beta = c$ = constant.

$$\Rightarrow \quad h = \frac{a\cos\frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)} \text{ and } k = \frac{b\sin\frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{h}{k} = \frac{a}{b}\cot\left(\frac{c}{2}\right)$$

$$\Rightarrow \quad k = \frac{b}{a}\tan\left(\frac{c}{2}\right)h$$

$\Rightarrow$ (h, k) lies on the straight line

30. a. Any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ having slope m is

$$y = mx + \sqrt{a^2m^2+b^2}$$

Points on the minor axis are $(0, ae)$, $(0, -ae)$.

$\therefore$ Sum of the squares of the perpendicular on the tangent from $(0, ae)$ and $(0, -ae)$

$$= \left[\frac{\sqrt{a^2m^2+b^2}-ae}{\sqrt{m^2+1}}\right]^2 + \left[\frac{\sqrt{a^2m^2+b^2}+ae}{\sqrt{m^2+1}}\right]^2$$

$$= \frac{2(a^2m^2+b^2+a^2e^2)}{m^2+1}$$

$$= \frac{2(a^2m^2+a^2-a^2e^2+a^2e^2)}{m^2+1}$$

$$= \frac{2a^2(m^2+1)}{m^2+1} = 2a^2$$

31. c. Equation of the tangent to the ellipse at P $(5\cos\theta, 4\sin\theta)$ is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{4} = 1$$

It meets the line $x = 0$ at Q $(0, 4\operatorname{cosec}\theta)$

Image of Q in the line $y = x$ is R $(4\operatorname{cosec}\theta, 0)$

$\therefore$ Equation of the circle is

$$x(x - 4\operatorname{cosec}\theta) + y(y - \operatorname{cosec}\theta) = 0$$

i.e., $\quad x^2+y^2-4(x+y)\operatorname{cosec}\theta = 0$

$\therefore$ Each member of the family passes through the intersection of $x^2+y^2 = 0$ and $x + y = 0$, i.e., the point $(0, 0)$.

32. c.

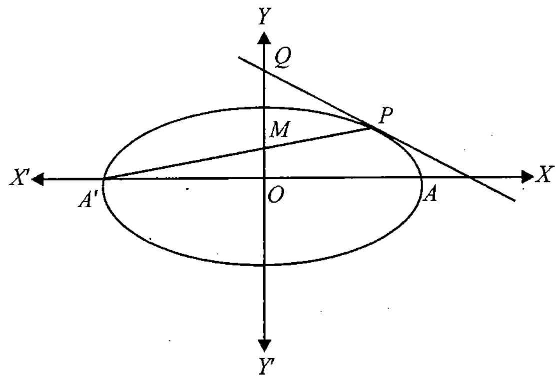


Fig. 4.58

Let point P be $(a\cos\theta, b\sin\theta)$.

Equation of the tangent at point P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

Then point Q is $(b\operatorname{cosec}\theta, 0)$

Equation of chord $A'P$ is

$$y - 0 = \frac{b\sin\theta}{a\cos\theta + a}(x + a)$$

Putting $x = 0$, we have $y = \frac{b\sin\theta}{\cos\theta+1}$

Then

$$OQ^2 - MQ^2 = b^2\operatorname{cosec}^2\theta - \left(b\operatorname{cosec}\theta - \frac{b\sin\theta}{\cos\theta+1}\right)^2$$

$$= \frac{2b^2}{\cos\theta+1} - \frac{b^2\sin^2\theta}{(\cos\theta+1)^2}$$

$$= \frac{b^2}{\cos\theta+1}\left(\frac{2\cos\theta+2-\sin^2\theta}{(\cos\theta+1)}\right)$$

$$= \frac{b^2}{\cos\theta+1}\left(\frac{2\cos\theta+1+\cos^2\theta}{(\cos\theta+1)}\right)$$

$$= b^2 = 4$$

33. b. One of the tangents of slope m to the given ellipse is

$$y = mx + \sqrt{18m^2 + 32}$$

For $m = -\frac{4}{3}$,

we have $y = -\frac{4}{3}x + 8$.

Then points on the axis where tangents meet are $A(6, 0)$ and $B(0, 8)$.

Then area of triangle ABC is $\frac{1}{2}(6)(8) = 24$ units.

34. b.

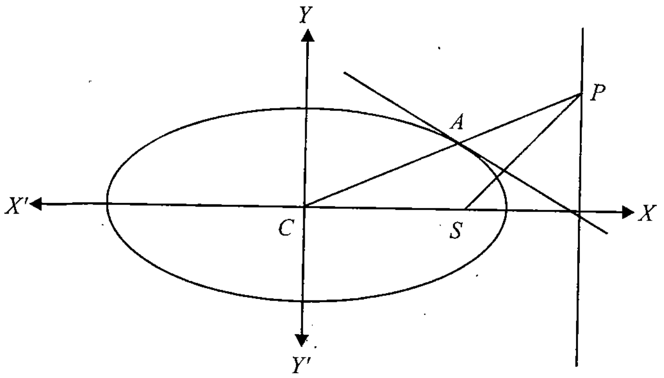


Fig. 4.59

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and let $A \equiv (a\cos\theta, b\sin\theta)$

Equation of AC will be $y = \frac{b}{a}\tan\theta\, x$

Solving with $x = \frac{a}{e}$, we get

$$P \equiv \left(\frac{a}{e}, \frac{b}{e}\tan\theta\right)$$

Slope of tangent at A is $-\frac{b}{a\tan\theta}$

Slope of PS

$$= \frac{\frac{b}{e}\tan\theta}{\frac{a}{e} - ae} = \frac{b\tan\theta}{a(1-e^2)} = \frac{a}{b}\tan\theta$$

So $\alpha = \frac{\pi}{2}$

35. a. A tangent of slope 2 to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = 2x \pm \sqrt{4a^2 + b^2} \quad \text{(i)}$$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

$\Rightarrow$ Eq. (i), passes through $(-2, 0)$

$$\Rightarrow \quad 0 = -4 \pm \sqrt{4a^2 + b^2}$$

$$\Rightarrow \quad 4a^2 + b^2 = 16$$

Using A.M. $\geq$ G.M., we get

$$\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2 b^2}$$

$$\Rightarrow \quad ab \leq 4$$

36. a. Equation of tangent $\frac{x}{a}\frac{\sqrt{3}}{2} + \frac{y}{b}\frac{1}{2} = 1$ (i)

and equation of tangent at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1 \quad \text{....(ii)}$$

Comparing (i) and (ii), we have $\cos\phi = \frac{\sqrt{3}}{2}$ and $\sin\phi = \frac{1}{2}$

Hence, $\phi = \frac{\pi}{6}$

37. d.

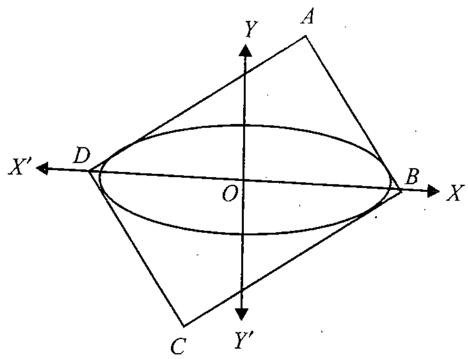


Fig. 4.60

Since sides of the square are tangent and perpendicular to each other, so the vertices lie on director circle

$$\Rightarrow \quad x^2 + y^2 = (a^2 - 7) + (13 - 5a)$$

$$= a^2 \ (\sqrt{2}\,a \text{ is side of the square})$$

$$\Rightarrow \quad (a^2 - 7) + (13 - 5a) = a^2$$

$$\Rightarrow \quad a = \frac{6}{5}$$

But for an ellipse to exist $a^2 - 7 > 0$ and $13 - 5a > 0$

$$\Rightarrow \quad a \in (-\infty, -\sqrt{7})$$

Hence, $a \neq \frac{6}{5}$

Hence, no such a exists.

38. d. Since locus of the point of intersection of the tangent at the end points of a focal chord is directrix

$\therefore$ Required locus is $x = \pm\frac{a}{e}$, which is pair of straight lines.

39. c. Normal at point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{(i)}$$

It meets axes at $Q\left(\frac{(a^2 - b^2)\cos\theta}{a}, 0\right)$

and $R\left(0, -\frac{(a^2 - b^2)\sin\theta}{b}\right)$

Let $T(h, k)$ is a midpoint of QR.

Then $2h = \frac{(a^2 - b^2)\cos\theta}{a}$

and $2k = -\frac{(a^2 - b^2)\sin\theta}{b}$

$\Rightarrow \cos^2\theta + \sin^2\theta = \frac{4h^2a^2}{(a^2 - b^2)^2} + \frac{4k^2b^2}{(a^2 - b^2)^2} = 1$

$\Rightarrow$ Locus is $\frac{x^2}{\frac{(a^2 - b^2)^2}{4a^2}} + \frac{y^2}{\frac{(a^2 - b^2)^2}{4b^2}} = 1$ (ii)

which is an ellipse, having eccentricity e', given by

$$e'^2 = 1 - \frac{\frac{(a^2 - b^2)^2}{4a^2}}{\frac{(a^2 - b^2)^2}{4b^2}} = 1 - \frac{b^2}{a^2} = e^2$$

$\Rightarrow e' = e$

Note:
In Eq.* (ii),** $\frac{(a^2 - b^2)}{4a^2} < \frac{(a^2 - b^2)}{4b^2}$. ***Hence, x-axis is minor axis.

40. c. Equation of normal to the ellipse at P is

$5x \sec\theta - 3y \operatorname{cosec}\theta = 16$ (i)

Equation of normal to the circle $x^2 + y^2 = 25$ at point Q is

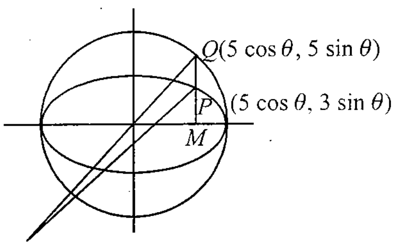


Fig. 4.61

$y = x \tan\theta$ (ii)

Eliminating θ from (i) and (ii), we get $x^2 + y^2 = 64$.

41. c. $\frac{x^2}{169} + \frac{y^2}{25} = 1$

Equation of normal at the point $(13\cos\theta, 5\sin\theta)$ is

$\frac{13x}{\cos\theta} - \frac{5y}{\sin\theta} = 144$, it passes through (0, 6)

$\Rightarrow (15 + 72\sin\theta) = 0$

$\Rightarrow \sin\theta = -\frac{5}{24}$

$\Rightarrow \theta = 2\pi - \sin^{-1}\left(\frac{5}{24}\right)$,

and $\pi + \sin^{-1}\frac{5}{24}$

Also y-axis is one of the normals.

42. c. Equation has three roots, hence three normal can be drawn.

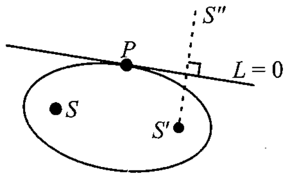


Fig. 4.62

Let image of S' be with respect to $x + y - 5 = 0$

$\Rightarrow \frac{h-2}{1} = \frac{k+1}{1} = \frac{-2(-4)}{2}$

$\Rightarrow S'' = (6, 3)$

Let P be the point of contact.

Because the line $L = 0$ is tangent to the ellipse, there exists a point P uniquely on the line such that $PS + PS' = 2a$.

Since $PS' = PS''$.

There exists one and only one point P on $L = 0$ such that $PS + PS'' = 2a$.

Hence, P should be the collinear with SS''.

Hence, P is a point of intersection of SS'' $(4x - 5y = 9)$, and $x + y - 5 = 0$, i.e. $P = \left(\frac{34}{9}, \frac{11}{9}\right)$.

43. c. A cyclic parallelogram will be a rectangle or square

So, $\angle QPR = 90°$

$\Rightarrow$ P lies on director circle of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$

$\Rightarrow x^2 + y^2 = 25$ is director circle of $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$

$\Rightarrow 16 + b^2 = 25$

$\Rightarrow b^2 = 9$

$\Rightarrow a^2(1 - e^2) = 9$

$\Rightarrow 1 - e^2 = \frac{9}{16}$

$\Rightarrow e^2 = \frac{7}{16}$

$\Rightarrow e = \frac{\sqrt{7}}{4}$

44. d.

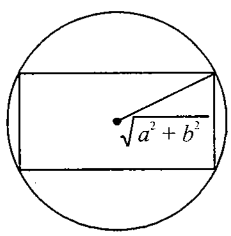


Fig. 4.63

Since mutually perpendicular tangents can be drawn from vertices of rectangle. So all the vertices of rectangle should lie on director circle $x^2 + y^2 = a^2 + b^2$.

Let breadth = $2l$ and length = $4l$, then

$$l^2 + (2l)^2 = a^2 + b^2$$

$$\Rightarrow \quad l^2 = \frac{a^2 + b^2}{5}$$

$$\Rightarrow \quad \text{Area} = 4l \times 2l = 8\,\frac{a^2 + b^2}{5}$$

45. b. Normal at $P(\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ (i)

Normal at $P\left(\frac{\pi}{2} + \theta\right)$ is $\frac{ax}{\cos\left(\frac{\pi}{2}+\theta\right)} - \frac{by}{\sin\left(\frac{\pi}{2}+\theta\right)} = a^2 - b^2$

or $$-\frac{ax}{\sin\theta} - \frac{by}{\cos\theta} = a^2 - b^2 \quad \text{(ii)}$$

Equations (i) and (ii) meet major axis at $G\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$ and $g\left(-\frac{(a^2-b^2)\sin\theta}{a}, 0\right)$

Now $PG^2 + Qg^2$

$$= \left(\frac{(a^2-b^2)\cos\theta}{a} - a\cos\theta\right)^2 + (0 - b\sin\theta)^2$$

$$+ \left(-\frac{(a^2-b^2)\sin\theta}{a} - a\sin\theta\right)^2 + (0 - b\cos\theta)^2$$

$$= \frac{(a^2-b^2)^2}{a^2} + b^2 + a^2$$

$$= a^2\left(\frac{(a^2-b^2)^2}{a^4} + \frac{b^2}{a^2} + 1\right)$$

$$= a^2\left(\left(1 - \frac{b^2}{a^2}\right)^2 + \frac{b^2}{a^2} + 1\right)$$

$$= a^2(e^4 + 2 - e^2)$$

46. c. The equation of the normal to the given ellipse at the point $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$.

$$\Rightarrow \quad y = \left(\frac{a}{b}\tan\theta\right)x - \frac{(a^2-b^2)}{b}\sin\theta \quad \text{(i)}$$

Let $\frac{a}{b}\tan\theta = m$, so that

$$\sin\theta = \frac{bm}{\sqrt{a^2 + b^2m^2}}$$

Hence, the equation of the normal Eq. (i) becomes

$$y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2 + b^2m^2}}$$

$\therefore$ $m \in R$, as $m = \frac{a}{b}\tan\theta \in R$

47. d.

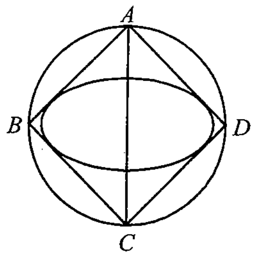


Fig. 4.64

Clearly, vertices of the square lie on the director circle of the ellipse $\frac{x^2}{7} + \frac{2y^2}{11} = 1$

which is $$x^2 + y^2 = 7 + \frac{11}{2} \text{ or } x^2 + y^2 = \frac{25}{2}$$

Clearly, $$AC = 2\sqrt{\frac{25}{2}}$$

Now $AB = BC = CD = AD$

and in ΔACD, $AC^2 = CD^2 + AD^2$

$$\Rightarrow \quad 2AD^2 = \left(2\sqrt{\frac{25}{2}}\right)^2$$

$$\Rightarrow \quad AD^2 = 25$$

$$\Rightarrow \quad AD = 5 \text{ units}$$

48. d. Equation of QR is $T = 0$ (chord of contact)

$$\frac{8x}{4} + \frac{27y}{9} = 1$$

$$\Rightarrow \quad 2x + 3y = 1 \quad \text{(i)}$$

Now, equation of the pair of lines passing through origin and points Q, R is given by

$\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = (2x + 3y)^2$ (Making equation of ellipse homogeneous using Eq. (i))

$$\Rightarrow \quad 9x^2 + 4y^2 = 36(4x^2 + 12xy + 9y^2)$$

$$\Rightarrow 135x^2 + 432xy + 320y^2 = 0$$

$\therefore$ Required angle is $\tan^{-1}\frac{2\sqrt{216^2 - 135 \times 320}}{455}$

$$= \tan^{-1}\frac{8\sqrt{2916 - 2700}}{455}$$

$$= \tan^{-1}\frac{8\sqrt{216}}{455}$$

$$= \tan^{-1}\frac{48\sqrt{6}}{455}$$

49. c. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Any point on the directrix is $P\left(\frac{a}{e}, k\right)$.

Chord of contact of P with respect to the ellipse is

$$\frac{a}{e}\frac{x}{a^2}+\frac{ky}{b^2}=1 \qquad \text{(i)}$$

Chord of contact of P with respect to the auxiliary circle is

$$\frac{a}{e}x+ky=a^2 \qquad \text{(ii)}$$

Equation (i) and (ii) intersect at $(ae, 0)$

50. a. Let (h, k) be the midpoint of the chord $7x+y-1=0$

$$\Rightarrow \quad \frac{hx}{1}+\frac{ky}{7}=\frac{h^2}{1}+\frac{k^2}{7} \qquad \text{(i)}$$

and $$7x+y=1 \qquad \text{(ii)}$$

represents same straight line

$$\Rightarrow \quad \frac{h}{7}=\frac{k}{7}\Rightarrow h=k$$

$\Rightarrow$ Equation of the line joining (0, 0) and (h, k) is $y-x=0$.

51. c. Given $m(n-1)=n$.

n is divisible by $n-1$

$$\Rightarrow \quad n=2\Rightarrow m=2$$

Hence, chord of contact of tangents drawn from (2, 2) to

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \text{ is } \frac{2x}{9}+\frac{2y}{4}=1$$

$$\Rightarrow \quad 4x+9y=18$$

52. b. The chord of contact of tangents from (h, k) is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$.

It meets the axes at points $\left(\frac{a^2}{h}, 0\right)$ and $\left(0, \frac{b^2}{k}\right)$.

Area of the triangle $=\frac{1}{2}\times\frac{a^2}{h}\times\frac{b^2}{k}=c$ (constant)

$$\Rightarrow \quad hk=\frac{a^2b^2}{2c} \text{ (c is constant)}$$

$xy=c^2$ is the required locus.

53. d. Since x-axis and y-axis are perpendicular tangents to the ellipse. (0, 0) lies on the director circle and midpoint of foci (2, 2) is centre of the circle.

Hence, radius $=2\sqrt{2}$

$\Rightarrow$ the area is 8π units.

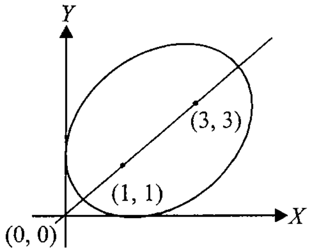


Fig. 4.65

54. a. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We know that the general equation of the tangent to the ellipse is

$$y=mx\pm\sqrt{a^2m^2+b^2} \qquad \text{(i)}$$

since $3x-2y-20=0$ or $y=\frac{3}{2}x-10$ is tangent to the ellipse.

Comparing with Eq. (i), $m=\frac{3}{2}$ and $a^2m^2+b^2=100$

$$\Rightarrow \quad a^2\times\frac{9}{4}+b^2=100$$

$$\Rightarrow \quad 9a^2+4b^2=400 \qquad \text{(ii)}$$

Similarly, since $x+6y-20=0$, i.e., $y=-\frac{1}{6}x+\frac{10}{3}$ is tangent to the ellipse, therefore comparing with Eq. (i),

$$m=\frac{1}{6} \text{ and } a^2m^2+b^2=\frac{100}{9}$$

$$\Rightarrow \quad \frac{a^2}{36}+b^2=\frac{100}{9}$$

$$\Rightarrow \quad a^2+36b^2=400 \qquad \text{(iii)}$$

Solving Eqs. (ii) and (iii), we get $a^2=40$ and $b^2=10$

Therefore, the required equation of the ellipse is

$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

55. b. We know that product of length of perpendiculars from foci upon any tangents to ellipse is b^2.

Hence, from the diagram, x_1 and x_2 are length of perpendiculars from foci upon tangent y-axis of the given ellipse, hence $x_1x_2=b^2$.

Similarly $y_1y_2=b^2$

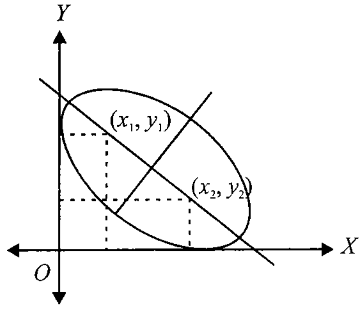


Fig. 4.66

56. c. $\sum_{i=1}^{10}(SP_i)(S'P'_i)=2560$

$$\Rightarrow \quad 10b^2=2560$$

$$\Rightarrow \quad b^2=256$$

$$\Rightarrow \quad b=16$$

$$\Rightarrow \quad 256=400(1-e^2)$$

$\Rightarrow \quad 1-e^2=\frac{16}{25}$

$\Rightarrow \quad e=\frac{3}{5}$

57. d. For the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, equation of director circle is $x^2+y^2=25$. The director circle will cut the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ at 4 points

Hence, number of points = 4.

58. c. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the parabola with focus $S(ae, 0)$ and directrix $x+ae=0$ is $y^2=4aex$

Now length of latus rectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

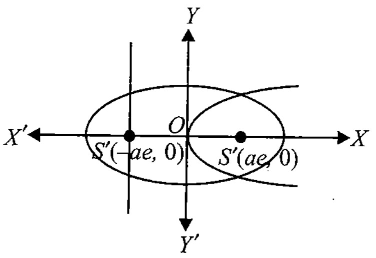


Fig. 4.67

For the two latus recta to be equal, we get

$$\frac{2b^2}{a}=4ae$$

$$\Rightarrow \quad \frac{2a^2(1-e^2)}{a}=4ae$$

$$\Rightarrow \quad 1-e^2=2e$$

$$\Rightarrow \quad e^2+2e-1=0$$

Therefore, $e=-\frac{2\pm\sqrt{8}}{2}=-1\pm\sqrt{2}$

Hence, $e=\sqrt{2}-1$

59. c. Here centre of the ellipse is (0, 0)

Let $P(r\cos\theta, r\sin\theta)$ be any point on the given ellipse then $r^2\cos^2\theta+2r^2\sin^2\theta+2r^2\sin\theta\cos\theta=1$

$$\Rightarrow \quad r^2=\frac{1}{\cos^2\theta+2\sin^2\theta+\sin 2\theta}$$

$$=\frac{1}{\sin^2\theta+1+\sin 2\theta}$$

$$=\frac{2}{1-\cos 2\theta+2+2\sin 2\theta}$$

$$=\frac{2}{3-\cos 2\theta+2\sin 2\theta}$$

$$\Rightarrow \quad r_{max}=\frac{\sqrt{2}}{\sqrt{3}-\sqrt{5}}$$

60. a. Combined equation of pair of lines through the origin joining the points of intersection of line $y=\sqrt{m}x+1$ with the given curve is $x^2+2xy+(2+\sin^2\alpha)y^2-(y-\sqrt{m}x)^2=0$

for the chord to subtend a right angle at the origin $(1-m)+(2+\sin^2\alpha-1)=0$ (as sum of the coefficients of $x^2+y^2=10$)

$$\Rightarrow \quad \sin^2\alpha=m-2$$

$$\Rightarrow \quad 0\le m-2\le 1$$

$$\Rightarrow \quad 2\le m\le 3$$

Multiple Correct Answers Type

1. a., c. $r^2-r-6>0$ and $r^2-6r+5>0$

$\Rightarrow (r-3)(r+2)>0$ and $(r-1)(r-5)>0$

$\Rightarrow (r<-2$ or $r>3)$ and $(r<1$ or $r>5)$

$\Rightarrow r<-2$ or $r>5$

Also $\quad r^2-r-6\neq r^2-6r+5$

$$\Rightarrow \quad r\neq\frac{11}{5}$$

2. a, c.

If both foci are fixed, then the ellipse is fixed, that is, both the directrices can be decided (eccentricity is given). Similar is the case for option (c). Thus (a) and (c) are the correct choices. In the remaining cases, size of the ellipse is fixed, but its position is not fixed.

3. a, b. Any point on this ellipse is $(\sqrt{6}\cos\phi, \sqrt{2}\sin\phi)$

Here centre is (0, 0), so $6\cos^2\phi+2\sin^2\phi=4$

$$\Rightarrow \quad 2\cos^2\phi=1$$

$$\Rightarrow \quad \cos^2\phi=\left(\frac{1}{\sqrt{2}}\right)^2=\cos^2\frac{\pi}{4}$$

$$\Rightarrow \quad \phi=\frac{\pi}{4}\text{ or }\frac{3\pi}{4}$$

4. a, b, c. $3x^2+2y^2+6x-8y+5=0$

$$\Rightarrow \quad \frac{(x+1)^2}{2}+\frac{(y-2)^2}{3}=1$$

Therefore, centre is $(-1, 2)$ and ellipse is vertical $(\because b>a)$

$$a^2=2,\ b^2=3$$

Now $\quad 2=3(1-e^2)$

$$\Rightarrow \quad e=\frac{1}{\sqrt{3}}$$

Foci are $(-1, 2 \pm be)$ and $(-1, 2 \pm 1) \equiv (-1, 3)$ and $(-1, 1)$ and directrix are $y = 2 \pm \frac{b}{e} \Rightarrow y = 5$ and $y = -1$

5. c, d. Ellipse is $16x^2 + 11y^2 = 256$

Equation of tangent at $\left(4 \cos \theta, \frac{16}{\sqrt{11}} \sin \theta\right)$ is $16x (4 \cos \theta) + 11y \left(\frac{16}{\sqrt{11}} \sin \theta\right) = 256$

It touches $(x-1)^2 + y^2 = 4^2$

$$\text{if } \left|\frac{4 \cos \theta - 16}{\sqrt{16 \cos^2 \theta + 11 \sin^2 \theta}}\right| = 4$$

$$\Rightarrow \quad (\cos \theta - 4)^2 = 16 \cos^2 \theta + 11 \sin^2 \theta$$

$$\Rightarrow 4 \cos^2 \theta + 8 \cos \theta - 5 = 0$$

$$\Rightarrow \quad \cos \theta = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

6. a, c. Tangent and normal are bisectors of $\angle SPS'$

Now equation of SP is $y = 3x/2$ and that of $S'P$ is $y = 5x$

Then their equations are $\frac{3x - 2y}{\sqrt{13}} = \pm \frac{5x - y}{\sqrt{26}}$

or $\quad 3x - 2y = \pm \frac{5x - y}{\sqrt{2}}$

$\Rightarrow$ lines are $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$ and $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$

Now (2, 3) and (1, 5) lie on the same side of $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$, which is equation of tangent.

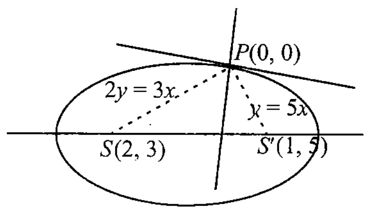


Fig. 4.68

Points (2, 3) and (1, 5) lie on the different sides of $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$, which is equation of normal.

7. c, d. The equation of the line joining θ and ϕ is

$$\frac{x}{5} \cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{3} \sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

If it passes through the point (4, 0), then

$$\frac{4}{5} \cos\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

$$\Rightarrow \quad \frac{4}{5} = \frac{\cos\left(\frac{\theta - \phi}{2}\right)}{\cos\left(\frac{\theta + \phi}{2}\right)}$$

$$\Rightarrow \quad \frac{4+5}{4-5} = \frac{\cos\left(\frac{\theta - \phi}{2}\right) + \cos\left(\frac{\theta - \phi}{2}\right)}{\cos\left(\frac{\theta - \phi}{2}\right) - \cos\left(\frac{\theta + \phi}{2}\right)}$$

$$= \frac{2 \cos \frac{\theta}{2} \cos \frac{\phi}{2}}{2 \sin \frac{\phi}{2} \sin \frac{\theta}{2}}$$

$$\Rightarrow \quad \tan \frac{\theta}{2} \tan \frac{\phi}{2} = -\frac{1}{9}$$

If it passes through the point (−5, 0),

then $\tan \frac{\phi}{2} \tan \frac{\theta}{2} = 9$

8. a, b, c, d.

$(\sqrt{3}x - 3\sqrt{3})^2 + (2y + 4)^2 = k$

so no locus for $k < 0$

ellipse for $k > 0$ and point for $k = 0$

9. a, c.

Let the point of equation of intersection of tangents A and B be $P(h, k)$, then equation of AB is

$$\frac{xh}{4} + \frac{yk}{1} = 1 \quad \text{(i)}$$

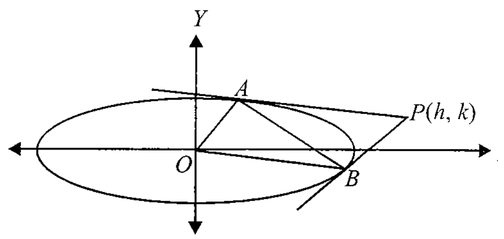


Fig. 4.69

Homogenizing the equation of ellipse using Eq. (i), we get

$$\frac{x^2}{4} + \frac{y^2}{1} = \left(\frac{xh}{4} + \frac{yk}{1}\right)^2$$

$$\Rightarrow x^2 \left(\frac{h^2 - 4}{16}\right) + y^2 (k^2 - 1) + \frac{2hk}{4} xy = 0 \quad \text{(ii)}$$

Given equation of OA and OB is

$$x^2 + 4y^2 + \alpha\, xy = 0 \quad \text{(iii)}$$

$\because$ Equation (ii) and (iii) represent same line,

Hence, $\frac{h^2 - 4}{16} = \frac{k^2 - 1}{4} = \frac{hk}{2a}$

$$\Rightarrow \quad h^2 - 4 = 4(k^2 - 1)$$

$\Rightarrow \quad h^2 - 4k^2 = 0$

$\Rightarrow$ Locus $(x - 2y)(x + 2y) = 0$

10. a, b, c.

a. The given equation is $\left(x - \frac{1}{13}\right)^2 + \left(y - \frac{2}{13}\right)^2$

$$= \frac{1}{a^2}\left(\frac{5x + 12y - 1}{13}\right)^2$$

It represents ellipse if $\frac{1}{a^2} < 1 \Rightarrow a^2 > 1 \Rightarrow a > 1$

b. $4x^2 + 8x + 9y^2 - 36y = -4$

$\Rightarrow 4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = 36$

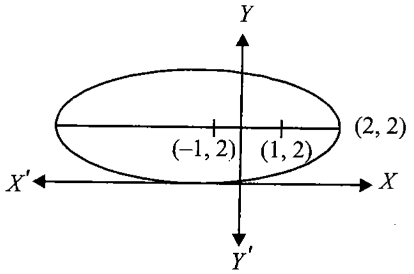


Fig. 4.70

$$\Rightarrow \quad \frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

Hence, $(-1, 2)$ is focus and $(1, 2)$ lies on the major axis. Then required minimum distance is 1.

c. Equation of normal at $P(\theta)$ is $5 \sec\theta\, x - 4 \operatorname{cosec}\theta\, y = 25 - 16$, and it passes through $P(0, \alpha)$

$$\therefore \quad \alpha = \frac{-9}{4 \operatorname{cosec}\theta}$$

$$\Rightarrow \quad \alpha = \frac{-9}{4}\sin\theta$$

$$\Rightarrow \quad |\alpha| < \frac{9}{4}$$

d. $\frac{2b^2}{a} = \frac{2a}{3} \Rightarrow 3b^2 = a^2$

$\Rightarrow$ From $b^2 = a^2(1 - e^2)$, $1 = 3(1 - e^2) \Rightarrow e = \sqrt{2/3}$

11. a, c, d.

$x^2 + 4y^2 - 2x - 16y + 13 = 0$

$\Rightarrow (x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 4$

$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$$

$\therefore$ Length of latus rectum $= \frac{2 \times 1}{2} = 1$

Also $\quad e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

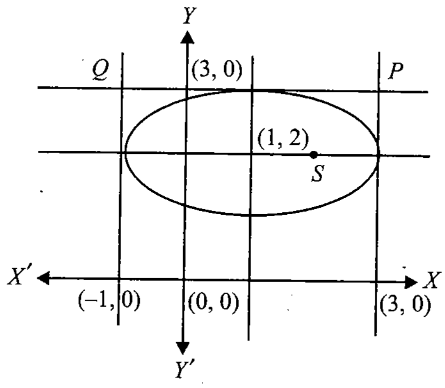


Fig. 4.71

$$\Rightarrow \quad 2ae = 2 \times 2 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Sum of the focal distance $= 2a = 4$

Tangents at the vertices are $x - 1 = \pm 2$

or $\quad x = 3, -1$

Therefore, the line $y = 3$ intersect these at points $P(3, 3)$ and $Q(-1, 3)$.

Coordinate of focus are $S(\sqrt{3} + 1, 2)$

Slope of PS is $\frac{1}{2 - \sqrt{3}}$, slope of QS is $\frac{1}{-2 - \sqrt{3}}$

$$\Rightarrow \quad \text{product of slopes} = \frac{1}{2 - \sqrt{3}} \times \frac{1}{-2 - \sqrt{3}} = -1$$

12. b, d.

Differentiating the equation of ellipse $x^2 + 3y^2 = 37$ w.r.t. x,

$$\frac{dy}{dx} = -\frac{x}{3y}$$

Slope of the given line is $\frac{6}{5}$, which is normal to the ellipse.

Hence, $\frac{3y}{5} = \frac{6}{5}$ or $2x = 5y$.

Points in the option (b) and (d) are satisfying the above relation.

13. a, c.

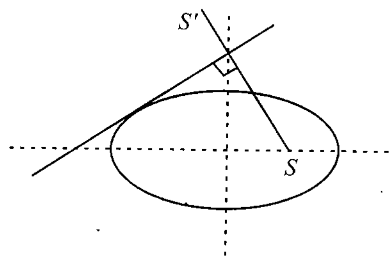


Fig. 4.72

Let $S'(h, k)$ be the image.

SS' cuts a tangent at a point which lies on the auxiliary circle of the ellipse

$$\Rightarrow \quad \left(\frac{h\pm 4}{2}\right)^2+\frac{k^2}{4}=25$$

$\Rightarrow$ locus is $(x \pm 4)^2 + y^2 = 100$

14. a, c.

Let $P(h, k)$ be the point of intersection of E_1 and E_2

$$\Rightarrow \quad \frac{h^2}{a^2}+k^2=1$$

$$\Rightarrow \quad h^2 = a^2(1-k^2) \qquad \text{(i)}$$

and $$\frac{h^2}{1}+\frac{k^2}{a^2}=1$$

$$\Rightarrow \quad k^2 = a^2(1-h^2) \qquad \text{(ii)}$$

Eliminating a from Eqs. (i) and (ii), we get

$$\frac{h^2}{1-k^2}=\frac{k^2}{1-h^2}$$

$$\Rightarrow \quad h^2(1-h^2)=k^2(1-k^2)$$

$$\Rightarrow (h-k)(h+k)(h^2+k^2-1)=0$$

Hence, the locus is a set of curves consisting of the straight lines

$y = x$, $y = -x$ and circle $x^2 + y^2 = 1$.

15. a, c.

$f(x)$ is a decreasing function and for major axis to be x-axis

$$f(k^2+2k+5) > f(k+11)$$

$$\Rightarrow \quad k^2+2k+5 < k+11$$

$$\Rightarrow \quad k \in (-3, 2)$$

Then for the remaining values of k, i.e., $k \in (-\infty, -3) \cup (2, \infty)$, major axis is y-axis.

16. a, b, c.

Clearly O is the midpoint of SS' and HH'

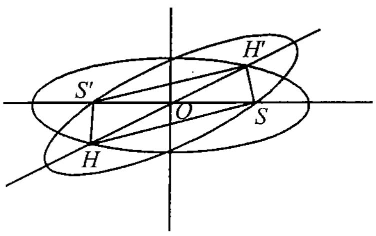


Fig. 4.73

$\Rightarrow$ Diagonals of quadrilateral $HSH'S'$ bisect each other, so it is a parallelogram.

Let $H'OH = 2r \Rightarrow OH = r = ae'$

H lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (suppose)

$$\therefore \quad \frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1$$

$$e'^2\cos^2\theta+\frac{e'^2\sin^2\theta}{1-e^2}=1 \qquad [\because b^2=a^2(1-e'^2)]$$

$$\Rightarrow e'^2\cos^2\theta-\frac{e'^2\cos^2\theta}{1-e^2}=1-\frac{e'^2}{1-e^2}$$

$$\Rightarrow \quad \cos^2\theta=\frac{1}{e^2}+\frac{1}{e'^2}-\frac{1}{e^2e'^2}$$

$$\Rightarrow \quad \theta=\cos^{-1}\sqrt{\frac{1}{e^2}+\frac{1}{e'^2}-\frac{1}{e^2e'^2}}$$

For $$\theta = 90°, \frac{e^2+e'^2}{e^2e'^2}=\frac{1}{e^2e'^2}$$

$$\Rightarrow \quad e^2+e'^2=1$$

17. a, d. The equation of the tangent at $(t^2, 2t)$ to the parabola

$y^2 = 4x$ is

$$2ty = 2(x+t^2)$$

$$\Rightarrow \quad ty = x+t^2$$

$$\Rightarrow \quad x-ty+t^2=0 \qquad \text{(i)}$$

The equation of the normal at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $5x^2 + 5y^2 = 20$ is

$\Rightarrow (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y = 5 - 4$

$\Rightarrow (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y = 1 \qquad \text{(ii)}$

Given that Eqs. (i) and (ii) represent the same line.

$$\Rightarrow \quad \frac{\sqrt{5}\sec\theta}{1}=\frac{-2\,\text{coses}\,\theta}{-t}=\frac{-1}{t^2}$$

$$\Rightarrow \quad t=\frac{2}{\sqrt{5}}\cot\theta \text{ and } t=-\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad \frac{2}{\sqrt{5}}\cot\theta=-\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad 4\cos\theta=-\sqrt{5}\sin^2\theta$$

$$\Rightarrow \quad 4\cos\theta=-\sqrt{5}(1-\cos^2\theta)$$

$$\Rightarrow \sqrt{5}\cos^2\theta-4\cos\theta-\sqrt{5}=0$$

$$\Rightarrow (\cos\theta-\sqrt{5})(\sqrt{5}\cos\theta+1)=0$$

$$\Rightarrow \quad \cos\theta=-\frac{1}{\sqrt{5}} \qquad [\because \cos\theta \neq -\sqrt{5}]$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$$

Putting $\cos\theta = -\frac{1}{\sqrt{5}}$ in $t = -\frac{1}{2}\sin\theta$, we get

$$t=-\frac{1}{2}\sqrt{1-\frac{1}{5}}=-\frac{1}{\sqrt{5}}$$

Hence, $$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \text{ and } t=-\frac{1}{\sqrt{5}}.$$

Reasoning Type

1. **d.** Statement 1 is false as locus of (x, y) is a line segment joining points (2, 0) and (–2, 0).
2. **c.** Statement 2 is false (locus of P may be a line segment also). statement 1 is true.
3. **a.** It is fundamental property of an ellipse.
4. **c.** $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$

   Ends of the major axis are (0, 6) and (0, 0).

   Equation of tangent at (0, 6) and (0, 0) is $y = 6$ and $y = 0$. Hence, statement 1 is true.

   But statement 2 is false, as tangents at the ends of major axis may be lines parallel to y-axis when $a < b$.
5. **a.** Locus of point of intersection of perpendicular tangents is director circle, which is $x^2 + y^2 = a^2 + b^2$.

   Now line $px + qy + r = 0$ may intersect this circle maximum at two points.

   Thus there can be maximum two points on the line from which perpendicular tangents can be drawn to the ellipse.
6. **a.** Statement 2 is true as it is one of the properties of ellipse.

   Ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$

   Focus $\equiv (\sqrt{5}, 0)$, $e = \frac{\sqrt{5}}{3}$

   One of the points on the ellipse $\equiv \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

   Equation of the circle as the diameter joining the points $(3/\sqrt{2}, 2/\sqrt{2})$ and focus $(\sqrt{5}, 0)$ is

   $(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$

   Hence, statement 1 is true and statement 2 is correct explanation of statement 1.
7. **d.** Let C_1, C_2 the centres and r_1, r_2 be the radii of the two circles. Let $S_1 = 0$ lies completely inside the circle. $S_2 = 0$. Let C and r be centre and radius of the variable circle, respectively.

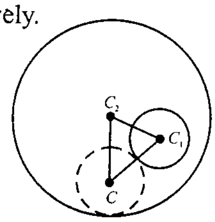


Fig. 4.74

   then $CC_2 = r_2 - r$ and $C_1C = r_1 + r$

   $\Rightarrow C_1C + C_2C = r_1 + r_2$ (constant)

   $\Rightarrow$ Locus of C is an ellipse

   $\Rightarrow S_2$ is true

   Statement 1 is false (two circles are intersecting).
8. **b.** By formula $p_1p_2 = b^2 = 3$.

   Also foot of perpendicular lies on auxiliary circle of the ellipse.

   Thus both the statements are true. But statement 2 is not correct explanation of statement 1.
9. **d.** Locus of point of intersection of perpendicular tangents is director circle.

   If there exists exactly one such point on the line $3x + 4y + 5\sqrt{5} = 0$, then it must touch the director circle

   $$x^2 + y^2 = a^2 + 1$$
   $$\Rightarrow \quad 5 = a^2 + 1$$
   $$\Rightarrow \quad a^2 = 4$$
   $$\Rightarrow \quad a = 2$$

   Hence, eccentricity $= \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$.
10. **a.** Let $y = mx$ be any chord through (0, 0). This will meet conic at points whose x-coordinates are given by $x^2 + m^2x^2 + mx^2 = 1$

    $$\Rightarrow \quad (1 + m + m^2)x^2 - 1 = 0$$
    $$\Rightarrow \quad x_1 + x_2 = 0$$
    $$\Rightarrow \quad \frac{x_1 + x_2}{2} = 0$$

    Also $\quad y_1 = mx_1, y_2 = mx_2$

    $$\Rightarrow \quad y_1 + y_2 = m(x_1 + x_2) = 0$$
    $$\Rightarrow \quad \frac{y_1 + y_2}{2} = 0$$

    $\Rightarrow$ Midpoint of chord is (0, 0) $\forall m$.

    Hence, statement 1 is true as (0, 0) is also centre of the ellipse.

    Statement 2 is fundamental property of the ellipse, hence statement 2 is correct explanation of statement 1.
11. **b.** Given ellipse is $\frac{x^2}{3} + \frac{y^2}{2} = 1$, whose area is $= \pi\sqrt{3}\sqrt{2} = \pi\sqrt{6}$.

    Circle is $x^2 + y^2 - 2x + 4y + 4 = 0$ or $(x - 1)^2 + (y - 2)^2 = 1$.

    Its area is π. Hence, statement 1 is true.

    Also statement 2 is true but it is not the correct explanation of statement 1.

    Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{1} = 1$, whose area is 5π and circle $x^2 + y^2 = 16$ whose area is 16π.

    Also here semi-major axis of ellipse (= 5) is more than the radius of the circle (= 4).
12. **d.** Statement 2 is true, see theory.

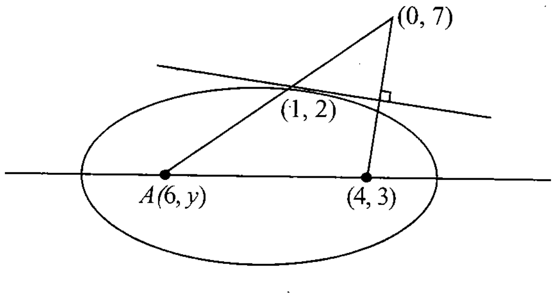


Fig. 4.75

Image of (4, 3) in the line $x + y - 3 = 0$ is

$$\frac{x-4}{1} = \frac{y-3}{1} = -2\frac{(4+3-3)}{2} = -4$$

$\Rightarrow$ $x = 0$ and $y = 7$

Now points (0, 7), (1, 2) and (6, y) are collinear.

$$\Rightarrow \quad \frac{7-2}{0-1} = \frac{y-2}{6-1}$$

$$\Rightarrow \quad y = 27$$

13. a. Statement 2 is correct as ellipse is a central conic and it also explains statement 1.

14. d. Statement 1: Locus of point of intersection of only perpendicular lines is a circle, and other vertices B and C do not form a circle

Statement 2 is obvious (standard definition).

15. a. Chord of contact of the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ with respect to point (8, 6) is

$$\frac{8x}{4} + \frac{6x}{2} = 1$$

or $\quad 2x + 3y = 1.$

Hence, statement 1 is correct, also statement 2 is correct and explains the statement 1.

16. a. Statement 2 is true as it is one of the properties of the ellipse.

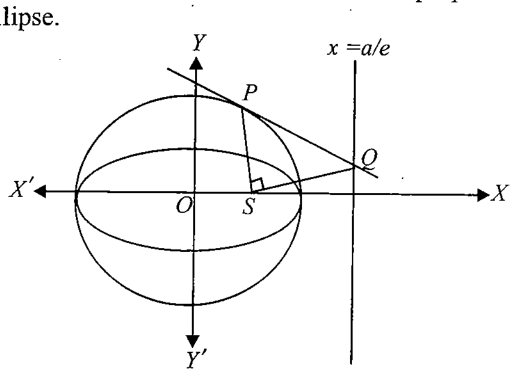


Fig. 4.76

Circle with minimum radius having PQ as chord when PQ is diameter of the circle, hence as shown in the figure it passes through the focus.

17. d. $|ay - bx| = c\sqrt{(x-a)^2 + (y-b)^2}$

$$\Rightarrow \quad \frac{|ay - bx|}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}\sqrt{(x-a)^2 + (y-b)^2}$$

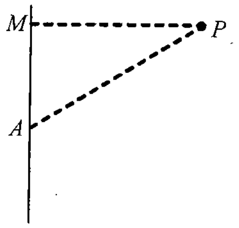


Fig. 4.77

or $PM = kPA$, where m is the length of perpendicular from P on the line $ay - bx = 0$ and PA is the length of line segment joining P to the point A (a, b) and A lies on the line, so the locus of P is a straight line through A inclined at an angle $\sin^{-1}\frac{c}{\sqrt{a^2+b^2}}$ to the given line (provided $c < \sqrt{a^2+b^2}$).

Linked Comprehension Type

For Problems 1–3

1. c., 2. a., 3. b.

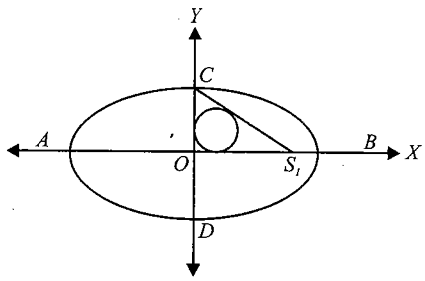


Fig. 4.78

1. a. $\because OS_1 = ae = 6, OC = b$

Sol.

Also $\quad CS_1 = a$

$\Rightarrow$ Area of $\Delta OCS_1 = \frac{1}{2}(OS_1) \times (OC) = 3b$

$\Rightarrow$ Semi-perimeter of $\Delta OCS_1 = \frac{1}{2}(OS_1 + OC + CS_1)$

$$= \frac{1}{2}(6 + a + b) \qquad \text{(i)}$$

$\Rightarrow$ In radius of $\Delta OCS_1 = 1$

$$\Rightarrow \quad \frac{3b}{\frac{1}{2}(6+a+b)} = 1$$

$$\Rightarrow \quad 5b = 6 + a \qquad \text{(ii)}$$

also $\quad b^2 = a^2 - a^2e^2$

$$= a^2 - 36 \qquad \text{(iii)}$$

$\Rightarrow$ From (ii), we get

$$25(a^2 - 36) = 36 + a^2 + 12a$$

$$\Rightarrow \quad 2a^2 - a - 78 = 0$$

$$\Rightarrow \quad a = \frac{13}{2}, -6$$

$$\Rightarrow \quad a = \frac{13}{2}$$

and $\quad b = \frac{5}{2}$

Area of ellipse $= \pi ab = \frac{65\pi}{4}$ sq. unit

Perimeter of $\Delta\, OCS_1 = 6 + a + b = 6 + \frac{13}{2} + \frac{5}{2} = 15$ units

Equation director circle is $x^2 + y^2 = a^2 + b^2$

or $$x^2 + y^2 = \frac{97}{2} = r^2$$

For Problems 4–6

4. c., 5. b., 6. a.

Sol.

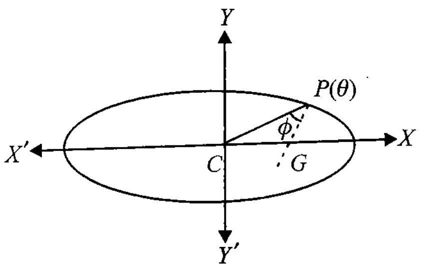


Fig. 4.79

Any point P on the ellipse at $(a\cos\theta, b\sin\theta)$

$\therefore$ Equation of CP is $y = \left(\frac{b}{a}\tan\theta\right)x$

The normal to the ellipse at P is $\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2$

Slope of the lines CP and the normal GP are $\frac{b}{a}\tan\theta$ and $\frac{a}{b}\tan\theta$, respectively

$$\therefore \quad \tan\phi = \frac{\frac{a}{b}\tan\theta - \frac{b}{a}\tan\theta}{1 + \frac{a}{b}\tan\theta\,\frac{b}{a}\tan 0}$$

$$= \frac{a^2 - b^2}{ab}\,\frac{\tan\theta}{\sec^2\theta}$$

$$= \frac{a^2 - b^2}{ab}\sin\theta\cos\theta = \frac{a^2 - b^2}{2ab}\sin 2\theta$$

Therefore, the greatest value of $\tan\phi = \frac{a^2 - b^2}{2ab} \times 1 = \frac{a^2 - b^2}{2ab}$

Given that $\frac{a^2 - b^2}{2ab} = \frac{3}{2}$. Let $\frac{a}{b} = t$

$$\Rightarrow \quad t - \frac{1}{t} = \frac{3}{2}$$

$$\Rightarrow \quad 2t^2 - 3t - 2 = 0$$

$$\Rightarrow \quad 2t^2 - 4t + t - 2 = 0$$

$$\Rightarrow \quad (2t + 1)(t - 2) \Rightarrow \frac{a}{b} = 2$$

$$\Rightarrow \quad e^2 = 1 - \frac{1}{4}$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

Rectangle inscribed in the ellipse whose one vertex is $(a\cos\theta, b\sin\theta)$ is $(2a\cos\theta)(2b\sin\theta) = 2ab\sin(2\theta)$ which has maximum value $2ab$. Given that $a = 10$, then $b = 5 \Rightarrow$ maximum area is 100.

Locus of intersection point of perpendicular tangents is $x^2 + y^2 = 10^2 + 5^2$ or $x^2 + y^2 = 125$ (director circle).

For Problems 7–9

7. b., 8. a., 9. c.

$$21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

$$3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$$

$$\Rightarrow \frac{(x - 3y + 3)^2}{60} + \frac{(3x + y - 1)^2}{90} = 1$$

$$\Rightarrow \left(\frac{x - 3y + 3}{\sqrt{1 + 3^2}\sqrt{6}}\right)^2 + \left(\frac{3x + y - 1}{\sqrt{1 + 3^2}3}\right)^2 = 1$$

Thus C is an ellipse whose lengths of axes are 6, $2\sqrt{6}$.

The minor and the major axes are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.

Their point of intersection gives the centre of the conic.

$\therefore$ Centre $\equiv (0, 1)$

For Problems 10–12

10. b., 11. d., 12. a.

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The line $y = mx \pm \sqrt{a^2m^2 + b^2}$ touches the ellipse for all m.

Hence, it is identical with

$$y = -\frac{2px}{\sqrt{1 - p^2}} + \frac{1}{\sqrt{1 - p^2}}$$

Hence, $$m = -\frac{2p}{\sqrt{1 - p^2}}$$

and $$a^2m^2 + b^2 = \frac{1}{1 - p^2}$$

$$\Rightarrow \quad a^2\frac{4p^2}{1 - p^2} + b^2 = \frac{1}{1 - p^2}$$

$$\Rightarrow p^2(4a^2 - b^2) + b^2 - 1 = 0$$

This equation is true for all real p if $b^2 = 1$ and $4a^2 = b^2$

$$\Rightarrow \quad b^2 = 1 \text{ and } a^2 = \frac{1}{4}$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{1/4} + \frac{y^2}{1} = 1$$

If e is its eccentricity, then

$$\frac{1}{4} = 1 - e^2 \Rightarrow e^2 = \Rightarrow e = \frac{\sqrt{3}}{2}$$

$be = \frac{\sqrt{3}}{2}$, hence foci are $\left(0, \pm\frac{\sqrt{3}}{2}\right)$

Equation of director circle is $x^2 + y^2 = \frac{5}{4}$.

For Problems 13–15

13. a., 14. b., 15. a.

Sol.

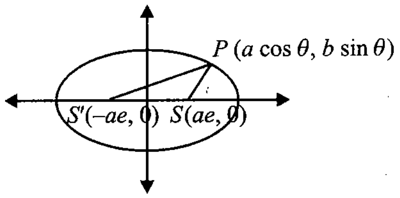


Fig. 4.80

Let the coordinates of P be $(a\cos\theta, b\sin\theta)$.

Here SP = focal distance of the point $P = a - ae\cos\theta$

$$S'P = a + ae\cos\theta$$

$$SS' = 2ae$$

If (h, k) be the coordinates of the incentre of $\Delta PSS'$, then

$$h = \frac{2ae(a\cos\theta) + a(1 - e\cos\theta)(-ae) + a(1 + e\cos\theta)ae}{2ae + a(1 - e\cos\theta) + a(1 + e\cos\theta)}$$

$$\Rightarrow \quad h = ae\cos\theta \quad \text{(i)}$$

and

$$k = \frac{2ae(b\sin\theta) + a(1 - e\cos\theta)\times 0 + a(1 + e\cos\theta)\times 0}{2ae + a(1 - e\cos\theta) + a(1 + e\cos\theta)}$$

$$\Rightarrow \quad k = \frac{eb\sin\theta}{(e+1)} \quad \text{(ii)}$$

Eliminating θ from (i) and (ii), we get

$$\frac{x^2}{a^2e^2} + \frac{y^2}{\left(\frac{be}{e+1}\right)^2} = 1$$

which clearly represents an ellipse. Let e_1 be its eccentricity. Then

$$\frac{b^2e^2}{(e+1)^2} = a^2e^2(1 - e_1^2)$$

$$\Rightarrow \quad e_1^2 = 1 - \frac{b^2}{a^2(e+1)^2}$$

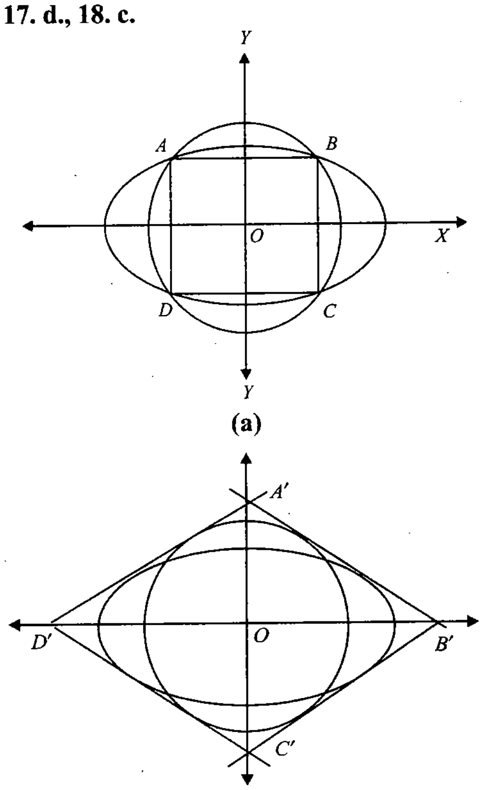
$$\Rightarrow \quad e_1^2 = 1 - \frac{1-e^2}{(e+1)^2} = 1 - \frac{1-e}{1+e}$$

$$\Rightarrow \quad e_1^2 = \frac{2e}{e+1} \Rightarrow e_1 = \sqrt{\frac{2e}{e+1}}$$

Maximum area of rectangle is $2(ae)\left(\frac{be}{e+1}\right) = \frac{2abe^2}{e+1}$

For Problems 16–18

16. a., 17. d., 18. c.



(a)

(b)

Fig. 4.81

16. a. Solving the curves given (eliminating x^2), we have

$$\frac{r^2 - y^2}{16} + \frac{y^2}{9} = 1$$

$$\Rightarrow \quad y^2 = \frac{144 - 9r^2}{7}$$

Solving the curves given (eliminating y^2), we have

$$\frac{x^2}{16} + \frac{r^2 - x^2}{9} = 1$$

$$\Rightarrow \quad x^2 = \frac{16r^2 - 144}{7}$$

If $ABCD$ is a square, then

$$x^2 = y^2$$

or $$\frac{144 - 9r^2}{7} = \frac{16r^2 - 144}{7}$$

or $$25r^2 = 288$$

or $$r = \frac{12}{5}\sqrt{2}$$

17. d. Tangent of slope m to circle and ellipse is

$$y = mx \pm \sqrt{r^2m^2 + r^2}$$

and $y = mx \pm \sqrt{16m^2 + 9}$, respectively.

For common tangent

$$r^2m^2 + r^2 = 16m^2 + 9$$

Also if A', B', C', D' is a square, then

$$m = \pm 1$$

$$\Rightarrow \quad r^2 + r^2 = 25$$

$$\Rightarrow \quad r = 5/\sqrt{2}$$

18. c. If $A'B'C'D'$ is a square, then tangents

$$y = \pm x \pm 5$$

for which diagonal length $A'C' = 10$.

Then area of $\Delta A'B'C'$ is 25π.

Also area of circle C_1 is $\frac{25\pi}{2}$. Hence, the required ratio is $\frac{1}{2}$.

For Problems 19–21

19. c., 20. b., 21. d.

19. c. $\lambda x - y + 2(1 + \lambda) = 0$

$$\Rightarrow \quad \lambda(x + 2) - (y - 2) = 0$$

This line passes through (−2, 2).

$$\mu x - y + 2(1 - \mu) = 0$$

$$\Rightarrow \quad \mu(x - 2) - (y - 2) = 0$$

This line passes through (2, 2).

Clearly these represent the foci of the ellipse. So $2ae = 4$.

The circle $x^2 + y^2 - 4y - 5 = 0 \Rightarrow x^2 + (y - 2)^2 = 9$ represents auxiliary circle. Thus $a^2 = 9 \Rightarrow e = \frac{2}{3}$ and $b^2 = 5$.

20. c. Required area $= \frac{1}{2}(2ae)(b) = 3\frac{2}{3}\sqrt{5} = 2\sqrt{5}$

For Problems 22–24

22. b., 23. d., 24. c.

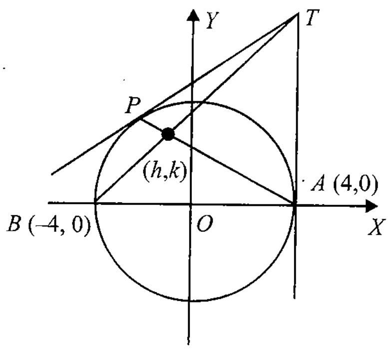


Fig. 4.82

Tangents at $P(4\cos\theta, 4\sin\theta)$ to $x^2 + y^2 = 16$ is

$$x\cos\theta + y\sin\theta = 4 \qquad \text{(i)}$$

Equation of AP is

$$y = \frac{\sin\theta}{\cos\theta - 1}(x - 4) \qquad \text{(ii)}$$

From (i), coordinates of the point T are given by

$$\left(4, \frac{4(1 - \cos\theta)}{\sin\theta}\right)$$

Equation of BT is

$$y = \frac{1 - \cos\theta}{2\sin\theta}(x + 4) \qquad \text{(iii)}$$

Let (h, k) be the point of intersection of the lines (ii) and (iii). Then we have

$$k^2 = -\frac{1}{2}(h^2 - 16)$$

$$\Rightarrow \quad \frac{h^2}{16} + \frac{k^2}{8} = 1$$

Therefore, locus of (h, k) is

$$\frac{x^2}{16} + \frac{y^2}{8} = 1$$

which is an ellipse with eccentrically $e = \frac{1}{\sqrt{2}}$.

Sum of focal distances of any point $= 2a = 8$.

Considering circle $x^2 + y^2 = a^2$, we find that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$ which is constant and does not change by changing the radius of the circle.

For Problems 25–27

25. a, 26. c., 27. b

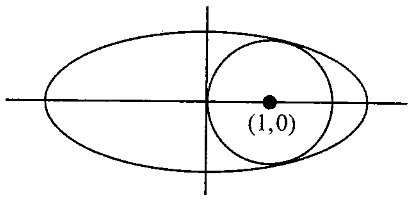


Fig. 4.83

Solving both equations, we have

$$\frac{x^2}{a^2} + \frac{1 - (x - 1)^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 + a^2[1 - (x - 1)^2] = a^2b^2$$

$$\Rightarrow \quad (b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0 \qquad \text{(i)}$$

for least area circle must touch the ellipse

$\Rightarrow$ Discriminant of (1) is zero

$$\Rightarrow \quad 4a^4 + 4a^2b^2(b^2 - a^2) = 0$$

$$\Rightarrow \quad a^2 + b^2(b^2 - a^2) = 0$$

$$\Rightarrow \quad a^2 + b^2(-a^2e^2) = 0$$

$$\Rightarrow \quad 1 - b^2e^2 = 0 \Rightarrow b = \frac{1}{e}$$

Also
$$a^2 = \frac{b^2}{1 - e^2} = \frac{1}{e^2(1 - e^2)}$$

$$\Rightarrow \quad a = \frac{1}{e\sqrt{1 - e^2}}$$

Let S be the area of the ellipse.

$$\Rightarrow \quad S = \pi ab = \frac{\pi}{e^2\sqrt{1 - e^2}}$$

$$= \frac{\pi}{\sqrt{e^4 - e^6}}$$

Area is minimum if $f(e) = e^4 - e^6$ is maximum

when $$f'(e) = 4e^3 - 6e^5 = 0$$

or $$e = \sqrt{\frac{2}{3}} \text{ (which is point of maxima for } f(e))$$

$\Rightarrow$ S is least when $e = \sqrt{\frac{2}{3}}$

$\Rightarrow$ Ellipse is $2x^2 + 6y^2 = 9$

Equation of auxiliary circle of ellipse is $x^2 + y^2 = 4.5$

Length of latus rectum of ellipse is $\frac{2b^2}{a} = \frac{2\frac{9}{4}}{\frac{9}{2}} = 1$

Matrix-Match Type

1. a. → p, q, r, s; b → p, q, r, s; c → q, r, s; d → p.

a. Equation of tangent at $\left(\frac{\cos\theta}{2}, \frac{\sin\theta}{3}\right)$ is

$2x\cos\theta + 3y\sin\theta = 1$

which is parallel to the given line $8x = 9y$

$$\therefore \cos\theta = \pm\frac{4}{5}, \sin\theta = \mp\frac{3}{5}$$

Hence, points are $\left(\frac{2}{5}, -\frac{1}{5}\right)$ and $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

Distance between the points is

$$\sqrt{\frac{16}{25} + \frac{4}{25}} = \frac{2}{\sqrt{5}}$$

which is less than 1.

b. The given equation is

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

$$\Rightarrow e^2 = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

Hence, the foci are $S, S' \equiv (-1, -2 \pm 4) \equiv S(-1, 2)$ and $S'(-1, -6)$

The required sum of distances = 2 + 6 = 8.

c. Equation of normal at $(3\cos\theta, 2\sin\theta)$ is

$3x\sec\theta - 2y\operatorname{cosec}\theta = 5$

which is parallel to the given line $2x + y = 1$. Therefore,

$$\cos\theta = \mp\frac{3}{5}, \sin\theta = \pm\frac{4}{5}$$

Hence, points are $\left(\frac{-9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

The required sum of distances $= \frac{16}{5}$.

d. Consider any point $(t, t+2)$, $t \in R$, on the line $x - y + 2 = 0$.

The chord of contact of ellipse w.r.t. this point is

$$xt + 2y(t+2) = 2$$

$$\Rightarrow (4y - 2) + t(x + 2y) = 0$$

This line passes through point of intersection of lines

$$4y - 2 = 0 \text{ and } x + 2y = 0$$

$$\therefore x = -1$$

Hence, the point is (–1, 1/2), whose distance from (2, 1/2) is 3.

2. a → r, s; b → p, q, r, s; c → r, s; d → r, s.

Equation of any tangent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(i)}$$

is given by

$$y = mx + \sqrt{a^2m^2 + b^2}$$

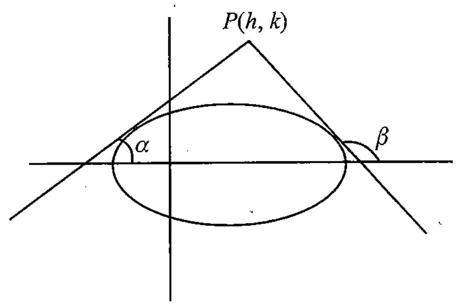


Fig. 4.84

Since it passes through $P(h, k)$

$$k = mh + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow m^2(h^2 - a^2) - 2kmh + (k^2 - b^2) = 0 \quad \text{(ii)}$$

As (ii) is quadratic in m, having two roots m_1 and m_2 (say), Therefore,

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}, \; m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2} \quad \text{(iii)}$$

$$\Rightarrow \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{m_1 + m_2}{1 - m_1m_2}$$

$$= \frac{\frac{2hk}{h^2 - a^2}}{1 - \frac{k^2 - b^2}{h^2 - a^2}}$$

$$= \frac{2hk}{h^2 - k^2}$$

a. $\alpha + \beta = \frac{c\pi}{2}$

When c is even,

$$m_1 + m_2 = 0$$

$$\frac{2kh}{h^2 - a^2} = 0 \Rightarrow 2kh = 0$$

$\Rightarrow xy = 0$, which is the equation of a pair of straight lines.

When c is odd,

$$1 - m_1 m_2 = 0$$

$$\Rightarrow \quad \frac{k^2 - b^2}{h^2 - b^2} = 1$$

Therefore, the locus of (h, k) is

$$y^2 - b^2 = x^2 - a^2$$

which is a hyperbola.

b. $m_1 m_2 = c$

$$\Rightarrow \quad \frac{k^2 - b^2}{h^2 - a^2} = c$$

When $c = 0$, $k = \pm b$, the locus is pair of straight lines.

When $c = 1$, $h^2 - k^2 = a^2 - b^2$, the locus is hyperbola.

When $c = -1$, $h^2 + k^2 = a^2 + b^2$, the locus is circle.

When $c = -2$, $2h^2 + k^2 = 2a^2 + b^2$, the locus is ellipse.

c. $\tan\alpha + \tan\beta = c$

$$\Rightarrow \quad m_1 + m_2 = c$$

$$\Rightarrow \quad \frac{2kh}{h^2 - a^2} = c$$

When $c = 0$, $kh = 0$, the locus is pair of straight lines.

When $c \neq 0$

$$c(h^2 - a^2) - 2kh = 0$$

Locus of (h, k) is

$$cx^2 - 2xy - ca^2 = 0$$

$$\Delta = -ca^2 \neq 0$$

Also,

$$h^2 - ab = 1 > 0$$

Therefore, the locus is a hyperbola for $c \neq 0$.

d. $\cot\alpha + \cot\beta = c$

$$\Rightarrow \quad \frac{1}{m_1} + \frac{1}{m_2} = c$$

$$\Rightarrow \quad \frac{m_1 + m_2}{m_1 m_2} = c$$

$$\Rightarrow \quad \frac{2kh}{k^2 - b^2} = c$$

$$\Rightarrow \quad c(k^2 - b^2) - 2kh = 0$$

When $c = 0$, locus is a pair of straight lines.

When $c \neq 0$, locus is a hyperbola (as in previous case c).

3. a. → p, r; b → r, s; c → p; d → q.

Tangent to ellipse at $P(\phi)$ is $\frac{x}{4}\cos\phi + \frac{y}{2}\sin\phi = 1$.

It must pass through the centre of the circle. Hence,

$$\frac{4}{4}\cos\phi + \frac{2}{2}\sin\phi = 1$$

$$\Rightarrow \quad \cos\phi + \sin\phi = 1$$

$$\Rightarrow \quad 1 + \sin 2\phi = 1$$

$$\text{or} \quad \sin 2\phi = 0$$

$$\Rightarrow \quad 2\phi = 0 \text{ or } \pi$$

$$\Rightarrow \quad \frac{\phi}{2} = 0 \text{ or } \frac{\pi}{4}$$

b. Consider any point $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$.

Given that $\quad OP = 2$

$$\Rightarrow \quad 6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow \quad 4\cos^2\theta = 2$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

c. Solving the equation of ellipse and parabola (eliminating x^2), we have

$$y - 1 + 4y^2 = 4$$

$$\Rightarrow \quad 4y^2 + y - 5 = 0$$

$$\Rightarrow \quad (4y + 5)(y - 1) = 0$$

$$\Rightarrow \quad y = 1, x = 0$$

The curves touch at (0, 1). So, the angle of intersection is 0.

d. The normal at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2$$

where $\quad a^2 = 14, b^2 = 5.$

It meets the curve again at $Q(2\theta)$, i.e. $(a\cos 2\theta, b\sin 2\theta)$. Hence,

$$\frac{a}{\cos\theta} a\cos 2\theta - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$$

$$\Rightarrow \quad \frac{14}{\cos\theta}\cos 2\theta - \frac{5}{\sin\theta}(\sin 2\theta) = 14 - 5$$

$$\Rightarrow \quad 28\cos^2\theta - 14 - 10\cos^2\theta = 9\cos\theta$$

$$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta - 2) = 0$$

$$\Rightarrow \quad \cos\theta = -\frac{2}{3}$$

4. a → s; b → p; c → q; d → r.

a. The locus is

$\Rightarrow \quad \frac{x^2}{16}+\frac{y^2}{36}=1$

$\Rightarrow \quad e=\sqrt{1-\frac{16}{36}}=\sqrt{\frac{20}{36}}=\frac{\sqrt{5}}{3}$

$\Rightarrow \quad 3e=\sqrt{5}$

b. $3(x^2+2x+1)+2(y^2-2y+1)=3+2+1$

$\Rightarrow \frac{(x+1)^2}{2}+\frac{(y-1)^2}{3}=1$

$\Rightarrow \quad e=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$

$\therefore \quad a=\sqrt{3},\ b=\sqrt{2}$

$\therefore \quad \text{Area}=\frac{1}{2}\times 2\sqrt{3}\times\sqrt{2}=\sqrt{6}$

c. Eliminating θ from $x=1+4\cos\theta, y=2+3\sin\theta$, we have

$\frac{(x-1)^2}{16}+\frac{(y-2)^2}{9}=1$

Hence, $a=4$ and $e=\frac{\sqrt{7}}{4}$.

$\Rightarrow \quad ae=\sqrt{7}$

$\Rightarrow$ Distance between the foci $=2\sqrt{7}$

d. $\frac{x^2}{16}+\frac{y^2}{7}=1$

$e=\sqrt{1-\frac{7}{16}}=\frac{3}{4}$

One end of latus rectum is

$\left(ae,\frac{b^2}{a}\right)=\left(3,\frac{7}{4}\right)$

Therefore, equation of tangent is

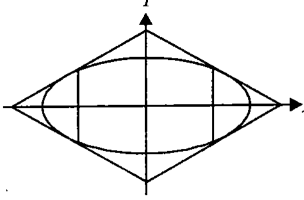


Fig. 4.85

$\frac{3x}{16}+\frac{7}{4}\frac{y}{7}=1$

or $\quad \frac{3x}{16}+\frac{y}{4}=1$

It meets x-axis at $\left(\frac{16}{3},0\right)$ and y-axis at $(0, 4)$.

Hence, the area of quadrilateral $=2\times\frac{16}{3}\times 4=\frac{128}{3}$.

5. a → q; b → r; c → r; d → p.

a. Points are $O(0, 0)$, $P(3, 4)$ and $Q(6, 8)$

$2a=OP+OQ$

$=5+10=15$

$\Rightarrow \quad a=\frac{15}{2}$

Also distance between foci,

$2ae=\sqrt{(6-3)^2+(8-4)^2}=5$

$\Rightarrow \quad e=\frac{1}{3}$

$\Rightarrow \quad b^2=\frac{225}{4}\left(1-\frac{1}{9}\right)=50$

$\Rightarrow \quad b=5\sqrt{2}$

$\Rightarrow \quad 2b=10\sqrt{2}$

b. We know that $\frac{1}{SP}+\frac{1}{SQ}=\frac{2a}{b^2}$

$\Rightarrow \quad \frac{1}{2}+\frac{1}{SQ}=\frac{10}{16}$

$\Rightarrow \quad SQ=8$

$\Rightarrow \quad PQ=10$

c. If the line $y=x+k$ touches the ellipse $9x^2+16y^2=144$, then

$k^2=16(1)^2+9$

$\Rightarrow \quad k=\pm 5$

d. Sum of the distances of a point on the ellipse from the foci $=2a=8$.

6. a → q; b → q; c → s; d → p.

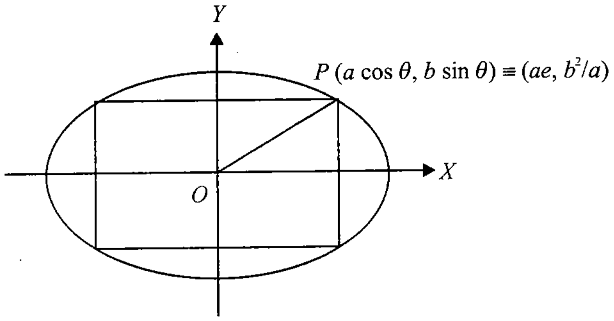


Fig. 4.86

a. Let one of the vertices of the rectangle be $P(a\cos\theta, b\sin\theta)$.

Then its area $A=(2a\cos\theta)(2b\sin\theta)=2ab\sin 2\theta$.

Hence, $\quad A_{max}=2ab$

Now area of rectangle formed by extremities of LR $=(2ae)(2b^2/a)=4eb^2$.

Given that

$2ab=4eb^2 \Rightarrow \frac{2b}{a}e=1$

$\Rightarrow \quad \frac{4b^2}{a^2}e^2=1$

$\Rightarrow \quad 4(1-e^2)e^2=1$

$\Rightarrow \quad 4e^4-4e^2+1=0$

$\Rightarrow \quad (2e^2-1)^2=0$

$\Rightarrow \quad e=\frac{1}{\sqrt{2}}$

b.

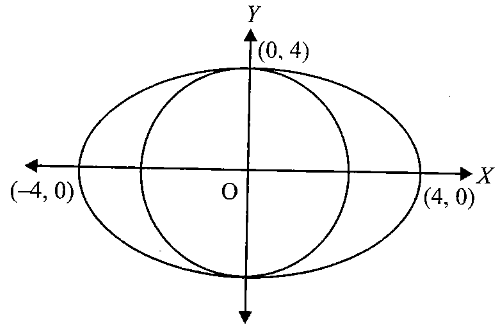


Fig. 4.87

For ellipse, distance c between the foci, $2ae = 8$ and length of semi-minor axis, is $b = 4$.

Now,

$$b^2 = a^2 - a^2e^2$$

$\Rightarrow$ $16 = a^2 - 16$

$\Rightarrow$ $a^2 = 32$

$\Rightarrow$ $e = \sqrt{1 - \frac{16}{32}} = \frac{1}{\sqrt{2}}$

c. Normal at point P (6, 2) to the ellipse passes through its focus Q (5, 2). Then P must be extremity of the major axis. Now $ae = QR = 1$ (where R is centre) and $a - ae = 1$.

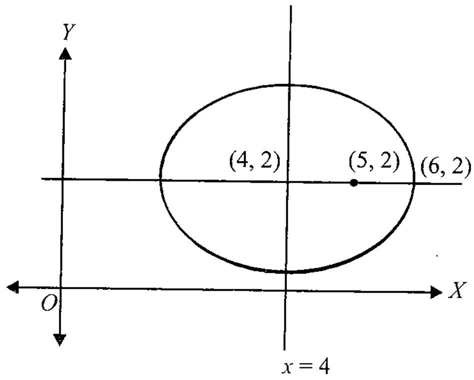


Fig. 4.88

$\therefore$ $a = 2$

$$b^2 = a^2 - a^2e^2 = 4 - 1 = 3$$

$\Rightarrow$ $e = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

d.

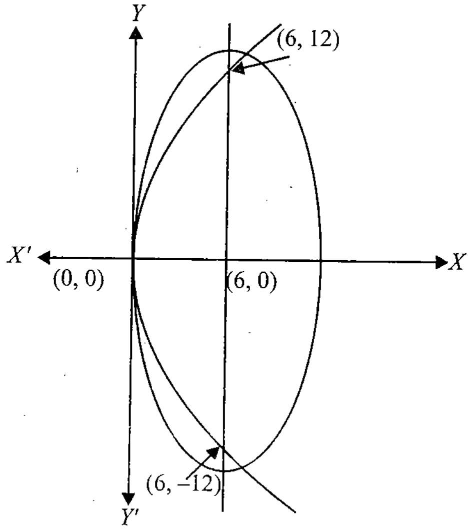


Fig. 4.89

Extremities of LR of parabola $y^2 = 24x$ are $(6, \pm 12)$.

For ellipse, $2be = 24$ and extremity of minor axis is (0, 0). Hence, $a = 6$.

Now, $a^2 = b^2 - b^2e^2$

$\Rightarrow$ $b^2 = 36 + 144 = 180$

$\Rightarrow$ $e^2 = \sqrt{1 - \frac{36}{180}} = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$

Integer type

1. (8)

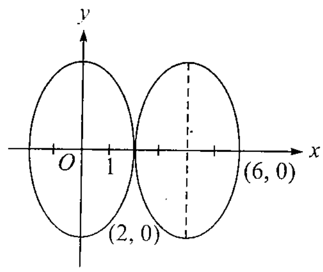


Fig. 4.90

Let $x - 4 = 2\cos\theta \Rightarrow x = 2\cos\theta + 4$

and $y = 3\sin\theta$

Now $E = \frac{x^2}{4} + \frac{y^2}{9}$

$$= \frac{(2\cos\theta + 4)^2}{4} + \sin^2\theta$$

$$= \frac{4\cos^2\theta + 16 + 16\cos\theta + 4\sin^2\theta}{4}$$

$= \dfrac{20+16\cos\theta}{4}$

$= 5 + 4\cos\theta$

Hence $E_{max} - E_{min} = (9 - 1) = 8$

2. (2) $\left(\pm ae, \dfrac{b^2}{a}\right)$ are extremities of the latus-rectum having positive ordinates.

$\Rightarrow a^2e^2 = -2\left(\dfrac{b^2}{a} - 2\right)$ (1)

But $b^2 = a^2(1 - e^2)$ (2)

$\therefore$ From (1) and (2), we get $a^2e^2 - 2ae^2 + 2a - 4 = 0$

$\Rightarrow ae^2(a - 2) + 2(a - 2) = 0$

$\therefore (ae^2 + 2)(a - 2) = 0$

Hence $a = 2$.

3. (7) By using condition of tangency, we get $4h^2 = 3k^2 + 2$

$\therefore$ Locus of $P(h, k)$ is $4x^2 - 3y^2 = 2$ (which is hyperbola)

Hence $e^2 = 1 + \dfrac{4}{3} \Rightarrow e = \sqrt{\dfrac{7}{3}}$

4. (9) Center of the given circle is $O(4, -3)$.

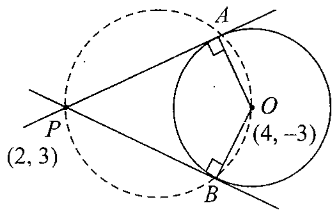


Fig. 4.91

The circumcircle of ΔPAB will circumscribe the quadrilateral $PBOA$ also, hence one of the diameters must be OP.

$\therefore$ Equation of circumcircle of ΔPAB will be

$(x - 2)(x - 4) + (y - 3)(y + 3) = 0$

$\Rightarrow x^2 + y^2 - 6x - 1 = 0$ (1)

Director circle of given ellipse will be

$(x + 5)^2 + (y - 3)^2 = 9 + b^2$

$\Rightarrow x^2 + y^2 + 10x - 6y + 25 - b^2 = 0$ (2)

$\therefore$ From (1) and (2), by applying condition of orthogonality, we get

$2[-3(5) + 0(-3)] = -1 + 25 - b^2 \Rightarrow -30 = 24 - b^2$

Hence $b^2 = 54$

5. (9) Equation of normal at $P(\theta)$ is $5\sec\theta x - 4\operatorname{cosec}\theta y = 25 - 16$ and it passes through $P(0, \alpha)$

$\alpha = \dfrac{-9}{4\operatorname{cosec}\theta}$ i.e. $\alpha = \dfrac{-9}{4}\sin\theta \Rightarrow |\alpha| < \dfrac{9}{4}$

6. (5) Points are $A(3, 4)$, $B(6, 8)$ and $O(0, 0)$.

$OA + OB = 2a$ (where a is semi-major axis.)

$2a = 5 + 10 = 15$

$\therefore a = \dfrac{15}{2}$

Now $2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$

$e = \dfrac{1}{3}$

$\therefore b^2 = \dfrac{225}{4}\left(1 - \dfrac{1}{9}\right) = 50$

7. (8) Equation of the chord whose mid point is (0, 3) is

$\dfrac{3y}{25} - 1 = \dfrac{9}{25} - 1$, i.e. $y = 3$

intersects the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

at $\dfrac{x^2}{16} = 1 - \dfrac{9}{25} = \dfrac{16}{25} \Rightarrow x = \pm\dfrac{16}{5}$

$\therefore$ length of the chord $= \dfrac{32}{5}$

thus $\dfrac{4k}{5} = \dfrac{32}{5} \therefore k = 8$

8. (8) $\dfrac{a}{e} - ae = 8 \Rightarrow 2a - \dfrac{a}{2} = 8$, i.e. $a = \dfrac{16}{3}$

$b^2 = a^2(1 - e^2) = \dfrac{256}{9}\left(1 - \dfrac{1}{4}\right) = \dfrac{64}{3}$

$\therefore$ length of minor axis $= 2b = \dfrac{16}{\sqrt{3}}$

$\therefore k = 8$

9. (4) $\because OS_1 = ae = 6$, $OC = b$ (let)

also $CS_1 = a$

$\because$ Area of $\Delta OCS_1 = \dfrac{1}{2}(OS_1) \times (OC) = 3b$

$\because$ semi-perimeter of $\Delta OCS_1 = ½ (OS_1 + OC + CS_1)$

$= ½ (6 + a + b)$ (1)

$\because$ Inradius of $\Delta OCS_1 = 1$

$\Rightarrow \dfrac{3b}{\frac{1}{2}(6 + a + b)} = 1 \Rightarrow 5b = 6 + a$ (2)

also $b^2 = a^2 - a^2e^2 = a^2 - 36$ (3)

$\Rightarrow$ from (2)

$25b^2 = 36 + 12a + a^2$

$\therefore 25(a^2 - 36) = 36 + a^2 + 12a$

$2a^2 - a - 78 = 0$

$\therefore a = \dfrac{13}{2}, -6$

$a = \dfrac{13}{2} \therefore b = \dfrac{5}{2}$

10. (4) Equation of tangent is $y = 2x \pm \sqrt{4a^2 + b^2}$

$\Rightarrow$ this is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

$\Rightarrow$ this tangent passes through (–2, 0)

$\Rightarrow 0 = -4 \pm \sqrt{4a^2 \cdot b^2} \quad \Rightarrow \quad 4a^2 + b^2 = 16$

$\Rightarrow$ Using A.M. ≥ G.M, we get

$$\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2 + b^2} \quad \Rightarrow \quad ab \leq 4$$

11. (8) $x^2 + 9y^2 - 4x + 6y + 4 = 0$

$\Rightarrow x^2 - 4x + 9y^2 + 6y + 4 = 0$

$\Rightarrow \quad (x - 2)^2 + (3y + 1)^2 = 1$

$$\Rightarrow \quad (x-2)^2 + \frac{\left(y + \frac{1}{3}\right)^1}{\frac{1}{9}} = 1$$

which is an equation of ellipse having centre at $\left(2, -\frac{1}{3}\right)$

General point on ellipse is.

$P(x, y) = (2 + a \cos \theta, -1/3 + b \sin \theta)$

$= (2 + \cos \theta, -1/3 + 1/3 \sin \theta)$

$x = 2 + \cos \theta$ and $y = -1/3 + 1/3 \sin \theta$

$\therefore 4x - 9y = 4(2 + \cos \theta) - 9\left(-\frac{1}{3} + \frac{1}{3}\sin\theta\right)$

$\Rightarrow f(\theta) = 8 + 4 \cos \theta + 3 - 3 \sin \theta$

$= 11 + 4 \cos \theta - 3 \sin \theta$

$\therefore f(\theta)_{max.} = 11 + 5 = 16$

12. (4) Let sides of rectangle be p and q

Area of rectangle $= pq = 200$ (1)

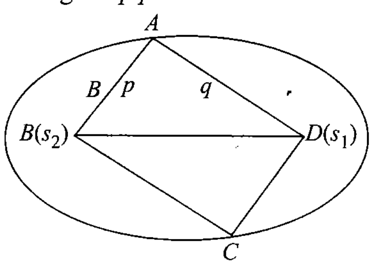


Fig. 4.92

Area of ellipse $= \pi ab = 200\pi$

$\therefore ab = 200$ (2)

we have to find the perimeter of rectangle $= 2(p + q)$

From triangle ABD

Distance $BD = \sqrt{p^2 + q^2}$ = distance between foci

or $p^2 + q^2 = 4a^2e^2$

or $(p + q)^2 - 2pq = 4(a^2 - b^2)$ (3)

Also from the definition of ellipse sum of focal lengths is $2a$,

Then $AB + AD = p + q = 2a$ (4)

putting value of $(p + q)$ in equation (3) from (4)

we have $(2a)^2 - 2pq = 4a^2 - 4b^2$ (using equation (1))

$\Rightarrow 4a^2 - 2 \times 200 = 4(a^2 - b^2)$

$\Rightarrow a^2 - 100 = a^2 - b^2$

$\Rightarrow b = 10$

from equation (2), $ab = 200 \Rightarrow a = 20$

since $p + q = 2a$ (from equation (4))

therefore perimeter $= 2(p + q) = 4a = 4 \times 20 = 80$

Archives

Subjective Type

1. Equation to the tangent at the point $P(a \cos \theta, b \sin \theta)$ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \text{(i)}$$

Perpendicular distance of (i) from the centre (0, 0) of the ellipse is given by

$$d = \frac{1}{\sqrt{\frac{1}{a^2}\cos^2\theta + \frac{1}{b^2}\cos^2\theta}}$$

$$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \quad 4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2\left\{1 - \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2}\right\}$$

$$= 4(a^2 - b^2)\cos^2\theta$$

$$= 4a^2e^2\cos^2\theta \quad \text{(ii)}$$

The coordinates of foci F_1 and F_2 are $F_1 \equiv (ae, 0)$ and $F_2 \equiv (-ae, 0)$.

$\therefore \quad PF_1 = e(1 - e \cos \theta)$

and $\quad PF_2 = a(1 + e \cos \theta)$

$$\therefore (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta \quad \text{(iii)}$$

Hence, from (ii) and (iii), we have

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

2.

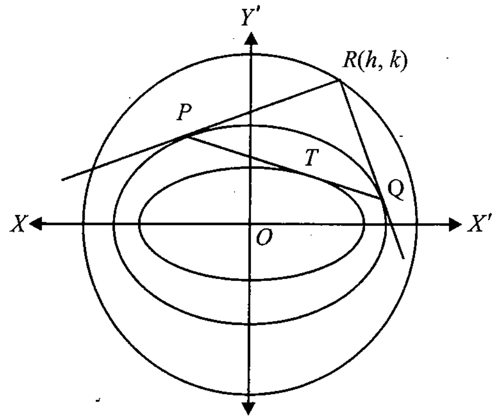


Fig. 4.93

The given ellipses are

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \quad \text{(i)}$$

and $$\frac{x^2}{6}+\frac{y^2}{3}=1 \quad \text{(ii)}$$

Then the equation of tangent to (i) at any point $T(2\cos\theta, \sin\theta)$ is given by

$$\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1 \quad \text{(iii)}$$

Let this tangent meet the ellipse (ii) at P and Q. Let the tangents drawn to ellipse (ii) at P and Q meet each other at $R(h, k)$.

Then PQ is chord of contact of ellipse (ii) with respect to the point $R(h, k)$ and is given by

$$\frac{xh}{6}+\frac{yk}{3}=1 \quad \text{(iv)}$$

Clearly, equations (iii) and (iv) represent the same line and hence should be identical.

Therefore, comparing the ratio of coefficients, we get

$$\frac{\cos\theta/2}{\frac{h}{6}}=\frac{\sin\theta}{k/3}=\frac{1}{1}$$

$\Rightarrow$ $$h=3\cos\theta, k=3\sin\theta$$

$\Rightarrow$ $$h^2+k^2=9$$

Therefore, the locus of (h, k) is

$$x^2+y^2=9$$

which is the director circle of the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ and we know that the director circle is the locus of point of intersection of the tangents which are at right angle.

Thus tangents at P and Q are perpendicular.

3. Let the midpoint of AB is (h, k), then coordinates of A and B are $(2h, 0)$ and $(0, 2k)$.

Then equation of line AB is $\frac{x}{2h}+\frac{y}{2k}=1$ or $y=-\frac{k}{h}+2k$

It touches the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$, if $4k^2=25\left(-\frac{k}{h}\right)^2+4$

or $\frac{25}{h^2}+\frac{4}{k^2}=4$

Therefore, locus of (h, k) is $\frac{25}{x^2}+\frac{4}{y^2}=4$.

(For the given tangent to the ellipse, radius of the circle is automatically fixed.)

4.

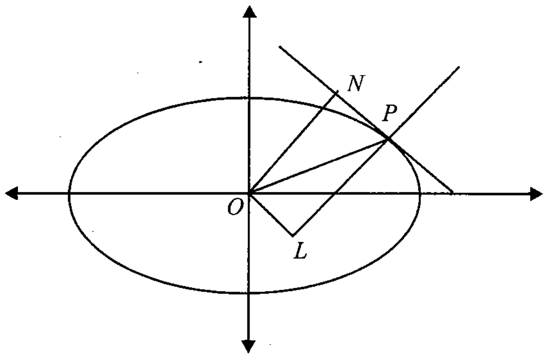


Fig. 4.94

The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since this ellipse is symmetrical in all four quadrants, either there exists no such point P or there are four points, one in each quadrant. Without loss of generality, we can assume that $a>b$ and P lies in the first quadrant.

Let P be $(a\cos\theta, b\sin\theta)$. Then equation of tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$\therefore$ $$ON=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Equation of ON is

$$\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0$$

Equation of normal at P is

$ax\sec\theta-by\,\text{cosec}\,\theta=a^2-b^2$

$\therefore$ $$OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\text{cosec}^2\theta}}$$

$$=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Now, $NP = OL$

$\therefore$ $$NP = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$\therefore$ Δ = Area of triangle OPN

$$= \frac{1}{2} \times ON \times NP$$

$$= \frac{1}{2}ab(a^2 - b^2)\frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{1}{2}ab(a^2 - b^2)\frac{1}{a^2\tan\theta + b^2\cot\theta}$$

$$= \frac{1}{2}ab(a^2 - b^2)\frac{1}{(a\sqrt{\tan\theta} - b\sqrt{\cot\theta})^2 + 2ab}$$

Now maximum Δ occurs when

$a\sqrt{\tan\theta} - b\sqrt{\cot\theta} = 0$ or $\tan\theta = \frac{b}{a}$

Therefore, P has coordinates $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)$.

By symmetry, we have four such points, i.e.

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right).$$

5. Let A, B, C be the points on circle whose coordinates are $A\,(a\cos\theta, a\sin\theta)$

$$B\left(a\cos\left(\theta + \frac{2\pi}{3}\right), a\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$C\left(a\cos\left(\theta + \frac{4\pi}{3}\right), a\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Further, $P\,(a\cos\theta, b\sin\theta)$ (given)

Hence,

$$Q\left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$R\left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

It is given that P, Q, R are on the same side of x-axis as A, B, C.

So required normals to the ellipse are

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 \quad \text{(i)}$$

$$ax\sec\left(\theta + \frac{2\pi}{3}\right) - by\,\text{cosec}\left(\theta + \frac{2\pi}{3}\right) = a^2 - b^2 \quad \text{(ii)}$$

$$ax\sec\left(\theta + \frac{4\pi}{3}\right) - by\,\text{cosec}\left(\theta + \frac{4\pi}{3}\right) = a^2 - b^2 \quad \text{(iii)}$$

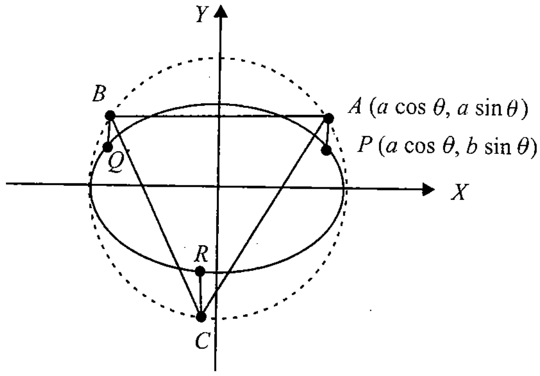


Fig. 4.95

Now, above three normals are concurrent. Hence, $\Delta = 0$.

$$\Rightarrow \Delta = \begin{vmatrix} \sec\theta & \text{cosec}\,\theta & 1 \\ \sec\left(\theta + \frac{2\pi}{3}\right) & \text{cosec}\left(\theta + \frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta + \frac{4\pi}{3}\right) & \text{cosec}\left(\theta + \frac{4\pi}{3}\right) & 1 \end{vmatrix}$$

Multiplying and dividing the different rows by $\sin\theta\cos\theta$, we get

$\sin\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{2\pi}{3}\right)$ and $\sin\left(\theta + \frac{4\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)$

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\pi - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

where $k = \sin\theta\cos\theta\sin\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{2\pi}{3}\right)\sin\left(\theta + \frac{4\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)$

[Operating $R_2 \to R_2 + R_3$ and simplifying R_2 we get $R_2 \equiv R_1$.] Hence, $\Delta = 0$.

6. Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Then coordinates of Q are $(a\cos\theta, a\sin\theta)$.

Let $R\,(h, k)$ divides PQ in the ratio $r{:}s$. Then

$$h = \frac{s\,(a\cos\theta) + r\,(a\cos\theta)}{(r+s)}$$

$$= a\cos\theta$$

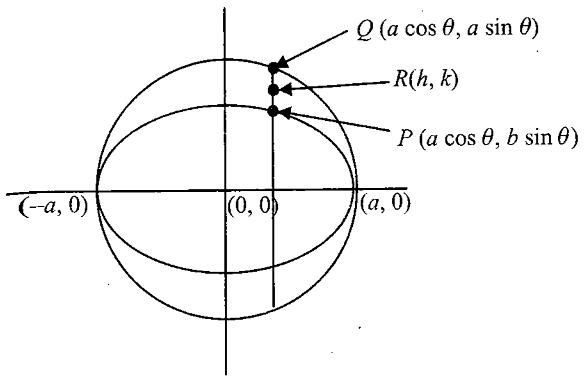


Fig. 4.96

$$\Rightarrow \quad \cos\theta = \frac{h}{a}$$

$$k = \frac{s\,(b\sin\theta) + r\,(a\sin\theta)}{(r+s)}$$

$$= \frac{\sin\theta\,(bs + ar)}{(r+s)}$$

$$\Rightarrow \quad \sin\theta = \frac{k(r+s)}{(bs+ar)}$$

We know that $\cos^2\theta + \sin^2\theta = 1$. Therefore,

$$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2\,(r+s)^2}{(bs+ar)^2} = 1$$

Hence, locus of R is

$$\frac{x^2}{a^2} + \frac{y^2\,(r+s)^2}{(bs+ar)^2} = 1$$

which is an ellipse.

7.

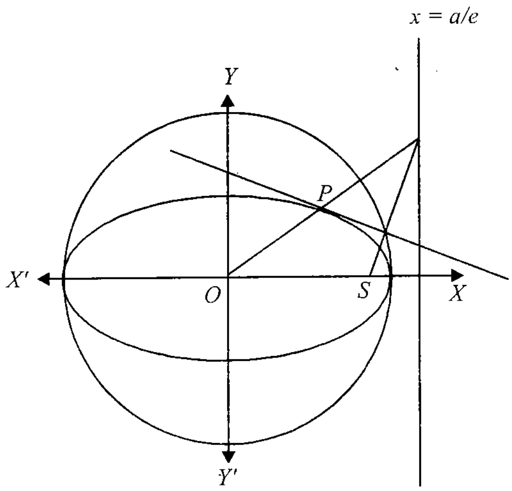


Fig. 4.97

Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and O be the centre.

Tangent at $P\,(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

whose slope is $-\dfrac{b^2x_1}{a^2y_1}$

Focus of the ellipse is $S\,(ae, 0)$.

Equation of the line through $S\,(ae, 0)$ perpendicular to tangent at P is

$$y = \frac{a^2y_1}{b^2x_1}(x - ae) \qquad \text{(i)}$$

Equation of OP is

$$y = \frac{y_1}{x_1}x \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\Rightarrow \quad \frac{y_1}{x_1}x = \frac{a^2y_1}{b^2x_1}(x - ae)$$

$$\Rightarrow \quad x(a^2 - b^2) = a^3e$$

$$\Rightarrow \quad x\,a^2e^2 = a^3e$$

$$\Rightarrow \quad x = a/e$$

This is the corresponding directrix.

8. Any tangent on ellipse is

$$y = mx \pm \sqrt{25m^2 + 4}$$

If this is also the tangent on the circle, then

$$\left|\frac{0 - m \times 0 \pm \sqrt{25m^2+4}}{\sqrt{1+m^2}}\right| = 4$$

$$\Rightarrow \quad m = \pm\frac{2}{\sqrt{3}}$$

Since common tangent is in first quadrant so

$$m = -\frac{2}{\sqrt{3}}$$

Hence, the common tangent in first quadrant is given by

$$\sqrt{3}\,y + 2x = 4\sqrt{7} \qquad \text{(i)}$$

The points of intersection of this tangent with x-axis and y-axis are $(2\sqrt{7}, 0)$ and $\left(0, \dfrac{4\sqrt{7}}{\sqrt{3}}\right)$.

Therefore, the length of intercept

$$= \sqrt{(2\sqrt{7} - 0)^2 + \left(0 - \frac{4\sqrt{7}}{\sqrt{3}}\right)^2}$$

$$= \frac{14}{\sqrt{3}}.$$

Objective Type

Fill in the blanks

1.

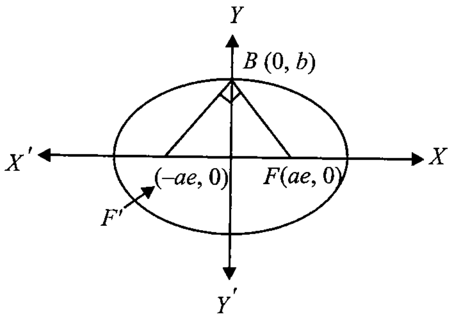


Fig. 4.98

$$m_{BF} \cdot m_{BF'} = -1$$

$$\Rightarrow \quad \frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\Rightarrow \quad \frac{b^2}{a^2e^2} = 1$$

$$\Rightarrow \quad e^2 = \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow \quad e^2 = \frac{1}{2}$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

Multiple choice questions with one correct answer

1. a. The given ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Here $\quad a^2 = 16$ and $b^2 = 9$

$\therefore \quad b^2 = a^2(1 - e^2) \Rightarrow 9 = 16(1 - e^2)$

$\Rightarrow \quad e = \frac{\sqrt{7}}{4}$

Hence, the foci are $(\pm\sqrt{7}, 0)$.

Radius of the circle = distance between $(\pm\sqrt{7}, 0)$ and $(0, 3) = \sqrt{7+9} = 4$.

2. c. For given slope there exists two parallel tangents to ellipse. Hence, there are two values of c.

3. c. The ellipse can be written as

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Here $\quad a^2 = 25, b^2 = 16$

Now, $\quad b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad \frac{16}{25} = 1 - e^2$$

$$\Rightarrow \quad e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\Rightarrow \quad e = \frac{3}{5}$$

Foci of the ellipse are $(\pm ae, 0) \equiv (\pm 3, 0)$, i.e. F_1 and F_2 are foci of the ellipse.

Therefore, we have $PF_1 + PF_2 = 2a = 10$ for every point P on the ellipse.

4. d. The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Then $\quad a^2 = 9, b^2 = 5$

$$\Rightarrow \quad e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

Hence, end point of latus rectum in first quadrant is $L(2, 5/3)$.

Equation of tangent at L is

$$\frac{2x}{9} + \frac{y}{3} = 1$$

The tangent meets x-axis at A (9/2, 0) and y-axis at B (0, 3).

Therefore, area of $\Delta OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$

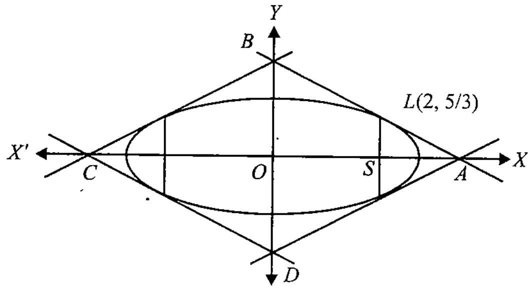


Fig. 4.99

By symmetry, area of quadrilateral

$$= 4 \times (\text{Area of } \Delta OAB)$$

$$= 4 \times \frac{27}{4} = 27 \text{ sq. units}$$

5. a. Any tangent to ellipse

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

is given by $\quad \frac{x\cos\theta}{\sqrt{2}} + y\sin\theta = 1$

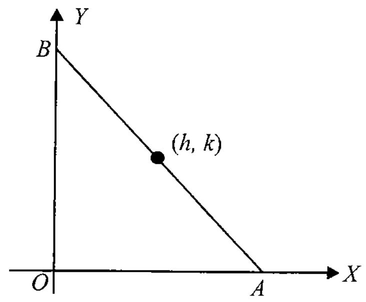


Fig. 4.100

Using midpoint formula, we have

$$A(\sqrt{2}\sec\theta, 0) \text{ and } B(0, \operatorname{cosec}\theta).$$

Hence, $\quad 2h = \sqrt{2}\sec\theta$ and $2k = \operatorname{cosec}\theta$

$$\Rightarrow \left(\frac{1}{\sqrt{2}h}\right)^2 + \left(\frac{1}{2k}\right)^2 = 1$$

$$\Rightarrow \quad \frac{1}{2h^2} + \frac{1}{4k^2} = 1$$

$$\Rightarrow \quad \frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

6. a. Any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$ is given by

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

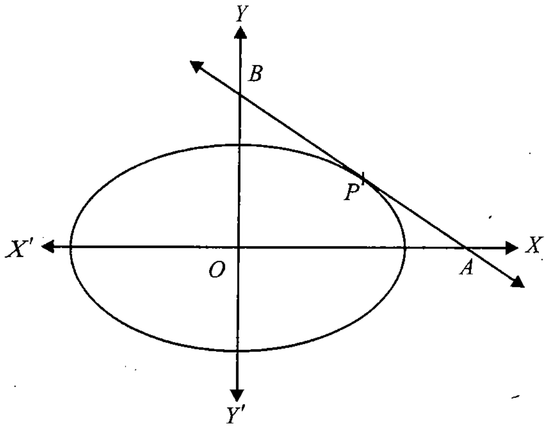


Fig. 4.101

It meets coordinate axes at $A(a\sec\theta, 0)$ and $B(0, b\operatorname{cosec}\theta)$

$\therefore \quad$ Area of $\Delta OAB = \frac{1}{2} \times a\sec\theta \times b\operatorname{cosec}\theta$

$$\Rightarrow \quad \Delta = \frac{ab}{\sin 2\theta}$$

For area to be minimum $\sin 2\theta$ should be maximum and we know maximum value of $\sin 2\theta = 1$.

$$\therefore \quad \Delta_{max} = ab$$

7. d.

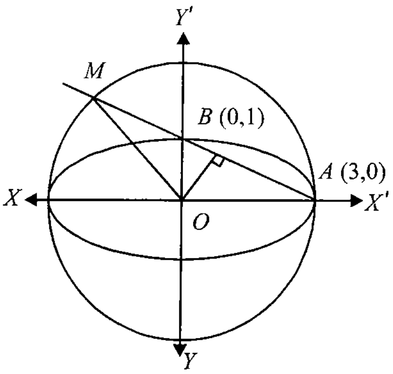


Fig. 4.102

Equation of line AM is

$$x + 3y - 3 = 0$$

Perpendicular distance of line from the origin $= \frac{3}{\sqrt{10}}$

Length of $\quad AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

$$\Rightarrow \quad \text{Area} = \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}}$$

$$= \frac{27}{10} \Rightarrow \text{q.units}$$

8. c. Normal is given by $4x\sec\phi - 2y\operatorname{cosec}\phi = 12$

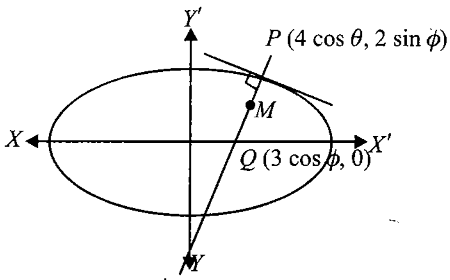


Fig. 4.103

$$Q \equiv (3\cos\phi, 0)$$

$$M \equiv (\alpha, \beta)$$

$$\Rightarrow \quad \alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$$

$$\Rightarrow \quad \cos\phi = \frac{2}{7}\alpha$$

$$\beta = \sin\phi$$

Using $\cos^2\phi + \sin^2\phi = 1$, we have

$$\frac{4}{49}\alpha^2 + \beta^2 = 1$$

$$\Rightarrow \quad \frac{4}{9}x^2 + y^2 = 1 \quad \text{(i)}$$

Now latus rectum,

$$x = \pm 2\sqrt{3} \quad \text{(ii)}$$

Solving (i) and (ii), we have $\frac{48}{49} + y^2 = 1$

$$\Rightarrow \quad y = \pm\frac{1}{7}$$

Points of intersection are $(\pm 2\sqrt{3}, \pm 1/7)$.

9. d. Since $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$ both P and Q lie inside C. Also $\frac{1^2}{9} + \frac{2^2}{4} = \frac{1}{9} + 1 > 1$ and $\frac{2^2}{9} + \frac{1}{4} = \frac{25}{36} < 1$. Hence, P lies outside E and Q lies inside E. Thus P lies inside C but outside E.

Multiple choice questions with one or more than one correct answer

1. b., d. Let (x_1, y_1) be the point at which tangents to ellipse $4x^2 + 9y^2 = 1$ are parallel to $8x = 9y$.

Then slope of the tangent $= \frac{8}{9}$.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{8}{9}$ (i)

Differentiating equation of ellipse w.r.t. x, we get

$$8x + 18y\frac{dx}{dx} = 0$$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-8x_1}{18y_1} = \frac{-4x_1}{9y_1}$

Substituting in equation (i), we get

$$\frac{-4x_1}{9y_1} = \frac{8}{9} \Rightarrow -x_1 = 2y_1 \quad \text{(ii)}$$

Also (x_1, y_1) is the point of contact which must be on curve. Hence,

$$4x_1^2 + 9y_1^2 = 1$$

$\Rightarrow \quad 4x \times 4x_1^2 + 9y_1^2 = 1$ [using (2)]

$\Rightarrow \quad y_1^2 = \frac{1}{25}$

$\Rightarrow \quad y_1 = \pm\frac{1}{5}$

$\Rightarrow \quad x_1 = \mp\frac{2}{5}$

Thus the required points are $\left(\frac{-2}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, -\frac{1}{5}\right)$.

Alternative Method:

Let $y = \frac{8}{9}x + c$ be the tangent to $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

where $c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{\frac{1}{4} \times \frac{64}{81} + \frac{1}{9}} = \pm\frac{5}{9}$

So, points of contact are $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right) = \left(\frac{2}{5}, \frac{-1}{5}\right)$ or $\left(\frac{-2}{5}, \frac{1}{5}\right)$.

2. b., c.

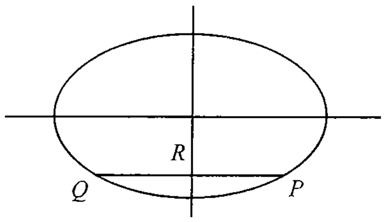


Fig. 4.104

The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$b^2 = a^2(1 - e^2)$$

$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$

Hence, the end points P and Q of the latus rectum are given by

$$P \equiv \left(\sqrt{3}, -\frac{1}{2}\right)$$

and $\quad Q \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$ (given $y_1, y_2 < 0$)

Coordinates of midpoint of PQ are

$$R \equiv \left(0, -\frac{1}{2}\right)$$

Length of latus rectum, $PQ = 2\sqrt{3}$

Hence, two parabolas are possible whose vertices are

$\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$.

The equations of the parabolas are

$$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$$

and $\quad x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. b., c.

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$

$\Rightarrow 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$

$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) = 2\sin\left(\frac{A}{2}\right)$

$\Rightarrow \quad \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} = 2$

$\Rightarrow \quad \frac{\sin B + \sin C}{\sin A} = 2$

$\Rightarrow \quad b + c = 2a$ (constant)

Hence, locus of vertex A is ellipse with B and C as foci.

Comprehension type

1. d. Equation of tangent having slope m is

$y = mx + \sqrt{9m^2 + 4}$

Tangent passes through the point (3, 4) then

$4 - 3m = \sqrt{9m^2 + 4}$

Squaring, we have

$16 + 9m^2 - 24m = 9m^2 + 4 \Rightarrow m = \frac{12}{24} = \frac{1}{2}$

$\therefore$ Equation of tangent is $y - 4 = \frac{1}{2}(x - 3)$ or $x - 2y + 5 = 0$

Let point of contact on the curve is $B(\alpha, \beta)$

$$\Rightarrow \frac{x\alpha}{9}+\frac{y\beta}{4}-1=0 \Rightarrow \frac{\alpha/9}{1}=\frac{\beta/4}{-2}=-\frac{1}{5}$$

$$\Rightarrow \alpha=-\frac{9}{5}, \beta=\frac{8}{5}$$

$$B \equiv\left(-\frac{9}{5}, \frac{8}{5}\right)$$

Another slope of tangent is ∞, then equation of tangent is $x=3$ and corresponding point of contact is $A(3, 0)$

2. c. Since slope of PA is ∞,

Slope of altitude through B must be 0, for which orthocenter is $\left(\frac{11}{5}, \frac{8}{5}\right)$

3. a. Locus is parabola

Equation of AB is $\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow \frac{x}{3}+y=1 \Rightarrow x+3y-3=0$

$\therefore$ Equation of locus is

$$(x-3)^2+(y-4)^2=\frac{(x+3y-3)^2}{10}$$

or $9x^2+y^2-6xy-54x-62y+241=0$

CHAPTER

5

Hyperbola

- Hyperbola: Definition 1
- Position of a Point (h, k) with respect to a Hyperbola
- Conjugate Hyperbola
- Auxiliary Circle and Eccentric Angle
- Comparison of Hyperbola and its Conjugate Hyperbola
- Hyperbola: Definition 2

HYPERBOLA: DEFINITION 1

The hyperbola is the set of all points in a plane, the difference of whose distance from two fixed points in the plane is a constant.

The term "*difference*" that means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The midpoint of the line segment joining the foci is called the centre of the hyperbola. The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis. The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola.

We denote the distance between the two foci by $2c$, the distance between the two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as $b = \sqrt{c^2 - a^2}$.

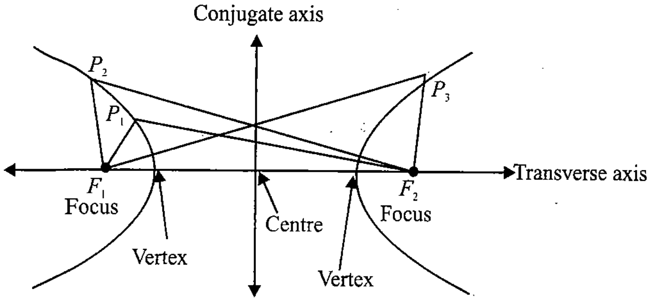


Fig. 5.1

$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Standard Equation of Hyperbola

Let the foci of a hyperbola be $(\pm c, 0)$. Then its centre is $(0, 0)$.

According to the definition of hyperbola,

$$PF_1 - PF_2 = 2a \ (2a < 2c, \text{ i.e. } c > a)$$

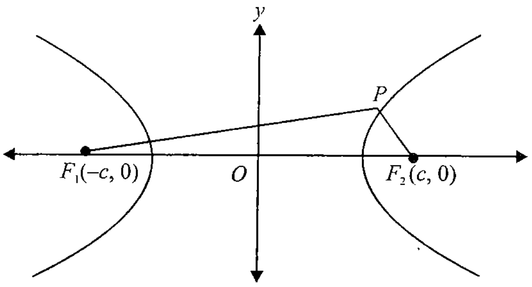


Fig. 5.2

$$\Rightarrow \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

i.e. $$\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Squaring both sides, we get

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

On simplifying, we get

$$\frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (\text{since } c^2 - a^2 = b^2)$$

Hence, any point on the hyperbola satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is same as the standard equation of ellipse except b^2 has been replaced by $-b^2$.

This is a second degree equation in which the powers of x and y are even, hence the curve is symmetric about both the axes. For each $x > a$ or $x < -a$ there are two values of y symmetrically situated on either side of x-axis and for each value of x lying in $(-a, a)$ the curve fails to exist. Hence, the curve denoted by Es. (i) consists of two symmetrical branches, each extending to infinity in two directions.

Eccentricity

$$e = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

$$= \frac{c}{a}$$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad a^2e^2 = a^2 + b^2$$

Therefore, equation of hyperbola in terms of eccentricity can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Coordinates of the foci are $(\pm ae, 0)$.

Two hyperbolas are said to be similar if they have the same value of eccentricity.

Rectangular or Equilateral Hyperbola

If $a = b$, hyperbola is said to be equilateral or rectangular and has the equation $x^2 - y^2 = a^2$.

Eccentricity for such a hyperbola is $\sqrt{2}$ as

$$e^2 = 1 + \frac{b^2}{a^2} = 1 + 1$$

$$= 2$$

Directrix

It is possible to define two lines, $x = \pm \frac{a}{e}$, corresponding to each focus, which satisfy the focal directrix property of the hyperbola, i.e. $PF_1 = e\,PM_1$ and $PF_2 = e\,PM_2$.

Hence, for any point P on the hyperbola,

$$\frac{PF_1}{PM_1} = e \text{ (constant)}$$

Focal Distance (Focal Radius)

The difference of the focal radii of any point on the hyperbola is equal to the length of the major axis. We have

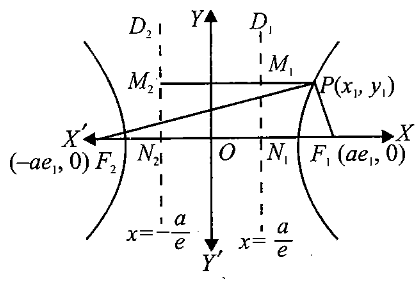


Fig. 5.3

$$PF_1 = ePM_1 = e\left(x_1 - \frac{a}{e}\right)$$
$$= ex_1 - a \quad \text{(i)}$$
$$PF_2 = e\left(x_1 + \frac{a}{e}\right)$$
$$= ex_1 + a \quad \text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$PF_2 - PF_1 = 2a$$

Equation of Hyperbola Whose Axes are Parallel to Coordinate Axes and Centre is (*h*, *k*)

Equation of such hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1 \quad (a > b)$$

Foci: $(h \pm ae, k)$

Directrix: $x = h \pm \frac{a}{e}$

Definition and Basic Terminology

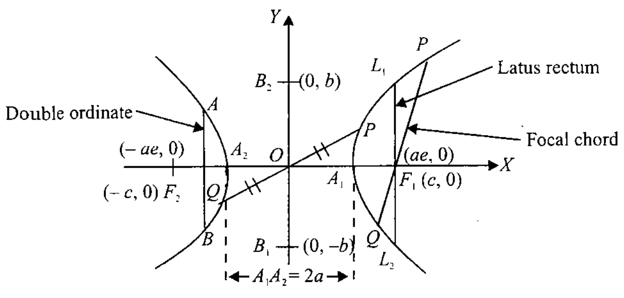


Fig. 5.4

- Line containing the fixed points F_1 and F_2 (called foci) is called transverse axis (TA) or focal axis and the distance between F_1 and F_2 is called focal length.
- The points of intersection A_1 and A_2 of the curve with the transverse axis are called vertices of the hyperbola. The length '$2a$' between the vertices is called the length of transverse axis (TA).
- The points B_1 (0, b) and B_2(0, $-b$) which have special significance, are known as the extremities of conjugate axis and the length '$2b$' is called the length of conjugate axis. The point of intersection of these two axes is called the centre 'O' of the hyperbola. (Transverse axis and conjugate axis together are called the principal axes).
- Any chord passing through centre is called diameter (PQ) and is bisected by it.
- Any chord passing through focus is called a focal chord and any chord perpendicular to the transverse axis is called a double ordinate (AB).
- A particular double ordinates which passes through focus and perpendicular to focal axis is called the latus rectum (L_1L_2)

Latus-Rectum Length

The two foci are ($\pm ae$, 0)

We have the equation of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Putting $x = ae$, we get

$$\Rightarrow \quad \frac{y^2}{b^2} = e^2 - 1 = \left(\frac{b^2}{a^2} + 1\right) - 1$$
$$= \frac{b^2}{a^2}$$
$$\Rightarrow \quad y = \pm \frac{b^2}{a}$$

Hence, coordinate of the extremities of LR

$$= \left(\pm ae, \pm b^2/a\right)$$

Length of LR $= \frac{2b^2}{a}$

$$= \frac{4b^2}{2a} = \frac{(\text{minor axis})^2}{\text{major axis}}$$

Also $L_1L_2 = 2a(e^2 - 1)$

$= 2e\left(ae - \frac{a}{e}\right)$

$= 2e(OF_1 - OA_1)$ (as shown in Fig. 5.4)

$= 2e \times$ (distance between focus and corresponding foot of the directrix)

POSITION OF A POINT (*h*, *k*) WITH RESPECT TO A HYPERBOLA

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative as the point (x_1, y_1) lies within, upon or outside the curve.

Explanation

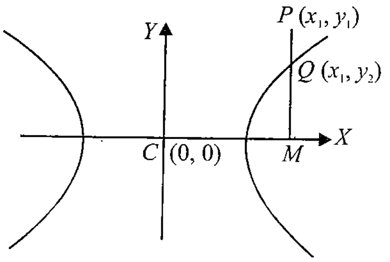


Fig. 5.5

If (x_1, y_1) lies on the hyperbola, then

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\Rightarrow \quad \left(\frac{x_1^2}{a^2} - 1\right) b^2 = y_1^2$$

Now if P lies outside the curve, then

$$y_1^2 > y_2^2$$

$$\Rightarrow \quad y_1^2 > \left(\frac{x_1^2}{a^2} - 1\right) b^2$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$$

CONJUGATE HYPERBOLA

Corresponding to every hyperbola there exists a hyperbola such that the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of the other. Such hyperbolas are known as conjugate to each other.

Hence, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(i)}$$

The conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{(ii)}$$

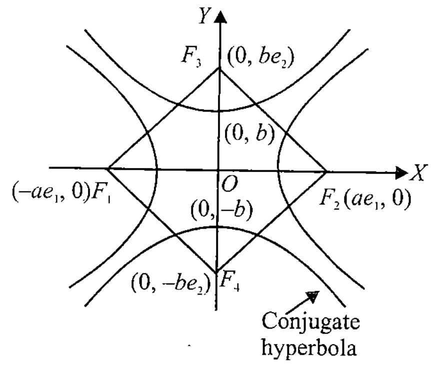


Fig. 5.6

Notes

- ***If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, respectively, then $e_1^{-2} + e_2^{-2} = 1$.***

  Proof:

  For hyperbola, $e_1^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$

  And for conjugate hyperbola,

  $$e_2^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2}$$

  $$\therefore \quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

- ***The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.***

  Proof:

  All the four sides of the quadrilateral $F_1F_3F_2F_4$ are obviously equal to their diagonals at right angles. Hence it is a rhombus.

  Now to prove that $F_1F_3F_2F_4$ is a square it is sufficient to prove that

  $$ae_1 = be_2$$

  or $\quad a^2e_1^2 = b^2e_2^2 = a^2(e_1^2 - 1)e_2^2$

  or $\quad e_1^2 = e_1^2e_2^2 - e_2^2$

  or $\quad e_1^2 + e_2^2 = e_1^2e_2^2$

  or $\quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ ***which is true***

  Hence $\quad ae_1 = be_2$

  $\Rightarrow$ ***$F_1F_3F_2F_4$ is a square.***

AUXILIARY CIRCLE AND ECCENTRIC ANGLE

Definition

A circle drawn with centre C and transverse axis as a diameter is called the *auxiliary circle* of the hyperbola. Hence, equation of the auxiliary circle is $x^2 + y^2 = a^2$.

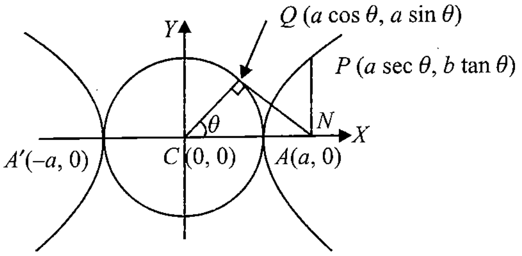


Fig. 5.7

Note: ***From Fig. 5.7, P and Q are called the "corresponding points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola ($0 \le \theta < 2\pi$).***

The equations $x = a \sec\theta$ and $y = b\tan\theta$ together represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where θ is a parameter.

θ	$Q(a\cos\theta, a\sin\theta)$	$P(a\sec\theta, b\tan\theta)$
$\theta \in \left[0, \frac{\pi}{2}\right)$	I	I
$\theta \in \left(\frac{\pi}{2}, \pi\right]$	II	III
$\theta \in \left[\pi, \frac{3\pi}{2}\right)$	III	II
$\theta \in \left(\frac{3\pi}{2}, 2\pi\right]$	IV	IV

The parametric form of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be represented as $x = a\sec\theta$, $y = b\tan\theta$.

For hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, parametric form is

$$x = h + a\sec\theta$$

$$y = k + b\tan\theta$$

COMPARISON OF HYPERBOLA AND ITS CONJUGATE HYPERBOLA

Fundamentals	Hyperbola	Conjugate Hyperbola
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinates	$(a\sec\theta, b\tan\theta)$,	$(b\sec\theta, a\tan\theta)$,
Focal radii of point $P(x_1, y_1)$	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii ($S'P - SP$)	$2a$	$2b$
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

Example 5.1 **Find the equation of hyperbola**

a. **whose axes are coordinate axes and the distances of one of its vertices from the foci are 3 and 1;**

b. **whose centre is (1, 0), focus is (6, 0) and transverse axis is 6;**

c. **whose centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2);**

d. **whose centre is (–3, 2), one vertex is (–3, 4) and eccentricity is $\frac{5}{2}$;**

e. **whose foci are (4, 2) and (8, 2) and eccentricity is 2.**

Sol.

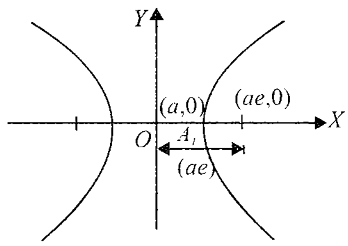


Fig. 5.8

$$ae - a = 1 \text{ and } ae + a = 3$$

$\therefore$ $\frac{e+1}{e-1} = 3$

$\Rightarrow$ $e = 2$ and $a = 1$

Also from $b^2 = a^2(e^2 - 1) = 3$, the equation is

$$x^2 - \frac{y^2}{3} = 1$$

or $$\frac{x^2}{3} - \frac{y^2}{1} = -1$$

b. Equation of hyperbola with centre (1, 0) is

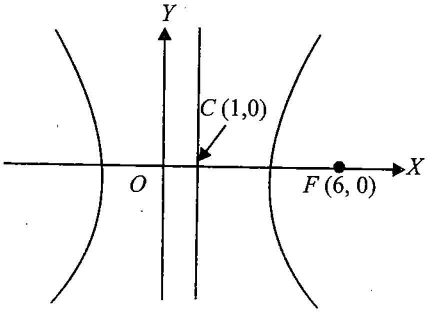


Fig. 5.9

$$\frac{(x-1)^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{(i)}$$

Given $a = 3$ and $ae = 5$; hence $e = \frac{5}{3}$.

Therefore, Eq. (i) becomes

$$\frac{(x-1)^2}{9} - \frac{y^2}{9\left(\frac{25}{9}-1\right)} = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$$

c. Equation of hyperbola with centre (3, 2) is

$$\frac{(x-3)^2}{a^2} - \frac{(y-2)^2}{a^2(e^2-1)} = 1, \text{ with axis parallel to } x\text{-axis}$$

a = distance between centre and vertex = 1

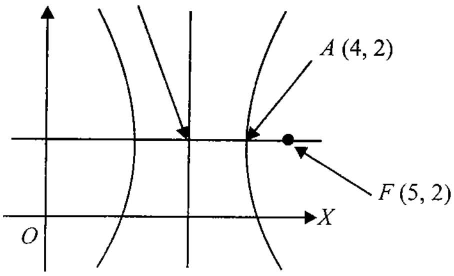


Fig. 5.10

ae = distance between centre and focus.

Hence, $\quad ae = 2 \Rightarrow e = 2$

Hence, the equation is

$$\frac{(x-3)^2}{1} - \frac{(y-2)^2}{(4-1)} = 1$$

$$\Rightarrow \quad (x-3)^2 - \frac{(y-2)^2}{3} = 1$$

d. Equation of the hyperbola is $\frac{(x+3)^2}{a^2} - \frac{(y-2)^2}{b^2} = -1$ (as the line joining centre to the vertex is parallel to y-axis)

Now $b = 2$ (distance between centre and vertex)

Focus $= be = 2 \times \frac{5}{2} = 5$

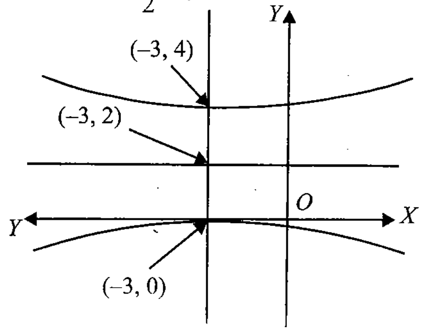


Fig. 5.11

Also, $\quad a^2 = b^2(e^2 - 1)$

$$= 4\left(\frac{25}{4} - 1\right)$$
$$= 21$$

Therefore, the required equation is

$$\frac{(x+3)^2}{21} - \frac{(y-2)^2}{4} = -1$$

e. Line joining the foci is parallel to x-axis.

Distance between the two foci = 4 = $2ae$

Hence, $\quad a = 1$ as $e = 2$

$\therefore \quad b^2 = a^2(e^2 - 1)$
$= 3$

Now centre is midpoint of the line joining the foci which is (6, 2).

Hence, equation of the hyperbola is

$$\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$$

Example 5.2 **If base of triangle and ratio of tangents of half of base angles are given, then identify the locus of opposite vertex.**

Sol. In ΔABC, base $BC = a$ is given

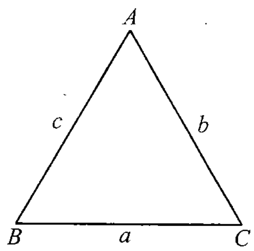


Fig. 5.12

Also, $$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = k \text{ (constant)}$$

$$\Rightarrow \quad \frac{\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}}{\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}} = k$$

$$\Rightarrow \quad \frac{s-b}{s-c} = k$$

$$\Rightarrow \quad \frac{(s-b)-(s-c)}{(s-b)+(s-c)} = \frac{k-1}{k+1}$$

$$\Rightarrow \quad \frac{c-b}{2s-(b+c)} = \frac{k-1}{k+1}$$

$$\Rightarrow \quad c - b = a\frac{k-1}{k+1} = \text{constant}$$

$$\Rightarrow \quad AB - AC = \text{constant}$$

So, locus of A is a hyperbola.

Example 5.3 **Two circles are given such that they neither intersect nor touch. Then identify the locus of centre of variable circle which touches both the circles externally.**

Sol.

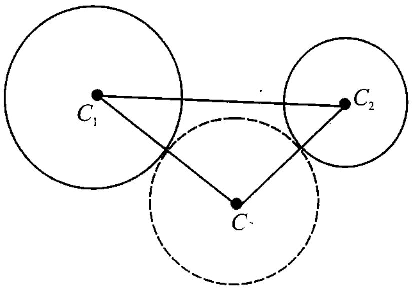


Fig. 5.13

In the figure circles with solid line have centres C_1 and C_2 and radii r_1 and r_2.

Let the circle with dotted line is variable circle which touches given two circles as explained in the question which has centre C and radius r.

Now, $\quad CC_2 = r + r_2$

and $\quad CC_1 = r_1 + r$

Hence, $\quad CC_1 - CC_2 = r_1 - r_2$ (= constant)

Then locus of C is hyperbola whose focii are C_1 and C_2.

Example 5.4 **Two rods are rotating about two fixed points in opposite directions. If they start from their position of co-incidence and one rotates at the rate double that of the other, then find the locus of point of intersection of the two rods.**

Sol. Suppose two rods are co-incident on the x-axis. One rotates about point O and the other about point $A(a, 0)$.

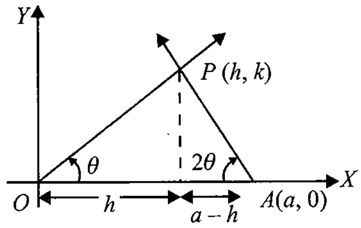


Fig. 5.14

If they rotate according to question, then at some time t, they are in the position as shown in the figure.

From the figure $\tan\theta = \frac{k}{h}$ and $\tan 2\theta = \frac{k}{a-h}$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{k}{a-h}$$

$$\Rightarrow \quad \frac{\frac{2k}{h}}{1-\frac{k^2}{h^2}} = \frac{k}{a-h}$$

$$\Rightarrow \quad \frac{2hk}{h^2-k^2} = \frac{k}{a-h}$$

$$\Rightarrow \quad 2h(a-h) = h^2 - k^2$$

$$\Rightarrow \quad 2ah - 2h^2 = h^2 - k^2$$

$$\Rightarrow \quad 3x^2 - y^2 - 2ax = 0$$

$\Rightarrow$ Locus is hyperbola.

Example 5.5 **Find the vertices of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$.**

Sol. The equation can be rewritten as

$$9(x^2 - 4x + 4 - 4) - 16(y^2 - 6y + 9 - 9) = 252$$

$$9(x-2)^2 - 16(y-3)^2 = 252 + 36 - 144 = 144$$

$$\Rightarrow \quad \frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

or $$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$$

Hence, vertices are $\quad X = \pm A,\ Y = 0$

$$\Rightarrow \quad (x-2) = \pm 4,\ y - 3 = 0$$

$$\Rightarrow \quad x = 6, -2 \text{ and } y = 3$$

$\Rightarrow$ Vertices are (6, 3), (–2, 3)

Example 5.6 **Find the coordinates of foci, the eccentricity and latus – rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.**

Sol. Equation can be rewritten as

$$\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$$

So $\quad a = 4, b = 3$

$b^2 = a^2(e^2 - 1)$ gives $e = \frac{5}{4}$.

Foci: $x - 4 = \pm ae$,

$y - 3 = 0$

give the foci as (9, 3), (–1, 3)

Centre: $x - 4 = 0, y - 3 = 0$,

i.e. centre is (4, 3)

Directrices: $x - 4 = \pm\frac{a}{e}$,

i.e. $x - 4 = \pm\frac{16}{5}$

Hence, directrices are $5x - 36 = 0$; $5x - 4 = 0$.

Latus rectum: $\frac{2b^2}{a} = 2 \times \frac{9}{4} = \frac{9}{2}$

Example 5.7 **If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then find the value of b^2.**

Sol. For hyperbola,

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \frac{81}{144} = \frac{225}{144}$$

$$\therefore \quad e = \frac{15}{12} = \frac{5}{4}$$

Also, $a^2 = \frac{144}{25}$

Hence, the foci are $(\pm ae, 0) \equiv \left(\pm\frac{12}{5} \times \frac{5}{4}, 0\right) \equiv (\pm 3, 0)$

Now for ellipse $ae = 3$ or $a^2e^2 = 9$

Now $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = a^2 - a^2e^2$

$= 16 - 9 = 7$

Example 5.8 **If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then find range of the eccentricity e of the hyperbola.**

Sol.

Let double ordinate PQ be such that $P \equiv (a \sec\theta, b \tan\theta)$, and $Q \equiv (a \sec\theta, -b\tan\theta)$ and O is centre (0, 0).

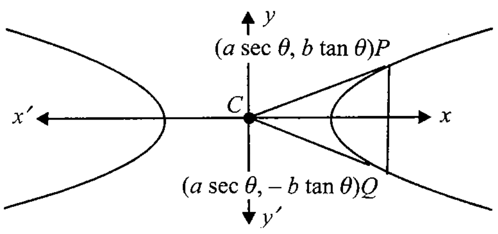


Fig. 5.15

ΔOPQ is equilateral

$\Rightarrow$ $\tan 30° = \frac{b\tan\theta}{a\sec\theta}$

$\Rightarrow$ $3\frac{b^2}{a^2} = \text{cosec}^2\theta$

$\Rightarrow$ $3(e^2 - 1) = \text{cosec}^2\theta$

Now, $\text{cosec}^2\theta \ge 1$

$\Rightarrow$ $3(e^2 - 1) \ge 1$

$\Rightarrow$ $e^2 \ge \frac{4}{3}$

$\Rightarrow$ $e > \frac{2}{\sqrt{3}}$

Example 5.9 **If the latus rectum subtends a right angle at the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find its eccentricity.**

Sol.

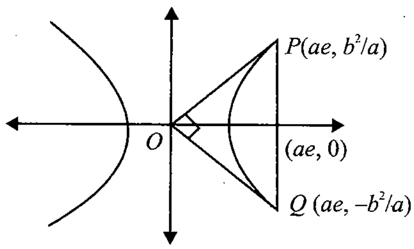


Fig. 5.16

$$m_{OP} \times m_{OQ} = -1$$

$\Rightarrow$ $\left(\frac{\frac{b^2}{a}}{ae}\right) \times -\left(\frac{\frac{b^2}{a}}{ae}\right) = -1$

$\Rightarrow$ $\frac{b^4}{a^4e^2} = 1$

$\Rightarrow$ $(e^2 - 1)^2 = e^2$

$\Rightarrow$ $e^4 - 3e^2 + 1 = 0$

$\Rightarrow$ $e^2 = \frac{3 \pm \sqrt{5}}{2}$

$\Rightarrow$ $e^2 = \frac{3 + \sqrt{5}}{2}$

$\Rightarrow$ $e = \frac{\sqrt{5} + 1}{2}$

Concept Application Exercise 5.1

1. If the latus rectum of a hyperbola forms an equilateral triangle with the vertex at the centre of the hyperbola, then find the eccentricity of the hyperbola.
2. The distance between the two directrix of a rectangular hyperbola is 10 units, then find the distance between its foci.
3. An ellipse and hyperbola are confocal (have the same focus) and the conjugate axis of the hyperbola is

equal to the minor axis of the ellipse. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, then prove that $\frac{1}{e_1^2}+\frac{1}{e_2^2}=2$.

4. Two straight lines pass through the fixed points $(\pm a, 0)$ and have slopes whose product is $p > 0$. Show that the locus of the points of intersection of the lines is a hyperbola.
5. Show that the locus represented by $x = \frac{1}{2}a\left(t+\frac{1}{t}\right)$, $y=\frac{1}{2}a\left(t-\frac{1}{t}\right)$ is a rectangular hyperbola.
6. Find the lengths of transverse axis and conjugate axis, eccentricity, the coordinates of foci, vertices, lengths of the latus rectum and equations of the directrices of the following hyperbola: $16x^2 - 9y^2 = -144$.
7. If AOB and COD are two straight lines which bisect one another at right angles, show that the locus of a point P which moves so that $PA \times PB = PC \times PD$ is a hyperbola. Find its eccentricity.
8. If S and S' be the foci, C the centre and P be a point on a rectangular hyperbola, show that $SP \times S'P = (CP)^2$.
9. Find the equation of the hyperbola whose foci are (8, 3), (0, 3) and eccentricity $=\frac{4}{3}$.

HYPERBOLA: DEFINITION 2

We can also define hyperbola w.r.t. one fixed point and a fixed line. Hyperbola is the locus of a point which moves in a plane such that ratio of its distances from a fixed point (i.e. focus) and the fixed line (i.e. directrix) is constant and more than 1. This ratio is called eccentricity and is denoted by e. For a hyperbola $e > 1$.

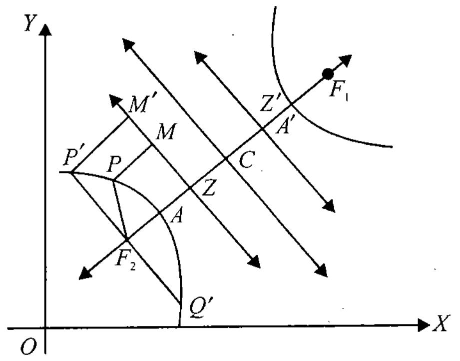


Fig. 5.17

From the diagram, for any point P on the curve, we have by definition,

$$\frac{F_2P}{PM} = e$$

or $F_2P = e\,PM$ (focal length or focal radius of point P)

Also, A and A' divide F_2Z in the ratio e:1 internally and externally, respectively.

If the focus F_2 has coordinates (α, β) and equation of directrix ZM is $lx + my + n = 0$, then equation of hyperbola is

$$\sqrt{(x-\alpha)^2+(y-\beta)^2} = \frac{e\,|lx+my+n|}{\sqrt{l^2+m^2}}$$

which is of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

and $\quad h^2 > ab$.

From the diagram length of latus rectum

$$= P'Q'$$
$$= 2F_2P'$$
$$= 2(e \times P'M')$$
$$= 2(e \times F_2Z)$$
$= 2e \times$ (distance of focus from corresponding directrix)

Note:
If $\phi(x, y) = 0$ is the equation of hyperbola and $\frac{\partial\phi}{\partial x}$ denotes the differential coefficient of ϕ w.r.t. x keeping y as constant and likewise $\frac{\partial\phi}{\partial y}$, then centre C is the solution of **$\frac{\partial\phi}{\partial x} = 0$, $\frac{\partial\phi}{\partial y} = 0$. (This is valid for all conic.)**

Example 5.10 **Find the equation of the hyperbola whose directrix is $x + 2y = 1$, focus is (2, 1) and eccentricity is 2.**

Sol. Let $P(x, y)$ be any point on the hyperbola, then by definition

$$SP = e \times PM$$

where S is focus and M is foot of ⊥ from P on directrix.

$$\Rightarrow \quad SP^2 = e^2\,PM^2$$

$$\Rightarrow \quad (x-2)^2 + (y-1)^2 = 4\left(\frac{x+2y-1}{\sqrt{5}}\right)^2$$

$$\Rightarrow \quad x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$$

This is the required equation of the hyperbola.

Example 5.11 **The equation of one of the directrices of a hyperbola is $2x + y = 1$, the corresponding focus is (1, 2) and $e = \sqrt{3}$. Find the equation of the hyperbola and coordinates of the centre and second focus.**

Sol. Let S be the focus and PM be perpendicular distance of a point $P(x, y)$ from the directrix, then

$$PS = e\,PM \text{ gives}$$

$$(x-1)^2+(y-2)^2=3\left[\frac{(2x+y-1)}{\sqrt{5}}\right]^2$$

$$\Rightarrow 5[x^2+y^2-2x-4y+5]$$

$$=3[4x^2+y^2+4xy-4x-2y+1]$$

$$\Rightarrow \quad 7x^2+12xy-2y^2-2x+14y-22=0 \qquad \text{(i)}$$

Equation of the perpendicular from S to directrix, i.e. of SZ is

$$x-2y=-3 \qquad \text{(ii)}$$

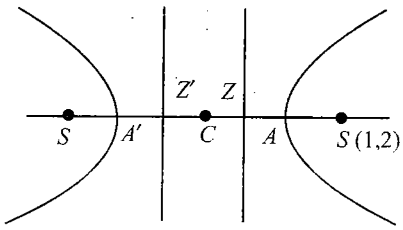


Fig. 5.18

Solving Eq. (ii) and $2x+y-1=0$, we get

$$Z=\left(-\frac{1}{5},\frac{7}{5}\right)$$

Since A and A' divide SZ in the ratio $\sqrt{3}:1$ internally and externally, we get

$$A=\left(\frac{5-\sqrt{3}}{5(\sqrt{3}+1)},\frac{10+7\sqrt{3}}{5(\sqrt{3}+1)}\right)$$

and

$$A'=\left(-\frac{5+\sqrt{3}}{5(\sqrt{3}-1)},\frac{7\sqrt{3}-10}{5(\sqrt{3}-1)}\right)$$

Since C is midpoint of AA', we get after simplification,

$$C=\left(-\frac{4}{5},\frac{11}{10}\right)$$

Now, if S' is the second focus, C is midpoint of SS' and $S'=(x_1,y_1)$, then

$$\frac{x_1+1}{2}=-\frac{4}{5}$$

and

$$\frac{y_1+2}{2}=\frac{11}{10}.$$

So,

$$S'=\left(-\frac{13}{5},\frac{1}{5}\right)$$

Example 5.12 **OA, OB are fixed straight lines, P is any point and PM, PN are the perpendiculars from P on OA, OB. Find the locus of P if the quadrilateral $OMPN$ is of constant area.**

Sol.

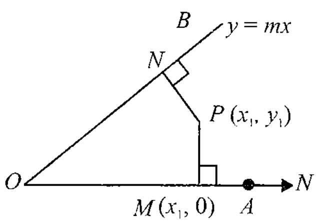


Fig. 5.19

Taking OA as x-axis, O as origin, let the equation of OB be $y=mx$.

Then $\qquad M=(x_1,0)$

Equation of the perpendicular PN is

$$my+x=my_1+x_1$$

Solving the equations OB and PN, we get

$$N=\left(\frac{my_1+x_1}{1+m^2},\frac{m(my_1+x_1)}{1+m^2}\right)$$

$\therefore$ Area of the quadrilateral $OMPN$ (by stair method)

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 \\ x_1 & 0 \\ x_1 & y_1 \\ \dfrac{my_1+x_1}{1+m^2} & \dfrac{m(my_1+x_1)}{1+m^2} \\ 0 & 0 \end{vmatrix}$$

$$=\frac{1}{2}x_1y_1+x_1\frac{m(my_1+x_1)}{1+m^2}$$

$$-y_1\frac{my_1+x_1}{1+m^2}=\pm k \text{ (say)}$$

Therefore, locus of P is

$$mx^2+2m^2xy-my^2\pm 2(1+m^2)k=0$$

Here $\qquad h=m^2,$

$a=m,$

$b=-m,$

$f=g=0$

and $\qquad c=\pm 2(1+m^2)k$

$\therefore \qquad \Delta=abc+2fgh-af^2-bg^2-ch^2$

$=\pm 2m^2(1+m^2)\,k\pm 2m^4(1+m^2)k^2\neq 0$

and $\qquad h^2>ab$

So the locus is a hyperbola.

Equation of a Hyperbola Referred to Two Perpendicular Lines

Consider the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as shown in figure.

Let $P(x,y)$ be any point on the hyperbola. Then, $PM=y$ and $PN=x$.

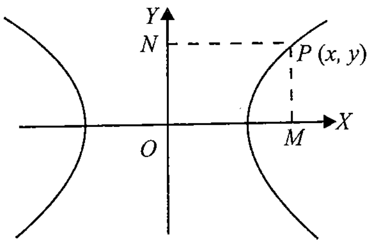


Fig. 5.20

$$\therefore \quad \frac{PN^2}{a^2} - \frac{PM^2}{b^2} = 1$$

It follows from this that if perpendicular distance p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv b_1x - a_1y + c_2 = 0$, respectively, satisfy the equation

$$\frac{p_1^2}{a^2} - \frac{p_2^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{\left(\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} - \frac{\left(\dfrac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1$$

then the locus of point P denotes an hyperbola in the plane of the given lines such that

i. the centre of the hyperbola is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$;

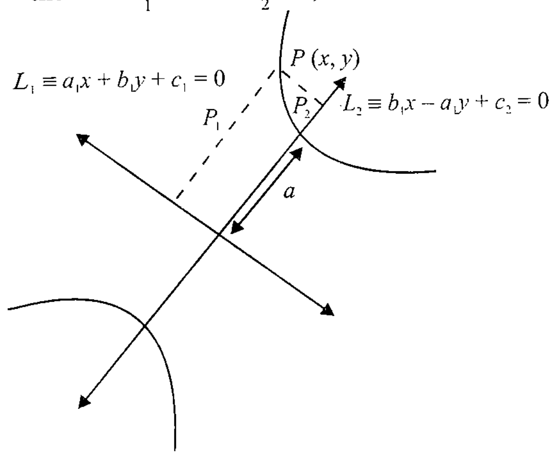


Fig. 5.21

ii. the transverse axis lies along $L_2 = 0$ and the conjugate axis lies along $L_1 = 0$;

iii. the length of the transverse and conjugate axes are $2a$ and $2b$, if $a > b$.

Example 5.13 **Find the coordinates of the foci and the centre of the hyperbola**

$$\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$$

Sol. Let $3x - 4y - 12 = X$ and $4x + 3y - 12 = Y$.

$\frac{X^2}{10^2} - \frac{Y^2}{15^2} = 1$ (note that X and Y are two perpendicular lines)

Centre is the point of intersection of $X = 0$ and $Y = 0$.

Hence, $\quad 3x - 4y = 12 \quad$ (i)

$\quad 4x + 3y = 12 \quad$ (ii)

Solving (i) and (ii), we get $x = \frac{84}{25}, y = -\frac{12}{25}$

Now, $\quad e^2 = 1 + \frac{225}{100} = \frac{325}{100}$

$\Rightarrow \quad e = \frac{\sqrt{13}}{2}$

Also, $\quad a = 10; b = 15.$

Focus is $(ae, 0)$.

Hence, $\quad X = ae$

and $\quad Y = 0$

$\Rightarrow \quad 3x - 4y - 12 = ae$

and $\quad 4x + 3y - 12 = 0$

Solving, we get $\quad x = \frac{84 + 3ae}{25} = \frac{84 + 15\sqrt{13}}{25}$

and $\quad y = \frac{-12 + 15\sqrt{13}}{25}$

Hence, focus is $\left(\frac{84 + 15\sqrt{3}}{25}, \frac{-12 + 15\sqrt{13}}{25}\right)$.

Example 5.14 **Find the eccentricity of the conic $4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$.**

Sol. Here $2y - x - 3 = 0$ and $2x + y - 1 = 0$ are perpendicular to each other.

Therefore, the equation of the conic can be written as

$$4 \times 5\left[\frac{2y - x - 3}{\sqrt{2^2 + 1^2}}\right]^2 - 9 \times 5\left[\frac{2x + y - 1}{\sqrt{2^2 + 1^2}}\right]^2 = 80$$

$$\Rightarrow \quad 4\left[\frac{2y - x - 3}{\sqrt{5}}\right]^2 - 9\left[\frac{2x + y - 1}{\sqrt{5}}\right]^2 = 16$$

On putting $\frac{2y - x - 3}{\sqrt{5}} = X$ and $\frac{2x + y - 1}{\sqrt{5}} = Y$, the given equation can be written as

$$4X^2 - 9Y^2 = 16$$

$$\Rightarrow \quad \frac{X^2}{4} - \frac{Y^2}{(4/3)^2} = 1$$

The eccentricity is given by

$$e = \sqrt{1 + \frac{4/3}{4}}$$

$$= \frac{2}{\sqrt{3}}$$

Intersection of a Line and Hyperbola

$$y = mx + c \tag{i}$$

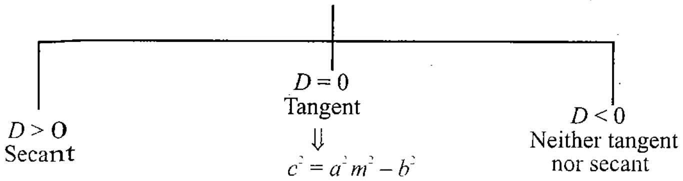
and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{ii}$$

Solving (i) and (ii), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow (b^2 - a^2m^2)\,x^2 - 2a^2\,mc\,x - a^2(b^2 + c^2) = 0 \tag{iii}$$



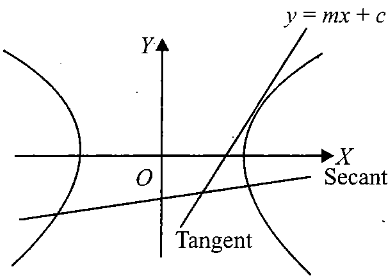


Fig. 5.22

Hence, $y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola.

In the above equation $a^2m^2 - b^2 \geq 0 \Rightarrow m^2 \geq \frac{b^2}{a^2}$

$$\Rightarrow m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right).$$

Hence, for given m, there can be two parallel tangents to the hyperbola.

If tangents passes through (h, k) then

$$(k - mh)^2 = a^2m^2 - b^2$$

$$(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0 \tag{iv}$$

Hence, passing through a given point (h, k) there can be a maximum of two tangents.

Now, $$m_1 + m_2 = \frac{2kh}{h^2 - a^2} \tag{v}$$

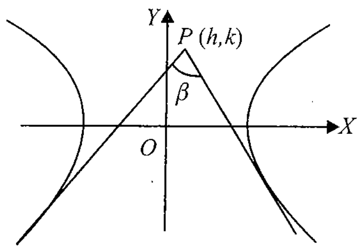


Fig. 5.23

$$m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2} \tag{vi}$$

β is the angle enclosed by the tangents as shown in the figure.

Now, $$\tan^2\beta = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$$

(substituting the value of $m_1 + m_2$ and m_1m_2 to get the locus of P)

If $\beta = 90°$ then $m_1m_2 = -1$

hence from (vi), we get

$$k^2 + b^2 = a^2 - h^2$$

$x^2 + y^2 = a^2 - b^2$ which is the equation to the *director circle* of the hyperbola.

If $a > b$, director circle is real with finite radius.

If $a = b$, director circle is a point circle which is origin.

If $a < b$, no real circle or no such point on the plane.

Note:

For hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, ***equation of tangent at point*** $P(x_1, y_1)$ **is**

$$y - y_1 = m(x - x_1) \pm \sqrt{a^2m^2 - b^2}$$

Equation of Tangent to the Hyperbola at Point (x_1, y_1)

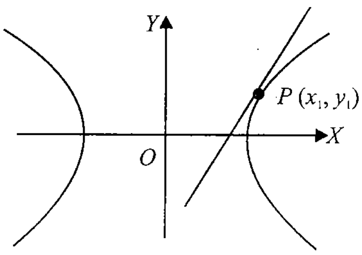


Fig. 5.24

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. x, we have

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2x_1}{a^2y_1}$$

Hence, equation of tangent is $y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

But (x_1, y_1) lies on the hyperbola. So,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

Hence, equation of tangent is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{(i)}$$

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ or $T = 0$

where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Note:

Equation of tangent at point P (x_1, y_1) to the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)}{b^2} = 1$ ***is*** $\frac{(x-h)(x_1-h)}{a^2} - \frac{(y-k)(y_1-k)}{b^2} = 1$

Equation of Tangent at Point ($a \sec\theta$, $b \tan\theta$)

Putting $x_1 = a \sec\theta$ and $y_1 = b \tan\theta$ in (i), we have

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad \text{(ii)}$$

Point of Contact When Line $y = mx + c$ Touches the Hyperbola

Line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when $c = \pm\sqrt{a^2m^2 - b^2}$.

Comparing lines $y = mx \pm \sqrt{a^2m^2 - b^2}$ with $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ we have

$$\frac{\frac{x_1}{a^2}}{m} = \frac{\frac{-y_1}{b^2}}{-1} = \frac{1}{\mp\sqrt{a^2m^2 - b^2}}$$

$$\Rightarrow \quad (x_1, y_1) \equiv \left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

or $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$, where $c = \sqrt{a^2m^2 - b^2}$

Point of Intersection of Tangent at Point $P(\alpha)$ and $Q(\beta)$

If point of intersection is $R(x_1, y_1)$, then

$$x_1 = a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$$

and $$y_1 = b\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$$

Example 5.15 **For all real values of m the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to which of the following certain hyperbolas:**

a. $9x^2 + 4y^2 = 36$ **b.** $4x^2 + 9y^2 = 36$
c. $9x^2 - 4y^2 = 36$ **d.** $4x^2 - 9y^2 = 36$

Sol. Comparing $y = mx + \sqrt{9m^2 - 4}$ with $y = mx + \sqrt{a^2m^2 - b^2}$, we have

$$a^2 = 9 \text{ and } b^2 = 4$$

Therefore, the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

or $4x^2 - 9y^2 = 36$

Example 5.16 **Find the equations of tangents to the curve $4x^2 - 9y^2 = 1$ which is parallel to $4y = 5x + 7$.**

Sol. Let m be the slope of the tangent to $4x^2 - 9y^2 = 1$.

Then, m = (slope of the line $4y = 5x + 7$)

$$= \frac{5}{4}$$

We have, $\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$

where $a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$

The equations of the tangents are $y = mx \pm \sqrt{a^2m^2 - b^2}$

or $y = \frac{5}{4}x \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$.

or $30x - 24y \pm \sqrt{161} = 0$

or $24y - 30x = \pm\sqrt{161}$

Example 5.17 **If the line $5x + 12y = 9$ touches the hyperbola $x^2 - 9y^2 = 9$, then find its point of contact.**

Sol. Solving line $5x + 12y = 9$ or $y = \frac{9 - 5x}{12}$ and $x^2 - 9y^2 = 9$, we have

$$x^2 - 9\left(\frac{9-5x}{12}\right)^2 = 9$$

$$\Rightarrow \quad x^2 - \frac{1}{16}(9-5x)^2 = 9$$

$$\Rightarrow 16x^2 - (25x^2 - 90x + 81) = 144$$

$$\Rightarrow \quad 9x^2 - 90x + 225 = 0$$

$$\Rightarrow \quad x^2 - 10x + 25 = 0$$

$$\Rightarrow \quad x = 5$$

$$\Rightarrow \quad y = \frac{9-25}{12} = -\frac{4}{3}$$

Example 5.18 **Find the equation of the tangent to the conic $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1).**

Sol. The equation of the tangent to $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1) is

$$2x - y - 4(x+2) + (y+1) + 11 = 0$$

$$\Rightarrow \quad x = 2$$

Example 5.19 **Find the value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$.**

Sol. If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

Here $c = 6,\ a^2 = 100,\ b^2 = 49$

$$\therefore \quad 36 = 100m^2 - 49$$

$$\Rightarrow \quad 100m^2 = 85$$

$$\Rightarrow \quad m = \sqrt{\frac{17}{20}}$$

Example 5.20 ***P* is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *N* is the foot of the perpendicular from *P* on the transverse axis. The tangent to the hyperbola at *P* meets the transverse axis at *T*. If *O* is the centre of the hyperbola, then find the value of $OT \times ON$.**

Sol. Let $P(x_1, y_1)$ be a point on the hyperbola.

Then the coordinates of N are $(x_1, 0)$.

The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

This meets x-axis at $T\left(\frac{a^2}{x_1}, 0\right)$

$$\therefore \quad OT \,.\, ON = \frac{a^2}{x_1} x_1 = a^2$$

Example 5.21 **On which curve does the perpendicular tangents drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ intersect ?**

Sol. The locus of the point of intersection of prependicular tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 - b^2.$$

Hence, the perpendicular tangents drawn to $\frac{x^2}{25} - \frac{y^2}{16} = 1$ intersect on the curve $x^2 + y^2 = 25 - 16$,

i.e. $\quad x^2 + y^2 = 9.$

Example 5.22 **Tangents drawn from the point (c, d) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make angles α and β with the x-axis. If $\tan\alpha \tan\beta = 1$, then find the value of $c^2 - d^2$.**

Sol. Any tangent to hyperbola is $y = mx + \sqrt{a^2m^2 - b^2}$. It passes through (c, d), so

$$d = mc + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad (d - mc)^2 = a^2m^2 - b^2$$

$$\Rightarrow (c^2 - a^2)m^2 - 2cdm + d^2 + b^2 = 0$$

$$\Rightarrow \quad \text{product of roots} = m_1 m_2 = \frac{d^2 + b^2}{c^2 - a^2}$$

$$\Rightarrow \quad \tan\alpha \tan\beta = \frac{d^2 + b^2}{c^2 - a^2} = 1$$

$$\Rightarrow \quad d^2 + b^2 = c^2 - a^2$$

$$\Rightarrow \quad c^2 - d^2 = a^2 + b^2$$

Example 5.23 **Tangents are drawn from the points on a tangent of the hyperbola $x^2 - y^2 = a^2$ to the parabola $y^2 = 4ax$. If all the chords of contact pass through a fixed point Q, prove that the locus of the point Q for different tangents on the hyperbola is an ellipse.**

Sol. Tangent at a point $(a\sec\theta, a\tan\theta)$ on the hyperbola $x^2 - y^2 = a^2$ is

$$x\sec\theta - y\tan\theta = a \qquad \text{(i)}$$

Any point on (i) will be of the form $\left(t, \frac{t\sec\theta - a}{\tan\theta}\right)$

Equation of chord of contact of the point w.r.t. parabola $y^2 = 4ax$ will be

$$y\left(\frac{t\sec\theta - a}{\tan\theta}\right) - 2a(x + t) = 0$$

$$\Rightarrow \left(\frac{-ay}{\tan\theta} - 2ax\right) + t\left(\frac{y\sec\theta}{\tan\theta} - 2a\right) = 0 \qquad \text{(ii)}$$

(ii) represents a family of straight lines each member of which passes through the point of intersection of straight lines $-\frac{ay}{\tan\theta} - 2ax = 0$ and $y\frac{\sec\theta}{\tan\theta} - 2a = 0$

$$\Rightarrow \quad y = 2a\sin\theta,\ x = -a\cos\theta$$

So the point Q is $(-a\cos\theta, 2a\sin\theta)$

Let $\quad \alpha = -a\cos\theta,\ \beta = 2a\sin\theta$

$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2}{4a^2} = 1$. So locus of Q is $\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1$, which is an ellipse.

Example 5.24 **Find the equations to the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.**

Sol. Tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=m_1x\pm\sqrt{a^2m_1^2-b^2}$$

The other hyperbola is

$$\frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1$$

Any tangent to this hyperbola is given by

$$y=m_2x\pm\sqrt{(-b^2)m_2^2-(-a^2)} \quad \text{(ii)}$$

If (i) and (ii) are same, then $m_1=m_2$ and

$$\Rightarrow \quad a^2m_1^2-b^2=a^2-b^2m_1^2$$

$$\Rightarrow \quad (a^2+b^2)m_1^2=a^2+b^2$$

$$\Rightarrow \quad m_1^2=1$$

$$\Rightarrow \quad m_1=\pm1$$

Equation of Pair of Tangents from Point (x_1, y_1)

Combined equation for pair of tangents PQ and PR is given by $T^2=SS_1$

where $$T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1,$$

$$S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$$

and $$S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

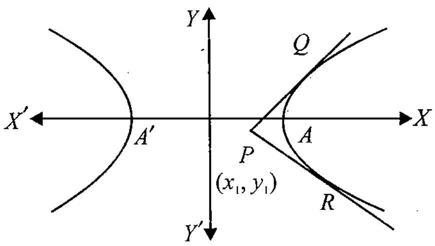


Fig. 5.25

Example 5.25 **How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$? Find the equation of these tangents and angle between them.**

Sol. Given point $P\equiv(4, 3)$

$$\text{Hyperbola } S\equiv\frac{x^2}{16}-\frac{y^2}{9}-1$$

$$\because \quad S_1\equiv\frac{16}{16}-\frac{9}{9}-1=-1<0$$

$\Rightarrow$ Point $P\equiv(4, 3)$ lies outside the hyperbola

Hence, two tangents can be drawn from the point P (4, 3).

Equation of pair of tangents is

$$SS_1=T^2$$

$$\Rightarrow\left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)(-1)=\left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \quad -\frac{x^2}{16}+\frac{y^2}{9}+1=\frac{x^2}{16}+\frac{y^2}{9}+1-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3}$$

$$\Rightarrow \quad \frac{x^2}{8}-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3}=0$$

$$\Rightarrow 3x^2-4xy-12x+16y=0$$

$$\theta=\tan^{-1}\left(\frac{4}{3}\right)$$

Concept Application Exercise 5.2

1. From the centre C of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ perpendicular CN is drawn on any tangent to it at the point P ($a\sec\theta$, $b\tan\theta$) in the first quadrant. Find the value of θ so that area of ΔCPN is maximum.
2. Find the common tangent to $9x^2-16y^2=144$ and $x^2+y^2=9$.
3. Find the equation of the tangent to the curve $4x^2-9y^2=1$ which is parallel to $4y=5x+7$.
4. Find the locus of a point P (α, β) moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
5. A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2=4ax$ touches the rectangular hyperbola $x^2-y^2=c^2$. Show that the locus of P is the ellipse $\frac{x^2}{c^2}+\frac{y^2}{(2a)^2}=1$.
6. PN is the ordinate of any point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and AA' is its transverse axis. If Q divides AP in the ratio $a^2:b^2$, then prove that NQ is perpendicular to $A'P$.
7. C is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

   The tangents at any point P on this hyperbola meet the straight lines $bx-ay=0$ and $bx+ay=0$ at points Q are R, respectively, Then prove that $CQ\cdot CR=a^2+b^2$.

Equation of Normal to the Hyperbola at Point (x_1, y_1)

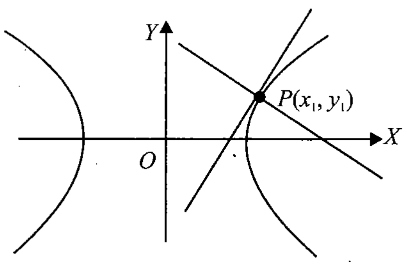


Fig. 5.26

Slope of normal at point (x_1, y_1) is $-\frac{a^2y_1}{b^2x_1}$.

Hence, equation of normal is

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \qquad \text{(i)}$$

Normal at Point P (a sec θ, b tan θ)

Putting $x_1 = a\sec\theta$ and $y_1 = b\tan\theta$ in (i), we get

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \qquad \text{(ii)}$$

Note:

- ***Normal other than transverse axis never passes through the focus.***
- ***Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle, i.e. $x^2 + y^2 = a^2$.***
- ***The product of the feet of these perpendiculars is b^2 (semi-conjugate axis)^2.***
- ***The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.***
- ***The tangent and normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as "an incoming light ray" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. If follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common points.***

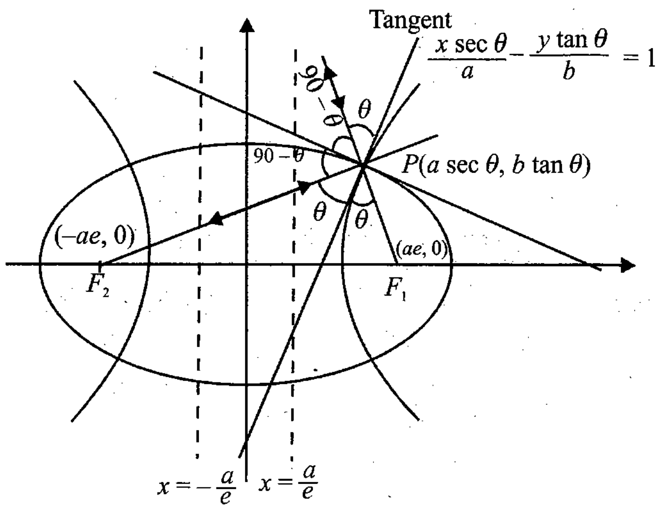


Fig. 5.27

Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ ($a > k > b > 0$) are confocal and therefore orthogonal.

- ***The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.***

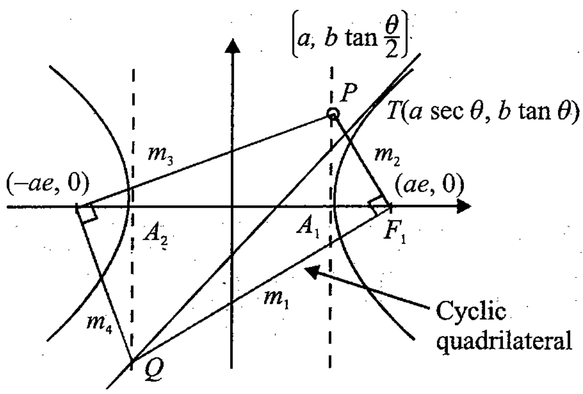


Fig. 5.28

Example 5.26 **If the normal at P (θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$ meets the transverse axis at G, then prove that $AG \times A'G = a^2(e^4\sec^2\theta - 1)$ (where A and A' are the vertices of the hyperbola).**

Sol. 4. The equation of the normal at P ($a\sec\theta$, $b\tan\theta$) to the given hyperbola is $ax\cos\theta + by\cot\theta = (a^2 + b^2)$.

This meets the transverse axis, i.e. x-axis at G.

So, the coordinates of G are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\theta, 0\right)$

The coordinates of the vertices A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

$$\therefore \quad AG \cdot A'G = \left(-a + \frac{a^2+b^2}{a}\sec\theta\right)\left(a + \frac{a^2+b^2}{a}\sec\theta\right)$$

$$= (-a + ae^2\sec\theta)(a + ae^2\sec\theta) = a^2(e^4\sec^2\theta - 1)$$

Example 5.27 **Normals are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at points θ_1 and θ_2 meeting the conjugate axis at G_1 and G_2 respectively. If $\theta_1 + \theta_2 = \frac{\pi}{2}$, prove that $CG_1 \cdot CG_2 = \frac{a^2e^4}{e^2-1}$ where C is the centre of the hyperbola and e is its eccentricity.**

Sol. Normal at point $P(a\sec\theta_1, b\tan\theta_1)$ is

$ax\cos\theta_1 + by\cot\theta_1 = (a^2+b^2)$.

It meets the conjugate axis at $G_1\left(0, \frac{a^2+b^2}{b}\tan\theta_1\right)$.

Normal at point $Q(a\sec\theta_2, b\tan\theta_2)$ is

$ax\cos\theta_2 + by\cot\theta_2 = (a^2+b^2)$.

It meets the conjugate axis at $G_2\left(0, \frac{a^2+b^2}{b}\tan\theta_2\right)$

$$\Rightarrow \quad CG_1 \cdot CG_2 = \frac{(a^2+b^2)^2}{b^2}\tan\theta_1\tan\theta_2$$

$$= \frac{(a^2+b^2)^2}{b^2} \quad \left(\because \theta_1 + \theta_2 = \frac{\pi}{2}\right)$$

$$= \frac{a^4\left(1+\frac{b^2}{a^2}\right)^2}{b^2}$$

$$= \frac{a^2e^4}{e^2-1}$$

Example 5.28 **Normal is drawn at one of the extremities of the latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which meets the axis at points A and B. Then find the area of triangle OAB (O being the origin).**

Sol. Normal at point $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

It meets the axes at $A\left(\frac{(a^2+b^2)x_1}{a^2}, 0\right)$ and $B\left(0, \frac{(a^2+b^2)y_1}{b^2}\right)$

$$\text{Area of} \quad \Delta OAB = \frac{1}{2}\left[\frac{(a^2+b^2)x_1}{a^2}\right]\left[\frac{(a^2+b^2)y_1}{b^2}\right]$$

$$= \frac{1}{2}\left[\frac{(a^2+b^2)^2x_1y_1}{a^2b^2}\right]$$

Now normal is drawn at the extremity of latus rectum.

Hence, $\quad (x_1, y_1) \equiv \left(ae, \frac{b^2}{a}\right)$

$$\Rightarrow \quad \text{Area} = \frac{1}{2}\left[\frac{(a^2+b^2)^2b^2e}{a^2b^2}\right]$$

$$= \frac{1}{2}\left[\frac{a^4\left(1+\frac{b^2}{a^2}\right)^2 e}{a^2}\right]$$

$$= \frac{1}{2}a^2e^5$$

Example 5.29 **A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection if point P lies in the first quadrant.**

Sol. Given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \text{(i)}$$

Now, x coordinate of point P is 8. Let y coordinate of P is α.

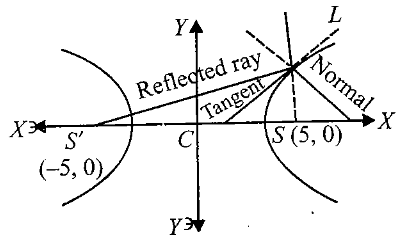


Fig. 5.29

As $(8, \alpha)$ lies on (i),

$$\therefore \quad \frac{64}{16} - \frac{\alpha^2}{9} = 1$$

$$\Rightarrow \quad \alpha^2 = 27$$

$$\Rightarrow \quad \alpha = 3\sqrt{3} \ (\because P \text{ lies in first quadrant})$$

Hence, coordinates of point P are $(8, 3\sqrt{3})$.

The equation of reflected ray passes through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$.

Therefore, its equation is

$$y - 0 = \frac{0 - 3\sqrt{3}}{-5-8}(x+5)$$

or $\quad 3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$

Equation of Chord Joining Points $P(\alpha)$ and $Q(\beta)$

Equation of chord passing through the points P ($a\sec\alpha$, $b\tan\alpha$) and Q ($a\sec\beta$, $b\tan\beta$) is given by

$$\begin{vmatrix} x & y & 1 \\ a\sec\alpha & b\tan\alpha & 1 \\ a\sec\beta & b\tan\beta & 1 \end{vmatrix} = 0.$$

$\Rightarrow$ $bx(\tan\alpha - \tan\beta) - ay(\sec\alpha - \sec\beta) + ab(\sec\alpha \tan\beta - \sec\beta\tan\alpha) = 0$

$\Rightarrow$ $bx\sin(\alpha-\beta) - ay(\cos\beta - \cos\alpha) + ab(\sin\beta - \sin\alpha) = 0$

$\Rightarrow$ $bx\cos\left(\frac{\alpha-\beta}{2}\right) - ay\sin\left(\frac{\alpha+\beta}{2}\right) - ab\cos\left(\frac{\alpha+\beta}{2}\right) = 0$

$\Rightarrow$ $\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$

Example 5.30 **If ($a\sec\theta$, $b\tan\theta$) and ($a\sec\phi$, $b\tan\phi$) are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $\tan\frac{x}{a}\tan\frac{\phi}{2} = \frac{1-e}{1+e}$.**

Sol. The equation of the chord joining ($a\sec\theta$, $b\tan\theta$) and ($a\sec\phi$, $b\tan\phi$) is

$$\frac{x}{a}\cos\frac{\theta-\phi}{2} - \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta+\phi}{2}$$

This passes through (ae, 0)

$$\Rightarrow \quad e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad e = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = \frac{\cos\left(\frac{\theta+\phi}{2}\right) - \cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right) + \cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\Rightarrow \quad \frac{e-1}{e+1} = -\tan\frac{\theta}{2}\tan\frac{\phi}{2}$$

$$\Rightarrow \quad \tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1-e}{1+e}$$

Chord of Contact

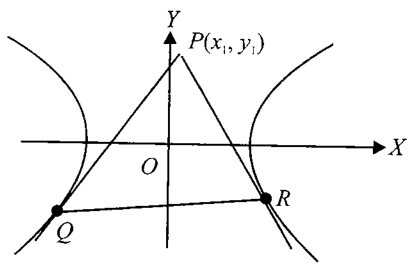


Fig. 5.30

In the diagram from point P tangents PQ and PR are drawn. Line QR is called chord of contact.

Its equation is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

or $\quad T = 0$

where

$$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

Example 5.31 **If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.**

Sol. 4 Let $P \equiv (h, k)$ be the point of intersection of tangents at A and B. Therefore, the equation of chord of contact AB of hyperbola is

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

or $\quad y = \frac{xb^2h}{ka^2} - \frac{b^2}{k}$ which touches the parabola $y^2 = 4ax$

Then $\quad -\frac{b^2}{k} = \frac{a}{\left(\frac{b^2h}{a^2k}\right)}$

$\Rightarrow \quad -\frac{b^2}{k} = \frac{ka^3}{b^2h}$

$\Rightarrow \quad y^2 = -\frac{b^4}{a^3}x$

Equation of the Chord of the Hyperbola Whose Midpoint is (x_1, y_1)

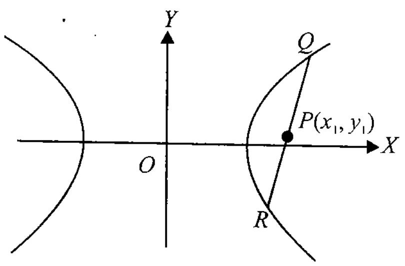


Fig. 5.31

Here chord QR is bisected at point P.

Its equation is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

or $T = S_1$, where

$$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

and

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

Example 5.32 **Find the locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.**

Sol. By $T = S_1$ the equation of chord whose midpoint is (h, k) is

$$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$$

$$\Rightarrow x(3h + 2) - y(2k + 3) + \cdots = 0$$

Its slope is $\frac{3h+2}{2k+3} = 2$(as it is parallel to $y = 2x$)

$$\Rightarrow \quad 3h - 4k = 4$$

$$\Rightarrow \quad 3x - 4y = 4$$

Example 5.33 **Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line $x + y = b$.**

Sol. Let the line $x + y = b$ bisect the chord at $P(\alpha, b - \alpha)$

Therefore, the equation of chord whose midpoint is $P(\alpha, b - \alpha)$ is given by

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b), therefore

$$\frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

$$\text{or } \alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0$$

$$\text{or} \quad \alpha = 0, \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

Hence, the required condition is $a \neq -b$.

Concept Application Exercise 5.3

1. If any line perpendicular to the transverse axis cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at points P and Q, then prove that normals at P and Q meet on the x-axis.
2. A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.
3. Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$, which is bisected at the point (5, 3).
4. Find the equation to the locus of the middle points of the chords of the hyperbola $2x^2 - 3y^2 = 1$, each of which makes an angle of 45° with the x-axis.
5. Prove that the locus of the point of intersection of the tangents at the ends of the normal chords of the hyperbola $x^2 - y^2 = a^2$ is $a^2(y^2 - x^2) = 4x^2y^2$.
6. If $\alpha + \beta = 3\pi$, then the chord joining the points α and β for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through which of the following points:

   a. focus
   b. centre
   c. one of the end points of the transverse axis
   d. one of the end points of the conjugates axis

Asymptotes of Hyperbola : Definition

If the length of the perpendicular from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the asymptote of the hyperbola.

The asymptote of the hyperbola can be found as follows,

Let $y = mx + c$ be the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two Eq. we get the quadratic as

$$(b^2 - a^2m^2)x^2 - 2a^2 mcx - a^2(b^2 + c^2) = 0 \quad \text{(i)}$$

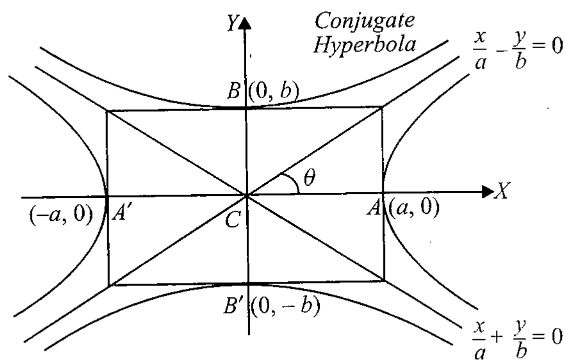


Fig. 5.32

In order that $y = mx + c$ be an asymptote, both roots of Eq. (i) must approach infinity, the conditions for which are as follows:

coefficient of $x^2 = 0$ and coefficient of $x = 0$

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ and $a^2 mc = 0 \Rightarrow c = 0$

Therefore, equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and

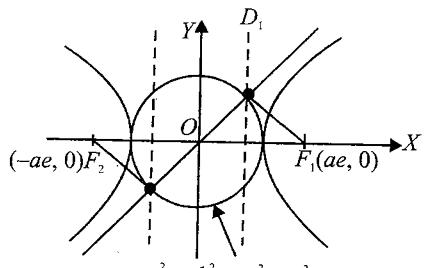
$$\frac{x}{a} - \frac{y}{b} = 0$$

The combined equation to the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Important Points

1. If the angle between the asymptotes of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ, then $e = \sec\theta$.

   Also acute angle between the asymptotes is $\theta = \tan^{-1}\left|\frac{2ab}{a^2 - b^2}\right|$
2. A hyperbola and its conjugate have the same asymptote.
3. The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.
4. The equation of the pair of asymptotes differs from the equation of the hyperbola and the conjugate hyperbola by some constant only.
5. The asymptotes of a hyperbola are the diagonals of the rectangle formed by the line drawn through the extremities of each axis parallel to the other axis.
6. For rectangular hyperbola we have $b = a$. Then the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angle.
7. If from any point on the asymptotes a straight line is drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi-conjugate axis.
8. Perpendicular from the foci on either asymptote meets it at the same point as the corresponding directrix and the common points of intersection lie on the auxiliary circle.



$$\frac{a^2}{e^2} + \frac{b^2}{e^2} = \frac{a^2 + b^2}{a^2} = a^2$$

Fig. 5.33

Proof:

$$\frac{x}{a} - \frac{y}{b} = 0$$

Asymptote: $\qquad y = \frac{b}{a}x \qquad$ (i)

Line through the focus and perpendicular to asymptote:

$$y - 0 = -\frac{a}{b}(x - ae)$$

or $\qquad by + ax = a^2e \qquad$ (ii)

Solving (i) and (ii) for x, we have

$$\left(\frac{b^2}{a} + a\right)x = a^2e$$

$\Rightarrow \qquad (b^2 + a^2)\,x = aa^2e$

$\Rightarrow \qquad (a^2e^2)x = a^2ae$

$\Rightarrow \qquad x = \frac{a}{e}$, hence $y = \frac{b}{a}\frac{a}{e} = \frac{b}{e}$

Now $\left(\frac{a}{e}, \frac{b}{e}\right)$ lies on the auxiliary circle.

9. The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C meets the asymptotes at Q and R and cuts off ΔCQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptotes is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing ΔCQR in case of a rectangular hyperbola is the hyperbola itself.

Example 5.34 **Find the eccentricity of the hyperbola with asymptotes $3x + 4y = 2$ and $4x - 3y = 2$.**

Sol. Since the asymptotes are perpendicular, hyperbola is rectangular and hence eccentricity is $\sqrt{2}$.

Example 5.35 **Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ as its asymptotes and which passes through the origin.**

Sol. Combined equation of the asymptotes is

$$(3x - 4y + 7)(4x + 3y + 1) = 0$$

or $12x^2 - 7xy - 12y^2 + 31x + 17y + 7 = 0 \qquad$ (i)

Since equation of hyperbola and combined equation of its asymptotes differ by a constant, therefore the equation of hyperbola may be taken as

$$12x^2 - 7xy - 12y^2 + 31x + 17y + k = 0 \qquad \text{(ii)}$$

As (ii) passes through origin (0, 0), we have $k = 0$.

Hence, equation to the required hyperbola is

$$12x^2 - 7xy - 12y^2 + 31x + 17y = 0$$

Example 5.36 **Find the equations of the asymptotes of the hyperbola $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$.**

Sol. Since equation of hyperbola and combined equation of its asymptotes differ by a constant, equations of asymptotes should be

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0 \qquad \text{(i)}$$

λ is to be chosen so that (i) represents a pair of straight lines.

Comparing (1) with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(ii)}$$

we have

$$a = 3, b = 8, h = 5, g = 7,$$
$$f = 11, c = \lambda$$

We know that (ii) represents pair of straight lines if $abc + 2hgf - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \quad 3{\cdot}8{\cdot}\lambda + 2{\cdot}7{\cdot}11{\cdot}5 - 3{\cdot}121 - 8{\cdot}49 - \lambda{\cdot}25 = 0$

$\Rightarrow \quad \lambda = 15$

Hence, combined equation of the asymptotes is $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$.

Example 5.37 **If a hyperbola passing through the origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes, then find the equations of its transverse and conjugate axes.**

Sol. Axes of hyperbola are bisectors of pair of asymptotes.

Transverse axis is the bisector which contains the origin and is given by

$$\frac{3x - 4y - 1}{5} = + \frac{4x - 3y - 6}{5}$$

or $\quad x + y - 5 = 0$

Conjugate axis is

$$\frac{3x - 4y - 1}{5} = - \frac{4x - 3y - 6}{5}$$

or $\quad x - y - 1 = 0$

Example 5.38 **Find the product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes**

Sol.

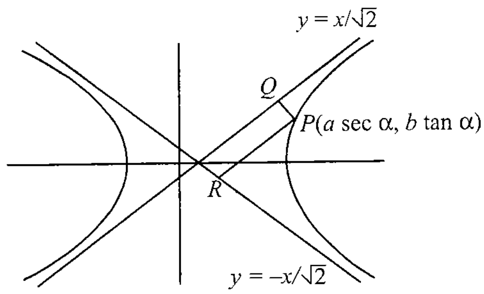


Fig. 5.34

Given hyperbola is $x^2 - 2y^2 - 2 = 0$

or $\quad \dfrac{x^2}{2} - \dfrac{y^2}{1} = 1$

$\Rightarrow \quad PQ{\cdot}PR = \dfrac{|a \sec\alpha - \sqrt{2} b \tan\alpha|}{\sqrt{3}} \dfrac{|a \sec\alpha + \sqrt{2} b \tan\alpha|}{\sqrt{3}}$

$= \dfrac{a^2 \sec^2\alpha - 2b^2 \tan^2\alpha}{3}$

$= \dfrac{2(\sec^2\alpha - \tan^2\alpha)}{3}$ (as $a = \sqrt{2}$ and $b = 1$)

$= \dfrac{2}{3}$

Concept Application Exercise 5.4

1. Find the angle between the asymptotes of the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.
2. Find the area of the triangle formed by any tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with its asymptotes.
3. Find the asymptotes of the curve $xy - 3y - 2x = 0$.
4. Show that the acute angle between the asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $(a^2 > b^2)$ is $2\cos^{-1}\left(\dfrac{1}{e}\right)$, where e is the eccentricity of the hyperbola.

Rectangular Hyperbola Referred to Its Asymptotes as the Axes of Coordinates

Referred to the transverse and conjugate axes along the axes of coordinates, the equation of the rectangular hyperbola is

$$x^2 - y^2 = a^2 \qquad \text{(i)}$$

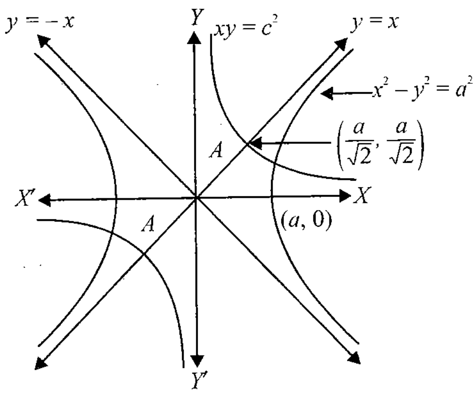


Fig. 5.35

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of $\frac{\pi}{4}$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola.

Now for new hyperbola equation of asymptotes is $xy = 0$.

Then equation of hyperbola is $xy = k$ (constant)

The hyperbola passes through the point $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

$$\therefore \quad k = \frac{a^2}{2}$$

Then equation of hyperbola is $xy = \frac{a^2}{2}$ or $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then its equation is $(x - \alpha)(y - \beta) = c^2$.

Important Points

For hyperbola $xy = c^2$

- Asymptotes: $x = 0$; $y = 0$
- Transverse axis: $y = x$; conjugate axis: $y = -x$
  - Vertex: $A(c, c)$ and $A'(-c, -c)$
  - Foci: $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$
  - Length of latus rectum = length of $AA' = 2\sqrt{2}c$
  - Equation of auxiliary circle: $x^2 + y^2 = 2c^2$
  - Equation of director circle: $x^2 + y^2 = 0$
  - $x^2 - y^2 = 1$ and $xy = c^2$ intersect at right angle

Properties of Rectangular Hyperbola $xy = c^2$

- Eccentricity of rectangular hyperbola is $\sqrt{2}$.
- Parametric form of rectangular hyperbola $xy = c^2$ is $P(ct, \frac{c}{t})$ where $t \in R - \{0\}$.
- Slope of chord joining the point $P(ct_1, \frac{c}{t_1})$ and $Q(ct_2, \frac{c}{t_2})$ is $-\frac{1}{t_1 t_2}$.
- Slope of tangent at point $(ct, \frac{c}{t})$ is $-\frac{1}{t^2}$.
- Equation of tangent at point whose parameter is 't' is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

or $\quad x + yt^2 - 2ct = 0$

- Equation of normal at point whose parameter is 't' is

$$y - \frac{c}{t} = t^2(x - ct)$$

or $\quad xt^3 - yt - ct^4 + c = 0$

- Equation of tangent at (x_1, y_1) is

$$xy_1 + x_1 y = 2c^2$$

or $\quad T = 0$

where

$$T = xy_1 + x_1 y - 2c^2$$

- Equation of normal at (x_1, y_1) is

$$xx_1 - yy_1 = x_1^2 - y_1^2$$

- Chord with a given middle point as (h, k) is

$$kx + hy = 2hk$$

Example 5.39 **A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.**

Sol. Let $A(t_1)$, $B(t_2)$, $C(t_3)$ be the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Coordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$, respectively.

Now slope of BC is $\dfrac{ct_3 - ct_2}{\frac{c}{t_3} - \frac{c}{t_2}} = -\dfrac{1}{t_2 t_3}$

Hence, slope of AD is $t_2 t_3$.

Equation of altitude AD is

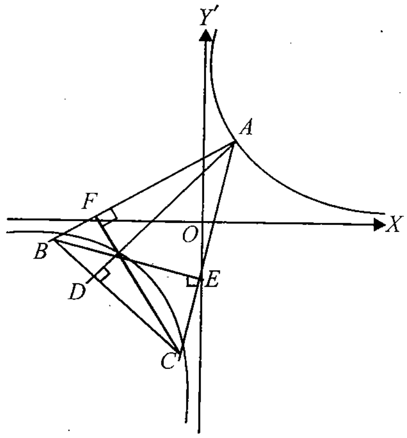


Fig. 5.36

$$y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$$

or $\quad t_1 y - c = x t_1 t_2 t_3 - c t_1^2 t_2 t_3 \qquad$ (i)

Similarly equation of altitude BE is

$$t_2 y - c = x t_1 t_2 t_3 - c t_1 t_2^2 t_3 \qquad \text{(ii)}$$

Solving Eqns. (i) and (ii) we get the orthocentre $\left(-\frac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$ which lies on $xy = c^2$.

Example 5.40 **If A, B and C be three points on the hyperbola $xy = c^2$ such that AB subtends a right angle at**

C. then prove that AB is parallel to normal to hyperbola at point C.

Sol.

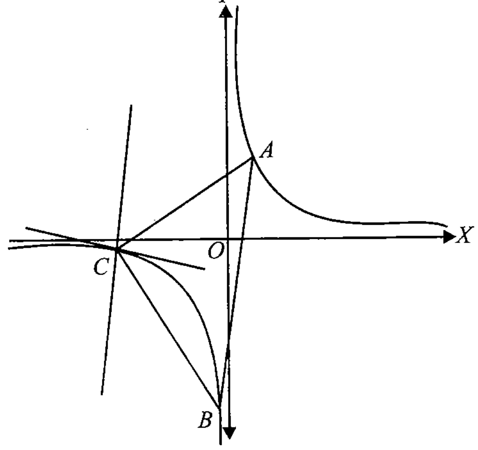


Fig. 5.37

Let coordinates of A, B and C be $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$

$$\text{Slope of } CA = -\frac{1}{t_1t_3}$$

$$\text{Slope of } CB = -\frac{1}{t_2t_3}$$

Given that $CA \perp CB$

$$\Rightarrow \quad \left(-\frac{1}{t_1t_3}\right) \times \left(-\frac{1}{t_2t_3}\right) = -1$$

$$\Rightarrow \quad \left(-\frac{1}{t_3^2}\right) \times \left(-\frac{1}{t_1t_2}\right) = -1$$

$\Rightarrow$ slope of tangent at point C × slope of $AB = -1$

$\Rightarrow$ Tangent at $C \perp AB$

$\Rightarrow$ Normal at C is parallel to AB

Example 5.41 **If PN is the perpendicular from a point on a rectangular hyperbola $xy = c^2$ to its asymptotes, then find the locus of the midpoint of PN.**

Sol. Let $xy = c^2$ be the rectangular hyperbola, and let $P(x_1, y_1)$ be the point on it.

Let $Q(h, k)$ be the midpoint of PN. Then the coordinates of Q are $\left(x_1, \frac{y_1}{2}\right)$.

$$\therefore \quad x_1 = h \text{ and } \frac{y_1}{2} = k$$

$$\Rightarrow \quad x_1 = h \text{ and } y_1 = 2k$$

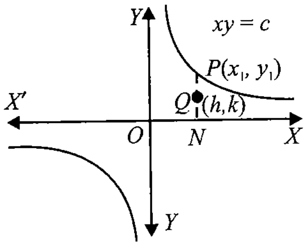


Fig. 5.38

But (x_1, y_1) lies on $xy = c^2$

$$\therefore \quad h(2k) = c^2$$

$$\Rightarrow \quad hk = \frac{c^2}{2}$$

Hence, the locus of (h, k) is $xy = \frac{c^2}{2}$, which is a rectangular hyperbola.

Example 5.42 **PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then find the value of product of the slopes of CP, CQ, CR and CS.**

Sol. Let coordinates of P, Q, R, S be $\left(ct_i, \frac{c}{t_i}\right)$, respectively (where $i = 1, 2, 3, 4$)

Now, $PQ \perp RS$

$$\Rightarrow \quad \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1$$

$$\Rightarrow \quad t_1t_2t_3t_4 = -1$$

$$\text{Now slope of } CP = \frac{\frac{c}{t_1}}{ct_1} = \frac{1}{t_1^2}$$

Hence, product of slopes of CP, CQ, CR and CS is

$$\frac{1}{t_1^2t_2^2t_3^2t_4^2} = 1$$

Concyclic Points on the Hyperbola $xy = c^2$

If a circle and the rectangular hyperbola $xy = c^2$ meet at the four points t_1, t_2, t_3 and t_4, then

a. $t_1t_2t_3t_4 = 1$

b. the centre of the mean position of the four points bisects the distance between the centres of the two curves

Proof:

a. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + d = 0 \quad \text{(i)}$$

Solving Eq. (i) and the equation of hyperbola, we have

$$x^2 + \frac{c^4}{x^2} + 2gx + 2f\frac{c^2}{x} + d = 0$$

$$\Rightarrow \quad x^4 + 2gx^3 + dx^2 + 2fc^2x + c^4 = 0 \qquad \text{(ii)}$$

From Eq. (i),

$$x_1x_2x_3x_4 = c^4$$

$$c^4[t_1t_2t_3t_4] = c^4$$

$$\Rightarrow \quad t_1t_2t_3t_4 = 1$$

b. Again, centre of the mean position of the four points of intersection is $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}\right)$

Now from Eq. (i),

$$x_1 + x_2 + x_3 + x_4 = -2g \qquad \text{(iii)}$$

$$\Rightarrow \quad \frac{\Sigma x_i}{4} = -\frac{g}{2}$$

using $xy = c^2$, we have

$$y_1 + y_2 + y_3 + y_4 = c^2\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right]$$

$$= \frac{c^2}{x_1x_2x_3x_4}\Sigma x_1x_2x_3$$

$$= \frac{c^2}{c^4}(-2fc^2) = -2f$$

$$\Rightarrow \quad \frac{\Sigma y_i}{4} = -\frac{f}{2}$$

Hence, $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}\right) \equiv \left(-\frac{g}{2}, -\frac{f}{2}\right)$

EXERCISES

Subjective Type

Solutions on page 5.36

1. A variable line $y = mx - 1$, cuts the lines $x = 2y$ and $y = -2x$ at points A and B. Prove that locus of the centroid of the triangle OAB (O being origin) is a hyperbola passing through origin.
2. Two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slopes m_1 and m_2 cut the axes in four concyclic points. Find the value of m_1m_2.
3. Let P be a point on the hyperbola $x^2 - y^2 = a^2$ where 'a' is a parameter, such that P is nearest to the line $y = 2x$. Find the locus of P.
4. Find the range of parameter a for which a unique circle will pass through the points of intersection of the rectangular hyperbola $x^2 - y^2 = a^2$ and the parabola $y = x^2$. Find also the equation of the circle.
5. Show that the midpoints of focal chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on another similar hyperbola.
6. A tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P and Q. Show that the locus of the midpoint of PQ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.
7. Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendicular drawn from the foci on the normal at the same point.
8. If one axis of varying central conic (hyperbola) be fixed in magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed point on the other axis is a parabola.
9. A transverse axis cuts the same branch of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P, P' and the asymptotes in Q. Q'. Prove that (i) $PQ = P'Q'$ and (ii) $PQ' = P'Q$.
10. A normal is drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P which meets the transverse axis (TA) at G. If perpendicular from G on the asymptote meets it at L, show that LP is parallel to conjugate axis.
11. Find the angle between the rectangular hyperbolas $(y - mx)(my + x) = a^2$ and $(m^2 - 1)(y^2 - x^2) + 4mxy = b^2$.

Objective Type

Solutions on page 5.39

Each question has four choices a, b, c, d, out of which *only one* answer is correct. Find the correct answer.

1. If the distance between two parallel tangents drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then their slope is equal to

   a. $\pm\frac{5}{2}$ **b.** $\pm\frac{4}{5}$
   c. $\pm\frac{7}{2}$ **d.** none of these

2. If the distance between the foci and the distance between the two directries of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are in the ratio 3:2, then $b:a$ is

   a. $1:\sqrt{2}$ **b.** $\sqrt{3}:\sqrt{2}$ **c.** 1:2 **d.** 2:1

3. A tangent drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square-units, with coordinate axes, then the square of its eccentricity is

a. 15 b. 24 c. 17 d. 14

4. The length of the transverse axis of the rectangular hyperbola $xy = 18$ is

a. 6 b. 12 c. 18 d. 9

5. A straight line has its extremities on two fixed straight lines and cuts off from them a triangle of constant area c^2. Then the locus of the middle point of the line is

a. $2xy = c^2$ b. $xy + c^2 = 0$
c. $4x^2y^2 = c$ d. none of these

6. If a variable line has its intercepts on the coordinate axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$

a. 1 b. 2
c. 3 d. cannot be decided

7. The equation of the transverse axis of the hyperbola $(x-3)^2 + (y+1)^2 = (4x+3y)^2$ is

a. $x + 3y = 0$ b. $4x + 3y = 9$
c. $3x - 4y = 13$ d. $4x + 3y = 0$

8. The family of curves for which the length of the normal at any point is equal to the radius vector of that point is

a. hyperbola b. straight line
c. parabola d. ellipse

9. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

a. 1 b. $\sqrt{2}$ c. 2 d. $\frac{1}{2}$

10. The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represents a hyperbola

a. the length of whose transverse axis is $4\sqrt{3}$
b. the length of whose conjugate axis is 4
c. whose centre is $(-1, 2)$
d. whose eccentricity is $\sqrt{\frac{19}{3}}$

11. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then ratio of square of its conjugate axis to the square of its transverse axis is

a. 2 b. 4 c. 6 d. 3

12. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is (axes are coordinate axes)

a. $\frac{4}{3}$ b. $\frac{4}{\sqrt{3}}$
c. $\frac{2}{\sqrt{3}}$ d. none of these

13. Let LL' be the latus rectum through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and A' be the farther vertex. If $\Delta A'LL'$ is equilateral, then the eccentricity of the hyperbola is (axes are coordinate axes)

a. $\sqrt{3}$ b. $\sqrt{3}+1$
c. $\frac{\sqrt{3}+1}{\sqrt{2}}$ d. $\frac{\sqrt{3}+1}{\sqrt{3}}$

14. The latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ is

a. $\frac{9}{4}$ b. 9 c. $\frac{3}{2}$ d. $\frac{9}{2}$

15. The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is

a. 2 b. $\frac{2}{\sqrt{3}}$ c. 4 d. $\frac{4}{5}$

16. The equations of the transverse and conjugate axes of a hyperbola are respectively $x + 2y - 3 = 0$, $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is

a. $\frac{2}{5}(x+2y-3)^2 - \frac{3}{5}(2x-y+4)^2 = 1$
b. $\frac{2}{5}(2x-y+4)^2 - \frac{3}{5}(x+2y-3)^2 = 1$
c. $2(2x-y+4)^2 - 3(x+2y-3)^2 = 1$
d. $2(x+2y-3)^2 - 3(2x-y+4)^2 = 1$

17. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}t = 0$ and $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

a. $\sqrt{3}$ b. 2 c. $\frac{2}{\sqrt{3}}$ d. $\frac{4}{3}$

18. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then a value of α is

a. $\frac{\pi}{6}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{2}$

19. With one focus of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

a. less than 2 b. 2
c. 1/3 d. none of these

20. If $ax + by = 1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2 - b^2$ equals to

a. $\frac{1}{a^2e^2}$ b. a^2e^2
c. b^2e^2 d. none of these

21. Locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a/an

a. ellipse b. circle
c. hyperbola d. parabola

22. The sides AC and AB of a ΔABC touch the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the vertex A lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the side BC must touch

a. parabola b. circle
c. hyperbola d. ellipse

23. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point $(0, -b)$ and the normal at P passes through the point $(2a\sqrt{2}, 0)$ then eccentricity of the hyperbola is

a. 2 b. $\sqrt{2}$ c. 3 d. $\sqrt{3}$

24. If values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are roots of the equation $x^2 - (a + b)x - 4 = 0$, then value of $(a + b)$ is equal to

a. 2 b. 4 c. zero d. none of these

25. Portion of asymptote of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (between centre and the tangent at vertex) in the first quadrant is cut by the line $y + \lambda(x - a) = 0$ (λ is a parameter) then

a. $\lambda \in R$ b. $\lambda \in (0, \infty)$
c. $\lambda \in (-\infty, 0)$ d. none of these

26. If angle between asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and product of perpendiculars drawn from foci upon its any tangent is 9, then locus of point of intersection of perpendicular tangents of the hyperbola can be

a. $x^2 + y^2 = 6$ b. $x^2 + y^2 = 9$
c. $x^2 + y^2 = 3$ d. $x^2 + y^2 = 18$

27. The co-ordinates of a point on the hyperbola, $\frac{x^2}{24} - \frac{y^2}{18} = 1$, which is nearest to the line $3x + 2y + 1 = 0$ are

a. (6, 3) b. (– 6, – 3) c. (6, – 3) d. (– 6, 3)

28. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is

a. zero b. 1 c. 2 d. 4

29. Locus of the feet of the perpendiculars drawn from either focus on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

a. $x^2 + y^2 = 9$ b. $x^2 + y^2 = \frac{1}{9}$
c. $x^2 + y^2 = \frac{7}{144}$ d. $x^2 + y^2 = \frac{1}{16}$

30. The locus of the foot of the perpendicular form the centre of the hyperbola $xy = 1$ on a variable tangent is

a. $(x^2 - y^2)^2 = 4xy$ b. $(x^2 + y^2)^2 = 2xy$
c. $(x^2 + y^2) = 4xy$ d. $(x^2 + y^2)^2 = 4xy$

31. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, the OT. ON is equal to

a. e^2 b. a^2 c. b^2 d. $\frac{b^2}{a^2}$

32. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the directrix in F. If PF subtends an angle θ at the corresponding focus, then θ equals

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{4}$ d. π

33. The locus of a point, from where tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ contain an angle of 45°, is

a. $(x^2 + y^2)^2 + a^2(x^2 - y^2) = 4a^2$
b. $2(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$
c. $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$
d. $(x^2 + y^2)^2 + a^2(x^2 - y^2) = a^4$

34. If tangents PQ and PR are drawn from variable point P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ so that the fourth vertex S of parallelogram $PQSR$ lies on circumcircle of triangle PQR, then locus of P is

a. $x^2 + y^2 = b^2$ b. $x^2 + y^2 = a^2$
c. $x^2 + y^2 = (a^2 - b^2)$ d. none of these

35. Number of points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$, from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, is /are

a. 0 b. 2 c. 3 d. 4

36. A normal to the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ has equal intercepts on positive x- and y-axes. If this normal touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $a^2 + b^2$ is equal to

a. 5 b. 25
c. 16 d. none of these

37. If the normal to the given hyperbola at the point $(ct, \frac{c}{t})$ meets the curve again at $(ct', \frac{c}{t'})$, then

a. $t^3t' = 1$ b. $t^3t' = -1$
c. $tt' = 1$ d. $tt' = -1$

38. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to λ ($\lambda \in R^+$), then locus of point P is

a. $x^2 = \lambda c^2$ **b.** $y^2 = \lambda c^2$
c. $xy = \lambda c^2$ **d.** none of these

39. If a ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line

a. $x\sqrt{2} - y + 5 = 0$ **b.** $\sqrt{2}y - x + 5 = 0$
c. $\sqrt{2}\,y - x - 5 = 0$ **d.** none of these

40. Let any double ordinate PNP' of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ be produced on both sides to meet the asymptotes in Q and Q', then $PQ \cdot P'Q$ is equal to

a. 25 **b.** 16
c. 41 **d.** none of these

41. For a hyperbola whose centre is at (1, 2) and asymptotes are parallel to lines $2x + 3y = 0$ and $x + 2y = 1$, then equation of hyperbola passing through (2, 4) is

a. $(2x + 3y - 5)(x + 2y - 8) = 40$
b. $(2x + 3y - 8)(x + 2y - 5) = 40$
c. $(2x + 3y - 8)(x + 2y - 5) = 30$
d. none of these

42. The chords of contact of a point 'P' w.r.t. a hyperbola and its auxiliary circle are at right angle, then the point P lies on

a. conjugate hyperbola
b. one of the directirx
c. one of the asymptotes
d. none of these

43. Asymptotes of the hyperbola $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$ and $\frac{x^2}{a_2^2} - \frac{y^2}{b_2^2} = 1$ are perpendicular to each other, then

a. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ **b.** $a_1a_2 = b_1b_2$
c. $a_1a_2 + b_1b_2 = 0$ **d.** $a_1 - a_2 = b_1 - b_2$

44. If $S = 0$ be the equation of the hyperbola $x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, then the value of k for which $S + K = 0$ represents its asymptotes is

a. 20 **b.** -16 **c.** -22 **d.** 18

45. If two distinct tangents can be drawn from the point (α, 2) on different branches of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then

a. $|\alpha| < \frac{3}{2}$ **b.** $|\alpha| > \frac{2}{3}$
c. $|\alpha| > 3$ **c.** none of these

46. A hyperbola passes through (2, 3) and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y - 40 = 0$, then the equation of its transverse axis is

a. $77x - 21y - 265 = 0$
b. $21x - 77y + 265 = 0$
c. $21x - 77y - 265 = 0$
d. $21x + 77y - 265 = 0$

47. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. Then area cut-off by the chord of contact on the asymptotes is equal to

a. $\frac{a}{2}$ **b.** ab **c.** $2ab$ **d.** $4ab$

48. From a point P (1, 2) two tangents are drawn to a hyperbola 'H' in which one tangent is drawn to each arm of the hyperbola. If the equations of asymptotes of hyperbola H are $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$, then eccentricity of 'H' is

a. 2 **b.** $\frac{2}{\sqrt{3}}$ **c.** $\sqrt{2}$ **d.** $\sqrt{3}$

49. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is

a. $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$
b. $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$
c. $2x^2 + 5xy + 2y^2 = 0$
d. none of these

50. The asymptotes of the hyperbola $xy = hx + ky$ are

a. $x - k = 0$ and $y - h = 0$
b. $x + h = 0$ and $y + k = 0$
c. $x - k = 0$ and $y + h = 0$
d. $x + k = 0$ and $y - h = 0$

51. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan\lambda$ in magnitude then its eccentricity is

a. $\sec\lambda$ **b.** $\operatorname{cosec}\lambda$ **c.** $\sec^2\lambda$ **d.** $\operatorname{cosec}^2\lambda$

52. The centre of a rectangular hyperbola lies on the line $y = 2x$. If one of the asymptotes is $x + y + c = 0$, then the other asymptote is

a. $x - y - 3c = 0$ **b.** $2x - y + c = 0$
c. $x - y - c = 0$ **d.** none of these

Hence, equation of other asymptote is $x + y - 3c = 0$.

53. Equation of a rectangular hyperbola whose asymptotes are $x = 3$ and $y = 5$ and passing through (7, 8) is

a. $xy - 3y + 5x + 3 = 0$
b. $xy + 3y + 4x + 3 = 0$
c. $xy - 3y + 5x - 3 = 0$
d. $xy - 3y - 5x + 3 = 0$

54. If foci of hyperbola lie on $y = x$ and one of the asymptote is $y = 2x$, then equation of the hyperbola, given that it passes through (3, 4) is

a. $x^2 - y^2 - \frac{5}{2}xy + 5 = 0$

b. $2x^2 - 2y^2 + 5xy + 5 = 0$

c. $2x^2 + 2y^2 - 5xy + 10 = 0$

d. none of these

55. $(x - 1)(y - 2) = 5$ and $(x - 1)^2 + (y + 2)^2 = r^2$ intersect at four points A, B, C, D and if centroid of ΔABC lies on line $y = 3x - 4$, then locus of D is

a. $y = 3x$ **b.** $x^2 + y^2 + 3x + 1 = 0$

c. $3y = x + 1$ **d.** $y = 3x + 1$

56. If tangents OQ and OR are dawn to variable circles having radius r and the centre lying on the rectangular hyperbola $xy = 1$, then locus of circumcentre of triangle OQR is (O being the origin)

a. $xy = 4$ **b.** $xy = \frac{1}{4}$

c. $xy = 1$ **d.** none of these

57. Four points are such that the line joining any two points is perpendicular to the line joining other two points. If three points out of these lie on a rectangular hyperbola then the fourth point will lie on

a. the same hyperbola

b. conjugate hyperbola

c. one of the directrix

d. one of the asymptotes

58. Equation of conjugate axis of hyperbola $xy - 3y - 4x + 7 = 0$ is

a. $y + x = 3$ **b.** $y + x = 7$

c. $y - x = 3$ **d.** none of these

59. If S_1 and S_2 are the foci of the hyperbola whose transverse axis length is 4 and conjugate axis length is 6, S_3 and S_4 are the foci of the conjugate hyperbola, then the area of the quadrilateral $S_1S_3S_2S_4$ is

a. 24 **b.** 26

c. 22 **d.** None of these

60. Suppose the circle having equation $x^2 + y^2 = 3$ intersects the rectangular hyperbola $xy = 1$ at the points A, B, C and D. The equation $x^2 + y^2 - 3 + \lambda(xy - 1) = 0$, $\lambda \in R$, represents

a. a pair of lines through origin for $\lambda = -3$

b. an ellipse through A, B, C and D for $\lambda = -3$

c. a parabola through A, B, C and D for $\lambda = -3$

d. a circle for any $\lambda \in R$

61. The family of the curves which intersects the family of rectangular hyperbola $xy = c^2$ orthogonally is

a. family of parabola

b. family of ellipse

c. family of circle

d. family of rectangular hyperbola

62. If two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose centre C be such that CP is perpendicular to CQ, $a < b$, then the value of $\frac{1}{CP^2} + \frac{1}{CQ^2}$ is

a. $\frac{b^2 - a^2}{2ab}$ **b.** $\frac{1}{a^2} + \frac{1}{b^2}$ **c.** $\frac{2ab}{b^2 - a^2}$ **d.** $\frac{1}{a^2} - \frac{1}{b^2}$

63. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

a. $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

b. $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

c. $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$

d. $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

64. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, then coordinates of the orthocentre of the triangle PQR is

a. $(x_4, -y_4)$ **b.** (x_4, y_4) **c.** $(-x_4, -y_4)$ **d.** $(-x_4, y_4)$

65. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A; C is the midpoint of PQ and 'O' is the origin. Then the ΔACO is

a. equilateral **b.** isosceles

c. right angled **d.** right isosceles

66. The curve $xy = c$ $(c > 0)$ and the circle $x^2 + y^2 = 1$ touch at two points, then distance between the points of contacts is

a. 1 **b.** 2 **c.** $2\sqrt{2}$ **d.** none of these

67 Let 'C' be a curve which is locus of the point of the intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $s \equiv (x - 2)^2 + (y + 1)^2 = 25$ intersects the curve C at four points P, Q, R and S. If O is centre of the curve 'C', then $OP^2 + OQ^2 + OR^2 + OS^2$ is

a. 50 **b.** 100 **c.** 25 **d.** $\frac{25}{2}$

68. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles then the equation of the circle through the points of intersection of two conic is

a. $x^2 + y^2 = 5$

b. $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$

c. $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$

d. $x^2 + y^2 = 25$

69. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms a triangle of constant area with the coordinate axe is

a. a straight line b. a hyperbola

c. an ellipse d. a circle

70. The angle between lines joining the origin to the points of intersection of the line $\sqrt{3}x+y=2$ and the curve $y^2-x^2=4$ is

a. $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ b. $\frac{\pi}{6}$

c. $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ d. $\frac{\pi}{2}$

71. A variable chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $(b>a)$ subtends a right angle at the centre of the hyperbola, if this chord touches

a. a fixed circle concentric with the hyperbola

b. a fixed ellipse concentric with the hyperbola

c. a fixed hyperbola concentric with the hyperbola

d. a fixed parabola having vertex at (0, 0)

72. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2+\alpha^2y^2=\alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2x^2-y^2=1$ subtends a right angle at the centre of the curves is

a. $\left[\frac{\sqrt{5}+1}{2}, 2\right]$ b. $(1, 2]$

c. $\left[\frac{\sqrt{5}-1}{2}, 1\right)$ d. $\left[\frac{\sqrt{5}-1}{2}, 1\right)\cup\left(1, \frac{\sqrt{5}+1}{2}\right]$

Multiple Correct Answers Type

Solutions on page 5.49

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. The equation $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right|=K$ will represent a hyperbola for

a. $K\in(0, 2)$ b. $K\in(-2, 1)$

c. $K\in(1, \infty)$ d. $K\in(0, \infty)$

2. If $x, y \in R$ then the equation $3x^4-2(19y+8)x^2+(361y^2+2(100+y^4)+64)=2(190y+2y^2)$ represents in rectangular Cartesian system

a. parabola b. hyperbola

c. circle d. ellipse

3. If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin, then

a. $e=\frac{\sqrt{386}}{12}$ b. $e=\frac{\sqrt{386}}{13}$

c. $LR=\frac{121}{6}$ d. $LR=\frac{121}{3}$

4. If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is

a. $\frac{\sqrt{386}}{12}$ b. $\frac{\sqrt{386}}{13}$

c. $\frac{\sqrt{386}}{25}$ d. $\frac{\sqrt{386}}{38}$

5. For which of the hyperbolas, we can have more than one pair of perpendicular tangents?

a. $\frac{x^2}{4}-\frac{y^2}{9}=1$ b. $\frac{x^2}{4}-\frac{y^2}{9}=-1$

c. $x^2-y^2=4$ d. $xy=44$

6. For the hyperbola $9x^2-16y^2-18x+32y-151=0$

a. one of the directrix is $x=\frac{21}{5}$

b. length of latus rectum $=\frac{9}{2}$

c. focii are (6, 1) and (− 4, 1)

d. eccentricity is $\frac{5}{4}$

7. If foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ coincide with the foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and eccentricity of the hyperbola is 2, then

a. $a^2+b^2=16$

b. there is no director circle to the hyperbola

c. centre of the director circle is (0, 0)

d. length of latus rectum of the hyperbola = 12

8. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$ then

a. $x_1+x_2+x_3+x_4=0$

b. $y_1+y_2+y_3+y_4=0$

c. $x_1x_2x_3x_4=c^4$

d. $y_1y_2y_3y_4=c^4$

9. The differential equation $\frac{dy}{dx}=\frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity

a. $\sqrt{\frac{3}{5}}$ b. $\sqrt{\frac{5}{3}}$

c. $\sqrt{\frac{2}{5}}$ d. $\sqrt{\frac{5}{2}}$

10. Circles are drawn on chords of the rectangular hyperbola $xy = 4$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose coordinates are

a. (2, 2) b. (2, –2) c. (–2, 2) d. (–2, –2)

11. The equation $(x - \alpha)^2 + (y - \beta)^2 = k(lx + my + n)^2$ represents

a. a parabola for $k < (l^2 + m^2)^{-1}$
b. an ellipse for $0 < k < (l^2 + m^2)^{-1}$
c. a hyperbola for $k > (1^2 + m^2)^{-1}$
d. a point circle for $k = 0$

12. If P is a point on a hyperbola, then

a. locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci) is tangents at vertex
b. locus of excentre of the circle described opposite to $\angle S'$ is hyperbola
c. locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci), is hyperbola
d. locus of excentre of the circle described opposite to $\angle S'$, is tangent at vertex

13. The lines parallel to normal to the curve $xy = 1$ is/are

a. $3x + 4y + 5 = 0$ b. $3x - 4y + 5 = 0$
c. $4x + 3y + 5 = 0$ d. $3y - 4x + 5 = 0$

14. From point (2, 2) tangents are drawn to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ then point of contact lies in

a. I quadrant b. II quadrant
c. quadrant d. IV quadrant

15. If the two intersecting lines intersect the hyperbola and neither of them is a tangent to it, then number of intersecting points are

a. 1 b. 2 c. 3 d. 4

16. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn.

a. if $n = 1$, then $e = \sqrt{2}$
b. if $n > 1$, then $0 < e < \sqrt{2}$
c. if $n = 0$, then $e > \sqrt{2}$
d. none of these

17. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then

a. $PG = PC$ b. $Pg = PC$
c. $PG = Pg$ d. $Gg = 2PC$

Reasoning Type

Solutions on page 5.52

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.
b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.
c. Statement 1 is True and Statement 2 is False.
d. Statement 1 is False and Statement 2 is True.

1. **Statement 1:** Asymptotes of hyperbola $3x + 4y = 2$ and $4x - 3y = 5$ are bisectors of transverse and conjugate axes of hyperbola.

Statement 2: Transverse and conjugate axes of hyperbola are bisectors of the asymptotes.

2. **Statement 1:** Every line which cuts the hyperbola in two distinct points has slope lies in $(-2, 2)$.

Statement 2: Slope of tangents of hyperbola lies in $(-\infty, -2) \cup (2, \infty)$.

3. **Statement 1:** A bullet is fired and it hits a target. An observer in the same plane heard two sounds, the crack of the rifle and the thud of the bullet striking the target at the same instant, then locus of the observer is hyperbola where velocity of sound is smaller than velocity of bullet.

Statement 2: If difference of distances of a point 'P' from the two fixed points is constant and less than the distance between the fixed points then locus of 'P' is a hyperbola.

4. **Statement 1:** Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $12x^2 - 4y^2 = 27$ intersect each other at right angle.

Statement 2: Given ellipse and hyperbola have same foci.

5. **Statement 1:** If a circle S = 0 intersects a hyperbola $xy = 4$ at four points. Three of them are (2, 2), (4, 1) and (6, 2/3), then coordinates of the fourth point are (1/4, 16).

Statement 2: If a circle S = 0 intersects a hyperbola $xy = c^2$ at t_1, t_2, t_3, t_4, then $t_1 - t_2 - t_3 - t_4 = 1$.

6. **Statement 1:** If a point (x_1, y_1) lies in the shaded region $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, shown in the figure, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 0$.

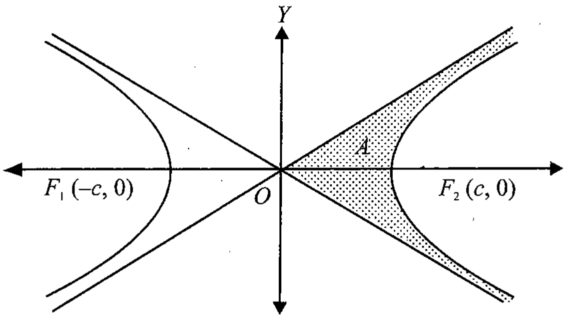


Fig. 5.39

Statement 2: If $P(x_1, y_1)$ lies outside the a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$.

7. **Statement 1:** Equations of tangents to the hyperbola $2x^2 - 3y^2 = 6$ which is parallel to the line $y = 3x + 4$ is $y = 3x - 5$ and $y = 3x + 5$.

   Statement 2: For given slope two parallel tangents can be drawn to the hyperbola.

8. **Statement 1:** There are infinite points from which two mutually perpendicular tangents can be drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

   Statement 2: The locus of point of intersection of perpendicular tangents lies on the circle.

9. **Statement 1:** If from any point $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, tangents are draws to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then corresponding chord of contact lies on another branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

   Statement 2: From any point outside the hyperbola two tangents can be drawn to the hyperbola.

10. **Statement 1:** If (3, 4) is a point on a hyperbola having focus (3, 0) and (λ, 0) and length of the transverse axis being 1 unit then λ can take the value 0 or 3.

    Statement 2: $|S'P - SP| = 2a$, where S and S' are the two foci, $2a$ = length of the transverse axis and P be any point on the hyperbola.

11. **Statement 1:** Given the base BC of the triangle and the ratio radius of the ex-circles opposite to the angles B and C. Then locus of the vertex A is hyperbola.

    Statement 2: $|S'P - SP| = 2a$, where S and S' are the two foci, $2a$ = length of the transverse axis and P be any point on the hyperbola.

Linked Comprehension Type

Solutions on page 5.53

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

A conic passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the coordinate axes is bisected at the point of tangency.

1. The eccentricity of the conic is
   a. 2 **b.** $\sqrt{2}$ **c.** $\sqrt{3}$ **d.** $\sqrt{\frac{3}{2}}$
2. The foci of the conic are
   a. $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$
   b. $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$
   c. (4, 4) and (–4, –4)
   d. $(4\sqrt{2}, 4\sqrt{2})$ and $(-4\sqrt{2}, -4\sqrt{2})$
3. The equations of directrix are
   a. $x + y = \pm 8$ **b.** $x + y = \pm 4$
   c. $x + y = \pm 4\sqrt{2}$ **d.** none of these

For Problems 4–6

The locus of foot of perpendicular from any focus of a hyperbola upon any tangent to the hyperbola is the auxiliary circle of the hyperbola. Consider the foci of a hyperbola as (–3, –2) and (5, 6) and the foot of perpendicular from the focus (5, 6) upon a tangent to the hyperbola as (2, 5).

4. The conjugate axis of the hyperbola is
   a. $4\sqrt{11}$ **b.** $2\sqrt{11}$ **c.** $4\sqrt{22}$ **d.** $2\sqrt{22}$
5. The directrix of the hyperbola corresponding to the focus (5, 6) is
   a. $2x + 2y - 1 = 0$ **b.** $2x + 2y - 11 = 0$
   c. $2x + 2y - 7 = 0$ **d.** $2x + 2y - 9 = 0$
6. The point of contact of the tangent with the hyperbola is
   a. $\left(\frac{2}{9}, \frac{31}{3}\right)$ **b.** $\left(\frac{7}{4}, \frac{23}{4}\right)$ **c.** $\left(\frac{2}{3}, 9\right)$ **d.** $\left(\frac{7}{9}, 7\right)$

For Problems 7–9

Let $P(x, y)$ is a variable point such that

$\left|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5) + (y-5)^2}\right| = 3$ which represents hyperbola.

7. The eccentricity e' of the corresponding conjugate hyperbola is
   a. $\frac{5}{3}$ **b.** $\frac{4}{3}$ **c.** $\frac{5}{4}$ **d.** $\frac{3}{\sqrt{7}}$
8. Locus of intersection of two perpendicular tangents to the hyperbola is

a. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{55}{4}$

b. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{25}{4}$

c. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{7}{4}$

d. none of these

9. If origin is shifted to point $\left(3, \frac{7}{2}\right)$ and the axes are rotated through an angle θ is clockwise sense so that equation of given hyperbola changes to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ is

a. $\tan^{-1}\left(\frac{4}{3}\right)$ b. $\tan^{-1}\left(\frac{3}{4}\right)$

c. $\tan^{-1}\left(\frac{5}{3}\right)$ d. $\tan^{-1}\left(\frac{3}{5}\right)$

For Problems 10–12

In hyperbola portion of tangent intercepted between asymptotes is bisected at the point of contact.

Consider a hyperbola whose centre is at origin. A line $x+y=2$ touches this hyperbola at $P(1, 1)$ and interests the asymptotes at A and B such that $AB=6\sqrt{2}$ units.

10. Equation of asymptotes are

a. $5xy+2x^2+2y^2=0$

b. $3x^2+4y^2+6xy=0$

c. $2x^2+2y^2-5xy=0$

d. none of these

11. Angle subtended by AB at centre of the hyperbola is

a. $\sin^{-1}\frac{4}{5}$ b. $\sin^{-1}\frac{2}{5}$ c. $\sin^{-1}\frac{3}{5}$ d. none of these

12. Equation of the tangent to the hyperbola at $\left(-1, \frac{7}{2}\right)$ is

a. $5x+2y=2$ b. $3x+2y=4$

c. $3x+4y=11$ d. none of these

For Problems 13–15

A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy=16$ is equal to the sum of ordinates of feet of normals. The locus of P is a curve C.

13. The equation of the curve C is

a. $x^2=4y$ b. $x^2=16y$ c. $x^2=12y$ d. $y^2=8x$

14. If the tangent to the curve C cuts the co-ordinate axes at A and B, then the locus of the middle point of AB is

a. $x^2=4y$ b. $x^2=2y$ c. $x^2+2y=0$ d. $x^2+4y=0$

15. Area of the equilateral triangle, inscribed in the curve C, having one vertex as the vertex of curve C is

a. $772\sqrt{3}$ sq. units b. $776\sqrt{3}$ sq. units

c. $760\sqrt{3}$ sq. units d. $768\sqrt{3}$ sq. units

For Problems 16–18

The vertices of ΔABC lie on a rectangular hyperbola such that the orthocentre of the triangle is (3, 2) and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point (1, 1).

16. The equation of the asymptotes is

a. $xy-1=x-y$ b. $xy+1=x+y$

c. $2xy=x+y$ d. none of these

17. Equation of the rectangular hyperbola is

a. $xy=2x+y-2$ b. $2xy=x+2y+5$

c. $xy=x+y+1$ d. none of these

18. Number of real tangents that can be drawn from the point (1, 1) to the rectangular hyperbola is

a. 4 b. 0 c. 3 d. 2

Matrix-Match Type

Solutions on page 5.56

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, b → q, b → r, c → p, c → q, and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Let the foci of the hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ be the vertices of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci of the ellipse be the vertices of the hyperbola. Let the eccentricities of the ellipse and hyperbola be e_E and e_H, respectively, then match the following.

Column I	Column II
a. $\frac{b}{B}$ is equal to	p. 1
b. e_H+e_E is always greater than	q. 2
c. if angle between the asymptotes of hyperbola is $\frac{2\pi}{3}$, then $4e_E$ is equal to	r. 3
d. If $e_E^2=\frac{1}{2}$ and (x, y) is point of intersection of ellipse and the hyperbola then $\frac{9x^2}{2y^2}$ is	s. 4

2.

Column I	Column II
a. The points common to the hyperbola $x^2 - y^2 = 9$ and circle $x^2 + y^2 = 41$ are	**p.** $(-5, -4)$
b. Tangents are drawn from point $\left(0, -\frac{9}{4}\right)$ to the hyperbola $x^2 - y^2 = 9$, then the point of tangency may have coordinate(s)	**q.** $(5, 4)$
c. The point which is diametrically opposite of point (5, 4) with respect to the hyperbola $x^2 - y^2 = 9$ is	**r.** $(-5, 4)$
d. If P and Q lie on the hyperbola $x^2 - y^2 = 9$ such that area of the isosceles triangle PQR where $PR = QR$ is 10 sq. units, where $R \equiv (0, -6)$, then P can have the co-ordinate(s)	**s.** $(5, -4)$

3. $A(-2, 0)$ and $B(2, 0)$ are the two fixed points and P is a point such that $PA - PB = 2$. Let S be the circle $x^2 + y^2 = r^2$, then match the following.

Column I	Column II
a. If $r = 2$, then the number of points P satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	**p.** 2
b. If $r = 1$, then he number of points satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	**q.** 4
c. For $r = 2$ the number of common tangents is	**r.** 0
d. For $r = 1/2$, the number of common tangents is	**s.** 1

4.

Column I	Column II
a. If z is a complex number such that Im $(z^2) = 3$, then eccentricity of the locus is	**p.** $\sqrt{3}$
b. If the latus rectum of a hyperbola through one focus subtends 60° angle at the other focus, then its eccentricity is	**q.** 2
c. If $A(3, 0)$ and $B(-3, 0)$ and $PA - PB = 4$, then eccentricity of conjugate hyperbola is	**r.** $\sqrt{2}$
d. If the angle between the asymptotes of hyperbola is $\pi/3$, then the eccentricity of its conjugate hyperbola is	**s.** $\frac{3}{\sqrt{5}}$

5. If e_1 and e_2 are the roots of the equation $x^2 - ax + 2 = 0$, then match the following.

Column I	Column II
a. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, respectively then the values of a are	**p.** 6
b. If both e_1 and e_2 are the eccentricities of the hyperbolas, then values of a are	**q.** $\frac{5}{2}$
c. If e_1 and e_2 are eccentricities of hyperbola and conjugate hyperbola, then values of a are	**r.** $2\sqrt{2}$
d. If e_1 is the eccentricity of the hyperbola for which there exists infinite points from which perpendicular tangents can be drawn and e_2 is the eccentricity of the hyperbola in which no such points exist then the values of a are	**s.** 5

Integer Type

Solutions on page 5.57

1. Eccentricity of the hyperbola $\left|\sqrt{(x-3)^2 + (y-2)^2} - \sqrt{(x+1)^2 + (y+1)^2}\right| = 1$ is
2. If $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having eccentricity 5, then the least positive integral value of m is
3. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and $x, y \in R$. Number of points in which the two graphs intersect is
4. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x-axis. If $\tan\theta . \tan\phi = 2$, then the value of $2\alpha^2 - \beta^2$ is
5. If tangents drawn from the point $(a, 2)$ to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are perpendicular, then the value of a^2 is
6. If hyperbola $x^2 - y^2 = 4$ is rotated by 45° in anticlockwise direction about its center keeping the axis intact then equation of hyperbola is $xy = a^2$, where a^2 is equal to
7. The area of triangle formed by the tangents from point (3, 2) to hyperbola $x^2 - 9y^2 = 9$ and the chord of contact w.r.t. point (3, 2) is

8. If a variable line has its intercepts on the co-ordinates axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$
9. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then ratio of square of its conjugate axis to the square of its transverse axis is
10. If distance between two parallel tangents having slope m drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then the value of $2|m|$ is
11. If L is the length of latus rectum of hyperbola for which $x = 3$ and $y = 2$ are the equations of asymptotes and which passes through the point (4, 6), then the value of $L/\sqrt{2}$ is
12. If the chord $x \cos\alpha + y \sin \alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, and the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is d then the value of $d/4$ is
13. A tangent drawn to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square units, with coordinate axes. If the eccentricity of hyperbola is e, then the value of $e^2 - 9$ is
14. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\theta = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$, then smallest positive value of θ is $\frac{\pi}{p}$, value of 'p' is
15. If locus of a point, whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is $xy = c^2$, then value of c^2 is

Archives

Solutions on page 5.59

Subjective Type

1. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid-point of the chord of contact. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. An ellipse has eccentricity $\frac{1}{2}$ and one focus at $S\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent (nearer to S) to the circle $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. The equation of the ellipse in standard form is_________. **(IIT-JEE, 1996)**

Multiple choice questions with one correct answer

1. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, $r > 1$ represents
   a. an ellipse b. a hyperbola
   c. a circle d. none of these
   which is not possible for any values of x and y. **(IIT-JEE, 1981)**
2. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and (x_2, y_2) in the region, the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is also in the region. The inequality defining this region is
   a. $x^2 + 2y^2 \le 1$
   b. $\max \{|x|, |y|\} \le 1$
   c. $x^2 - y^2 \le 1$
   d. $y^2 - x \le 0$ **(IIT-JEE, 1981)**
3. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
   a. no locus if $k > 0$ b. an ellipse if $k > 0$
   c. a point if $k = 0$ d. a hyperbola if $k > 0$
   (IIT-JEE, 1994)
4. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
   a. no locus if $k > 0$
   b. an ellipse if $k < 0$
   c. a point if $k = 0$
   d. a hyperbola if $k > 0$
   (IIT-JEE, 1994)
5. Let P ($a \sec \theta$, $b \tan \theta$) and Q ($a \sec \phi$, $b \tan \phi$), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to
   a. $\frac{a^2 + b^2}{a}$ b. $-\left(\frac{a^2 + b^2}{a}\right)$
   c. $\frac{a^2 + b^2}{b}$ d. $-\left(\frac{a^2 + b^2}{b}\right)$ **(IIT-JEE, 1999)**
6. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is
   a. $9x^2 - 8y^2 + 18x - 9 = 0$
   b. $9x^2 - 8y^2 - 18x + 9 = 0$

c. $9x^2 - 8y^2 - 18x - 9 = 0$

d. $9x^2 - 8y^2 + 18x + 9 = 0$ **(IIT-JEE, 1999)**

7. Which of the following is independent of α in the hyperbola $(0 < \alpha < \frac{\pi}{2})\ \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$?

a. eccentricity b. abscissa of foci

c. directrix d. vertex

8. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

a. $(-2, \sqrt{6})$ b. $(-5, 2\sqrt{6})$

c. $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ d. $(4, -\sqrt{6})$ **(IIT-JEE, 2004)**

9. A hyperbola, having the transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

a. $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

b. $x^2 \sec^2 - y^2 \operatorname{cosec}^2\theta = 1$

c. $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$

d. $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$ **(IIT-JEE, 2007)**

10. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

a. $1 - \sqrt{\frac{2}{3}}$ b. $\sqrt{\frac{3}{2}} - 1$

c. $1 + \sqrt{\frac{2}{3}}$ d. $\sqrt{\frac{3}{2}} + 1$ **(IIT-JEE, 2008)**

11. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

a. four straight lines, when $c = 0$ and a, b are of the same sign

b. two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

c. two straight lines and hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

d. a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

(IIT-JEE, 2009)

12. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x–axis at $(9, 0)$, then the eccentricity of the hyperbola is

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$

(C) $\sqrt{2}$ (D) $\sqrt{3}$ **(IIT-JEE, 2011)**

Multiple choice questions with one or more than one correct answer

1. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then

a. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

b. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

c. focus of hyperbola is $(5, 0)$

d. vertex of hyperbola is $(5\sqrt{3}, 0)$ **(IIT-JEE, 2006)**

2. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then **(IIT-JEE, 2009)**

a. equation of ellipse is $x^2 + 2y^2 = 2$

b. the foci of ellipse are $(\pm 1, 0)$

c. equation of ellipse is $x^2 + 2y^2 = 4$

d. the foci of ellipse are $(\pm\sqrt{2}, 0)$

3. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

a. equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

b. a focus of hyperbola is $(2, 0)$

c. eccentricity of hyperbola is $\frac{2}{\sqrt{3}}$

d. equation of hyperbola is $x^2 - 3y^2 = 3$

(IIT-JEE, 2011)

Comprehension Type

For Problems 1–2

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

1. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

   a. $2x - \sqrt{5}y - 20 = 0$ b. $2x - \sqrt{5}y + 4 = 0$
   c. $3x - 4y + 8 = 0$ d. $4x - 3y + 4 = 0$

   (IIT-JEE, 2010)

2. Equation of the circle with AB as its diameter is

   a. $x^2 + y^2 - 12x + 24 = 0$ b. $x^2 + y^2 + 12x + 24 = 0$
   c. $x^2 + y^2 + 24x - 12 = 0$ d. $x^2 + y^2 - 24x - 12 = 0$

   (IIT-JEE, 2010)

Match the following

1. Match the conic in Column I with the statements/ expressions in Column II. **(IIT-JEE, 2009)**

Column I	Column II
a. Circle	p. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b. Parabola	q. Point z in the complex plane satisfying $\lvert z + 2\rvert - \lvert z - 2\rvert = \pm 3$
c. Ellipse	r. Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
d. Hyperbola	s. The eccentricity of the conic lies in the interval $1 \le x < \infty$
	t. Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

Integer type

1. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **(IIT-JEE, 2010)**

ANSWERS AND SOLUTIONS

Subjective Type

1.

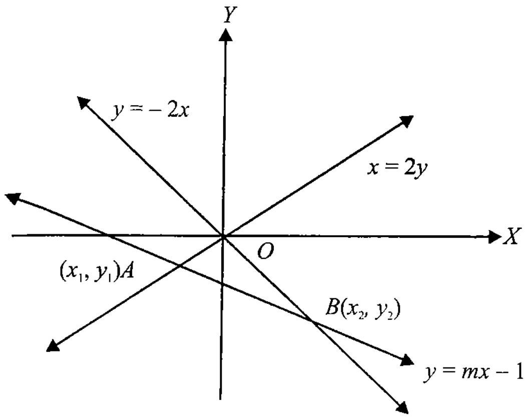


Fig. 5.40

Solving the variable line $y = mx - 1$ with $x = 2y$, we get

$$x_1 = \frac{2}{2m-1} \quad \text{(i)}$$

Solving with $y = -2x$, we get

$$x_2 = \frac{1}{m+2}$$

Now, $y_1 + y_2 = m(x_1 + x_2) - 2$

Let centroid of the triangle OAB be (h, k)

So, $$h = \frac{x_1 + x_2}{3}$$

and $$k = \frac{y_1 + y_2}{3} = \frac{m(x_1 + x_2) - 2}{3}$$

$$\Rightarrow \quad m = \frac{3k+2}{3h}$$

So, $$3h = x_1 + x_2 = \frac{2}{2\left(\frac{3k+2}{3h}\right) - 1} + \frac{1}{\left(\frac{3k+2}{3h}\right) + 2}$$

[using (i)]

$$\Rightarrow \frac{2}{6k - 3h + 4} + \frac{1}{6h + 3k + 2} = 1$$

Simplifying we get the final locus as $6x^2 - 9xy - 6y^2 - 3x - 4y = 0$ which is a hyperbola passing through origin, as $h^2 > ab$ and $\Delta \neq 0$.

2.

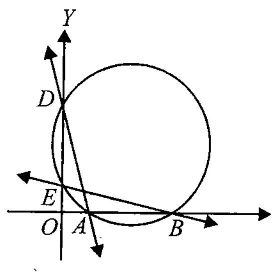


Fig. 5.41

Let the tangent be $y = m_1x + \sqrt{a^2m_1^2 - b}$

$y = m_2x + \sqrt{a^2m_2^2 - b^2}$

Points of intersection of these tangents with axes are

$A\left(\dfrac{-\sqrt{a^2m_1^2-b^2}}{m_1}, 0\right), C\left(0, \sqrt{a^2m_1^2-b^2}\right),$

$B\left(-\dfrac{\sqrt{a^2m_2^2-b^2}}{m_2}, 0\right) D\left(0, \sqrt{a^2m_2^2-b^2}\right).$

Now as the four points are concyclic

$$OA\cdot OB = OC\cdot OD$$

$\Rightarrow \left(\dfrac{-\sqrt{a^2m_1^2-b^2}}{m_1}\right)\left(\dfrac{-\sqrt{a^2m_2^2-b^2}}{m_2}\right) = \sqrt{a^2m_1^2-b^2}\sqrt{a^2m_2^2-b^2}$

$\Rightarrow \qquad m_1m_2 = 1$

3. Consider any point P $(a \sec\theta, a\tan\theta)$ on $x^2 - y^2 = a^2$. This point will be nearest to $y = 2x$, if tangent at this point is parallel to $y = 2x$.

Differentiating $x^2 - y^2 = a^2$ w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\therefore \left[\dfrac{dy}{dx}\right]_{(a\sec\theta,\, a\tan\theta)} = \operatorname{cosec}\theta$

Slope of $y = 2x$ is 2,

$\Rightarrow \qquad \operatorname{cosec}\theta = 2$ or $\theta = \pi/6$

Thus $\qquad P \equiv (a\sec\pi/6, a\tan\pi/6)$

$$\equiv \left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) \equiv (h, k)$$

$$h = \frac{2a}{\sqrt{3}} \text{ and } k = \frac{a}{\sqrt{3}} \Rightarrow k = \frac{h}{2}$$

So required locus is $2y - x = 0$.

4. Equation of family of curves, passing through the points of intersection of $x^2 - y^2 = a^2$ and $y = x^2$ is

$x^2 - y^2 - a^2 + \lambda(x^2 - y) = 0$

$\Rightarrow x^2(1 + \lambda) - y^2 - a^2 - \lambda y = 0$

It will be a circle if $\quad \lambda = -2$

$\Rightarrow \quad -x^2 - y^2 - a^2 + 2y = 0$

$\Rightarrow \quad x^2 + y^2 - 2y = -a^2$

$\Rightarrow \quad x^2 + (y-1)^2 = 1 - a^2$

$\Rightarrow \quad 1 - a^2 > 0 \Rightarrow a^2 < 1$

$\Rightarrow \quad a \in (-1, 1) \qquad$ (i)

Also both the curves will intersect at real points if $y^2 - y + a^2 = 0$ for some real y,

if $\qquad -\dfrac{1}{2} < a < \dfrac{1}{2}$

(i) and (ii), $a \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

5.

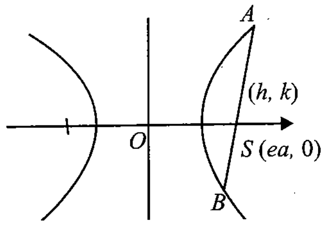


Fig. 5.42

Equation of chord AB with $T = S_1$ is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

If passes through $(ae, 0)$

$$\Rightarrow \quad \frac{he}{a} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

Locus is $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{ex}{a}$

$$\Rightarrow \quad \frac{1}{a^2}[x^2 - aex] - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \quad \frac{\left(x - \frac{ae}{2}\right)^2 - \frac{a^2e^2}{4}}{a^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \quad \frac{\left(x - \frac{ae}{2}\right)^2}{a^2} - \frac{y^2}{b^2} = \frac{e^2}{4}$$

which is also a hyperbola with eccentricity e.

6.

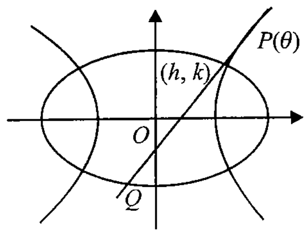


Fig. 5.43

Tangent to hyperbola at $P(\theta)$ is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \qquad \text{(i)}$$

Also the chord of the ellipse with middle point (h, k) is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} = \lambda \text{ (say)} \qquad \text{(ii)}$$

Comparing (i) and (ii), we get

$$\frac{\sec\theta}{a}\frac{a^2}{h} = -\frac{\tan\theta}{b}\frac{b^2}{k} = \frac{1}{\lambda}$$

$\Rightarrow$ $\sec\theta = \frac{h}{\lambda a}$ and $\tan\theta = -\frac{k}{\lambda b}$

$\therefore$ $\frac{h^2}{\lambda^2 a^2} - \frac{k^2}{\lambda^2 b^2} = 1$ (eliminating θ)

$\Rightarrow$ $\frac{h^2}{a^2} - \frac{k^2}{b^2} = \lambda^2 = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 = 1$

Hence, locus is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

7. Tangent at point P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$;

Normal at point P is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2e^2$

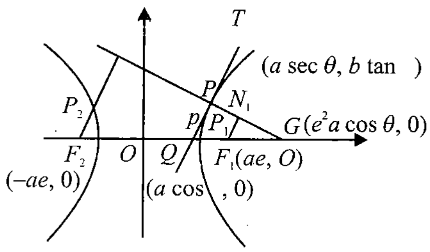


Fig. 5.44

From the diagram $\frac{p}{p_1} = \frac{QG}{F_1G}$

$$= \frac{e^2 a\sec\theta - a\cos\theta}{e^2 a\sec\theta - ae}$$

$$= \frac{e^2 - \cos^2\theta}{e^2 - e\cos\theta}$$

$$= \frac{e + \cos\theta}{e}$$

$\Rightarrow$ $\frac{p}{p_1} = 1 + \frac{\cos\theta}{e}$

Similarly, $\frac{p}{p_2} = 1 - \frac{\cos\theta}{e}$

$\Rightarrow$ $\frac{p}{p_1} + \frac{p}{p_2} = 2$

$\Rightarrow$ $\frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

8. Equation of tangent to hyperbola at point P ($a\sec\theta$, $b\tan\theta$) is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \quad \text{(i)}$$

Since (i) passes through the point $(0, c)$, therefore

$$-c\tan\theta = b \quad \text{(ii)}$$

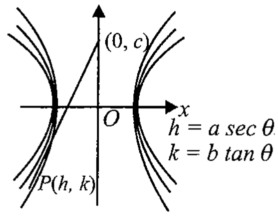


Fig. 5.45

Now substituting the value of b in (ii), we get

$$\frac{k}{c} = -\tan^2\theta \text{ and } \frac{h^2}{a^2} = \sec^2\theta$$

Adding, we have

$$\frac{k}{c} + \frac{h^2}{a^2} = 1$$

$\Rightarrow$ $\frac{h^2}{a^2} = 1 - \frac{k}{c}$

$\Rightarrow$ $h^2 = \frac{a^2}{c}(c - k)$

$x^2 = -\frac{a^2}{c}(y - c)$ which is a parabola

9. Hyperbola is $b^2x^2 - a^2y^2 = a^2b^2$ (i)

Let the transversal be $y = mx + c$ (ii)

Solving (i) and (ii), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$\Rightarrow (b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$

$\Rightarrow$ $\frac{x_1 + x_2}{2} = \frac{2a^2mc}{b^2 - a^2m^2}$ (iii)

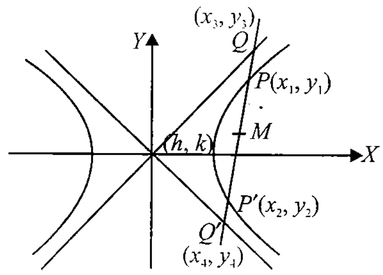


Fig. 5.46

Solving $y = mx + c$ with pair of asymptotes $b^2x^2 - a^2y^2 = 0$, we have

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2c^2 = 0 \quad \text{(iv)}$$

$\Rightarrow$ $\frac{x_3 + x_4}{2} = \frac{2a^2mc}{b^2 - a^2m^2}$ (v)

$\Rightarrow$ $MQ = MQ'$ and $MP = MP'$

$\Rightarrow$ $PQ = P'Q'$

10. Equation of GL with slope $-a/b$ and passing through $(e^2x_1, 0)$ is

$$y - 0 = -\frac{a}{b}(x - e^2x_1)$$

$$ax + by = ae^2x_1 \quad \text{(i)}$$

Put $$y = \frac{b}{a}x$$

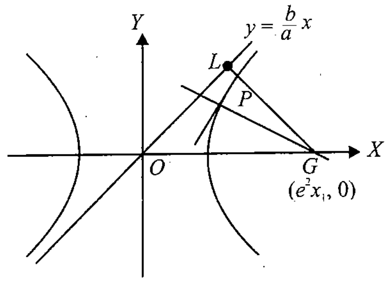


Fig. 5.47

$$ax + b\frac{b}{a}x = ae^2x_1$$

$$x\left[\frac{a^2+b^2}{a}\right] = ae^2x_1$$

$$x = x_1$$

Thus abscissa of point L is x_1 which is same as that of point P.

Hence, LP is parallel to conjugate axis.

11. For first hyperbola,

$$(y - mx)\left(m\frac{dy}{dx} + 1\right) + (my + x)\left(\frac{dy}{dx} - m\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1$$

For second hyperbola,

$$(m^2 - 1)\left(2y\frac{dy}{dx} - 2x\right) + 4m\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2$$

$\therefore \quad m_1m_2 = -1$

$\Rightarrow$ Angle between the hyperbolas $= \frac{\pi}{2}$

Objective Type

1. a. According to the question $\frac{2\sqrt{9m^2-49}}{\sqrt{1+m^2}} = 2$

$$\Rightarrow \quad 9m^2 - 49 = 1 + m^2$$

$$\Rightarrow \quad 8m^2 = 50$$

$$\Rightarrow \quad m = \pm\frac{5}{2}$$

2. a. Given that

$$\frac{\text{Distance between foci}}{\text{Distance between two directrix}} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{2ae}{2\frac{a}{e}} = \frac{3}{2}$$

$$\Rightarrow \quad e^2 = \frac{3}{2}$$

$$\Rightarrow \quad 1 + \frac{b^2}{a^2} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{1}{\sqrt{2}}$$

3. c. $P\left(a\sec\frac{\pi}{2}, b\tan\frac{\pi}{6}\right) \equiv P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

Therefore, equation of tangent at P is $\frac{x}{\frac{\sqrt{3}a}{2}} - \frac{y}{\sqrt{3}b} = 1$

$\therefore$ Area of the triangle $= \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$

$$\therefore \quad \frac{b}{a} = 4$$

$$\therefore \quad e^2 = 1 + \frac{b^2}{a^2} = 17$$

4. b. Transverse axis is along the line $y = x$.

Solving $y = x$ and $xy = 18$, we have $x^2 = 18$ or $x = \pm3\sqrt{2}$.

Then two vertices of the hyperbola are $(\pm3\sqrt{2}, \pm3\sqrt{2})$.

Distance between them $= \sqrt{72 + 72} = 12$.

5. a. Let the given straight line be axis of coordinates and let the equation of the variable line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line cuts the coordinate axis at the point A $(a, 0)$ and B $(0, b)$.

Therefore, the area of ΔAOB is

$$\frac{1}{2}ab = c^2$$

$$\Rightarrow \quad ab = 2c^2 \quad \text{(i)}$$

If (h, k) be the coordinates of the middle point of AB, then

$$h = \frac{a}{2} \text{ and } k = \frac{b}{2}$$

On eliminating a and b from Eqns. (i) and (ii), we get

$$2hk = c^2$$

Hence, the locus of (h, k) is $2xy = c^2$.

6. b. Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate hyperbola, therefore

$$\frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\Rightarrow \quad 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

The line passing through the points $(e, 0)$ and $(0, e')$ is

$$e'x + ey - ee' = 0$$

It is tangent to the circle $x^2 + y^2 = r^2$.

Hence, $$\frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$\Rightarrow \quad r^2 = \dfrac{e^2e'^2}{e^2 + e'^2} = 4$

$\Rightarrow \quad r = 2$

7. **c.** $(x-3)^2 + (y+1)^2 = (4x+3y)^2$

$\Rightarrow \quad (x-3)^2 + (y+1)^2 = 25\left(\dfrac{4x+3y}{5}\right)^2$

$\Rightarrow \quad PS = 5PM$

$\Rightarrow$ directrix is $4x + 3y = 0$ and focus $(3, -1)$

So equation of transverse axis is $y + 1 = \frac{3}{4}(x-3)$

$\Rightarrow \quad 3x - 4y = 13$

8. **a.** Given that

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{x^2 + y^2}$$

$\Rightarrow \quad y^2\left(\dfrac{dy}{dx}\right) = x^2$

$\Rightarrow \quad ydy = \pm xdx$

$\Rightarrow \quad y^2 \pm x^2 = k^2$

$\Rightarrow$ family of curves may be a circle or rectangular hyperbola

9. **b.** We have

$x^2 - y^2 - 4x + 4y + 16 = 0$

$\Rightarrow \quad (x-2)^2 - (y-2)^2 = -16$

$\Rightarrow \quad \dfrac{(x-2)^2}{4^2} - \dfrac{(y-2)^2}{4^2} = -1$

This is a rectangular hyperbola, whose eccentricity is always $\sqrt{2}$.

10. **d.** We have

$16(x^2 - 2x) - 3(y^2 - 4y) = 44$

$\Rightarrow \quad 16(x-1)^2 - 3(y-2)^2 = 48$

$\Rightarrow \quad \dfrac{(x-1)^2}{3} - \dfrac{(y-2)^2}{16} = 1$

This equation represents a hyperbola with eccentricity

$$e = \sqrt{1 + \frac{16}{3}} = \sqrt{\frac{19}{3}}$$

11. **c.** Let the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

then $\quad 2a = ae$, i.e., $e = 2$

$\therefore \quad \dfrac{b^2}{a^2} = e^2 - 1 = 3$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2} = 3$

12. **c.** We have

$\dfrac{2b^2}{a} = 8$

and $\quad 2b = \frac{1}{2}(2ae)$

$\therefore \quad \frac{2}{a}\left(\frac{ae}{2}\right)^2 = 8$

$\Rightarrow \quad ae^2 = 16 \qquad$ (i)

Also, $\quad 2\dfrac{b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

$\Rightarrow \quad a^2(e^2 - 1) = 4a$

$\Rightarrow \quad ae^2 - a = 4 \qquad$ (ii)

From (i) and (ii), we have

$16 - \dfrac{16}{e^2} = 4$

$\Rightarrow \quad \dfrac{16}{e^2} = 12$

$\Rightarrow \quad e = \dfrac{2}{\sqrt{3}}$

13. **d.**

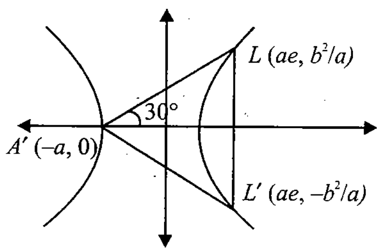


Fig. 5.48

$\tan 30° = \dfrac{\frac{b^2}{a}}{a + ae}$

$\Rightarrow \quad \dfrac{1+e}{\sqrt{3}} = e^2 - 1$

$\Rightarrow \quad e - 1 = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad e = \dfrac{\sqrt{3}+1}{\sqrt{3}}$

14. **d.** Hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ can be written as

$9(x^2 - 2x) - 16(y^2 + 2y) = 151$

$\Rightarrow \quad 9(x-1)^2 - 16(y+1)^2 = 151 + 9 - 16 = 144$

$\Rightarrow \quad \dfrac{(x-1)^2}{16} - \dfrac{(y+1)^2}{9} = 1$

or $\quad \dfrac{X^2}{16} - \dfrac{Y^2}{9} = 1$

[where $X = x - 1$, $Y = y + 1$]

Here $\quad a^2 = 16, b^2 = 9$

Latus rectum $= 2\dfrac{b^2}{a} = \dfrac{2(9)}{4} = \dfrac{9}{2}$

15. **a.** Given hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{\frac{1}{3}} = 1$$

Its eccentricity 'e' is given by

$$\frac{1}{3} = 1\,(e^2 - 1)$$

Hence, eccentricity e' of the conjugate hyperbola is given by

$$1 = \frac{1}{3}(e'^2 - 1)$$

$\Rightarrow$ $e'^2 = 4$

$\Rightarrow$ $e' = 2$

16. b. The equation of the hyperbola is

$$\frac{\left(\frac{2x - y + 4}{\sqrt{5}}\right)^2}{\frac{1}{2}} - \frac{\left(\frac{x + 2y - 3}{\sqrt{5}}\right)^2}{\frac{1}{3}} = 1$$

or $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$

17. b. Eliminating t from the given two equation, we have

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

whose eccentricity is

$$e = \sqrt{1 + \frac{48}{16}} = 2$$

18. b. For hyperbola

$$\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$$

We have

$$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5}$$

$$= 1 + \cos^2\alpha$$

For ellipse

$$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$$

We have

$$e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha$$

Given that

$$e_1 = \sqrt{3}e_2$$

$\Rightarrow$ $e_1^2 = 3e_2^2$

$\Rightarrow$ $1 + \cos^2\alpha = 3\sin^2\alpha$

$\Rightarrow$ $2 = 4\sin^2\alpha$

$\Rightarrow$ $\sin\alpha = \frac{1}{\sqrt{2}}$

19. b. Given hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{(i)}$$

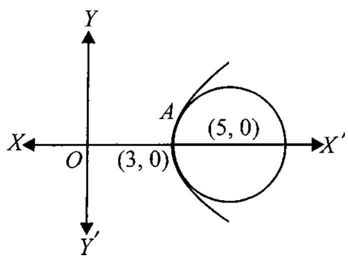


Fig. 5.49

$\Rightarrow$ $e^2 = 1 + \frac{16}{9} = \frac{25}{9}$

$\Rightarrow$ $e = \frac{5}{3}$

Hence, its foci are $(\pm 5, 0)$.

The equation of the circle with $(5, 0)$ as centre is

$$(x - 5)^2 + y^2 = r^2 \quad \text{(ii)}$$

Solving (i) and (ii), we have

$$16x^2 - 9[r^2 - (x - 5)^2] = 144$$

or $25x^2 - 90x - 9r^2 + 81 = 0$

Since the circle touches the hyperbola, above equation must have equal roots. Hence,

$$90^2 - 4(25)(81 - 9r^2) = 0$$

$\Rightarrow$ $9 - (9 - r^2) = 0$

$\Rightarrow$ $r = 0$ which is not possible.

Hence, the circle cannot touch at two points.

It can only be tangent at the vertex. Hence, $r = 5 - 3 = 2$.

20. a. Any tangent to hyperbola is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad \text{(i)}$$

Given tangent is

$$ax + by = 1 \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we have

$$\sec\theta = a^2 \text{ and } \tan\theta = -b^2$$

Eliminating θ, we have

$$a^4 - b^4 = 1$$

$\Rightarrow$ $(a^2 - b^2)(a^2 + b^2) = 1$

Also $a^2 + b^2 = a^2e^2$

$\Rightarrow$ $a^2 - b^2 = \frac{1}{a^2e^2}$

21. c. Let the point be (h, k).

Then equation of the chord of contact is $hx + ky = 4$.

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore$ $x\left(\frac{4 - hx}{k}\right) = 1$ has two equal roots

$\Rightarrow$ $hx^2 - 4x + k = 0$

$\Rightarrow$ $hk = 4$

$\Rightarrow$ locus of (h, k) is $xy = 4$

$\Rightarrow$ $c^2 = 4$

22. d. Let the vertex A be $(a\cos\theta, b\sin\theta)$

Since AC and AB touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

BC is the chord of contact. Its equation is

$$\frac{x\cos\theta}{a} - \frac{y\sin\theta}{b} = -1$$

or $-\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

or $\frac{x\cos(\pi-\theta)}{a}+\frac{y\sin(\pi-\theta)}{b}=1$

which is the equation of the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the point $(a\cos(\pi-\theta), b\sin(\pi-\theta))$.

Hence, BC touches the given ellipse.

23. b. Equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

It passing through $(0, -b)$. So,

$$0+\frac{y_1}{b}=1 \Rightarrow y_1=b$$

Equation of normal is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2e^2$$

which passes through $(2a\sqrt{2}, 0)$. Hence,

$$\frac{a^2 2a\sqrt{2}}{x_1}=a^2e^2$$

$$\Rightarrow \quad x_1=\frac{2a\sqrt{2}}{e^2}$$

Now (x_1, y_1) lies on the hyperbola

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \quad \frac{8}{e^4}-1=1$$

$$\Rightarrow \quad e^2=2$$

24. c. Equation of hyperbola is

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

Equation of tangent is

$$y=mx+\sqrt{9m^2-16}$$

$$\Rightarrow \quad \sqrt{9m^2-16}=2\sqrt{5}$$

$$\Rightarrow \quad m=\pm 2$$

$$\Rightarrow \quad a+b=\text{sum of roots}=0$$

25. b.

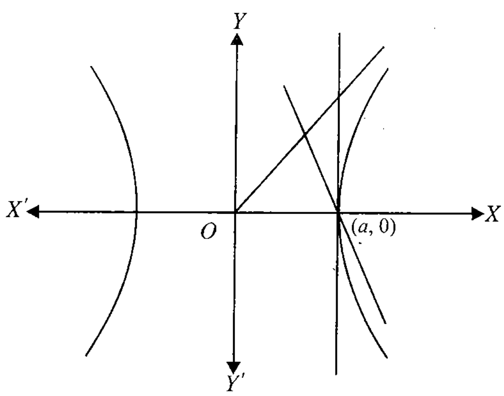


Fig. 5.50

The line $y+\lambda(x-a)=0$ will intersect the portion of the asymptote in the first quadrant only if its slope is negative. Hence,

$$-\lambda<0$$

$$\Rightarrow \quad \lambda>0$$

$$\therefore \quad \lambda\in(0,\infty)$$

26. d. Product of perpendiculars drawn from foci upon any of its tangents = 9

$$\Rightarrow \quad b^2=9$$

Also, $\frac{b}{a}=\tan 30°=\frac{1}{\sqrt{3}}$

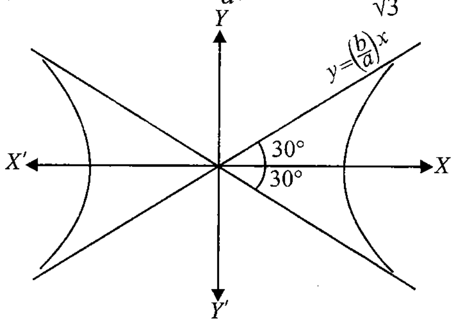


Fig. 5.51

$$\therefore \quad a^2=3b^2=27$$

Therefore, required locus is the director circle of the hyperbola which is given by $x^2+y^2=27-9$

or $\quad x^2+y^2=18$

if $\quad \frac{b}{a}=\tan 60°$, then

$$a^2=\frac{b^2}{3}=\frac{9}{3}=3$$

Hence, the required locus is $x^2+y^2=3-9=-6$ which is not possible.

27. c.

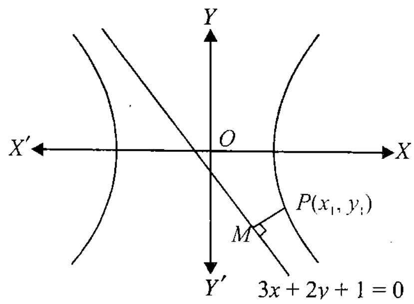


Fig. 5.52

Point P is nearest to the given line if tangent at P is parallel to the given line.

Now slope of tangent at $P(x_1, y_1)$ is

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{18y_1}{24x_1}=\frac{3}{4}\frac{y_1}{x_1}$ which must be equal to $-\frac{3}{2}$

$$\Rightarrow \quad \frac{3}{4}\frac{y_1}{x_1}=-\frac{3}{2}$$

$$\Rightarrow \quad y_1 = -2x_1 \quad \text{(i)}$$

Also (x_1, y_1) lies on the curve. Hence,

$$\frac{x_1^2}{24} - \frac{y_1^2}{18} = 1 \quad \text{(ii)}$$

Solving (i) and (ii) we get two points (6, −3) and (−6, 3) of which (6, −3) is nearest.

28. a. Tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at $P(3 \sec \theta, 2 \tan \theta)$ is

$$\frac{x}{3} \sec \theta - \frac{y}{2} \tan \theta = 1$$

This is perpendicular to

$$5x + 2y - 10 = 0$$

$$\Rightarrow \quad \frac{2 \sec \theta}{3 \tan \theta} = \frac{2}{5}$$

$$\Rightarrow \quad \sin \theta = \frac{5}{3} \text{ which is not possible.}$$

Hence, there is no such tangent.

29. d. $\dfrac{y^2}{\frac{1}{16}} - \dfrac{x^2}{\frac{1}{9}} = 1$

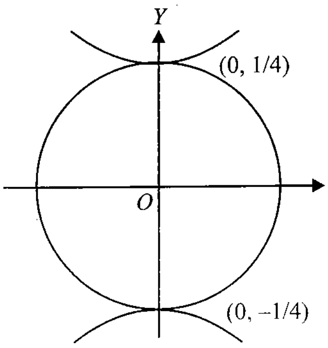


Fig. 5.53

Locus will be the auxiliary circle

$$x^2 + y^2 = \frac{1}{16}$$

30. d. Let the foot of perpendicular from $O(0, 0)$ to tangent to hyperbola is $P(h, k)$. Slope of $OP = \frac{k}{h}$

Then equation of tangent to hyperbola is

$$y - k = -\frac{h}{k}(x - h)$$

or $\quad hx + ky = h^2 + k^2$

Solving it with $\quad xy = 1$, we have

$$hx + \frac{k}{x} = h^2 + k^2$$

or $\quad hx^2 - (h^2 + k^2)x + k = 0$

This equation must have real and equal roots. Hence,

$$D = 0$$

$$\Rightarrow \quad (h^2 + k^2)^2 - 4hk = 0$$

$$\Rightarrow \quad (x^2 + y^2)^2 = 4xy$$

31. b.

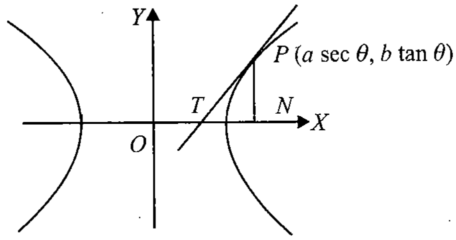


Fig. 5.54

Tangent at point P is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

It meets the x-axis at point T $(a \cos \theta, 0)$ and foot of perpendicular from P to x-axis is N $(a \sec \theta, 0)$

From the diagram,

$$OT = a \cos \theta \text{ and } ON = a \sec \theta$$

$$\Rightarrow \quad OT \cdot ON = a^2$$

32. b. Let directrix be $x = \frac{a}{e}$ and focus be S $(ae, 0)$. Let $P(a \sec \theta, b \tan \theta)$ be any point on the curve.

Equation of tangent at P is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$$

Let F be the intersection point of tangent of directrix.

Then $\quad F = \left(\frac{a}{e}, \frac{b(\sec \theta - e)}{e \tan \theta}\right)$

$$\Rightarrow \quad m_{SF} = \frac{b(\sec \theta - e)}{-e \tan \theta (a^2 - 1)},$$

$$m_{PS} = \frac{b \tan \theta}{a(\sec \theta - e)}$$

$$\Rightarrow \quad m_{SF} \cdot m_{PS} = -1$$

33. c. Let $y = mx \pm \sqrt{m^2 a^2 - a^2}$ be two tangents and passes through (h, k). Then

$$(k - mh)^2 = m^2 a^2 - a^2$$

$$\Rightarrow m^2(h^2 - a^2) - 2khm + k^2 + a^2 = 0$$

$$\Rightarrow \quad m_1 + m_2 = \frac{2kh}{h^2 - a^2}$$

and $\quad m_1 m_2 = \dfrac{k^2 + a^2}{h^2 - a^2},$

Now, $\quad \tan 45° = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$$\Rightarrow \quad 1 = \frac{(m_1 + m_2)^2 - 4m_1 m_2}{(1 + m_1 m_2)^2}$$

$\Rightarrow \quad \left(1+\frac{k^2+a^2}{h^2-a^2}\right)^2=\left(\frac{2kh}{h^2-a^2}\right)^2-4\left(\frac{k^2+a^2}{h^2-a^2}\right)$

$\Rightarrow \quad (h^2+k^2)^2=4h^2k^2-4(k^2+a^2)(h^2-a^2)$

$\Rightarrow \quad (x^2+y^2)^2=4(a^2y^2-a^2x^2+a^4)$

$\Rightarrow (x^2+y^2)^2+4a^2(x^2-y^2)=4a^2$

34. c. Fourth vertex of parallelogram lies on circumcircle

$\Rightarrow$ parallelogram is cyclic

$\Rightarrow$ parallelogram is a rectangle

$\Rightarrow$ tangents are perpendicular

$\Rightarrow$ locus of P is the director circle

35. a. Director circle of circle $x^2+y^2=a^2$ is $x^2+y^2=2a^2$.

The semi-transverse axis is $\sqrt{3}a$.

Radius of the circle is $\sqrt{2}a$.

Hence, director circle and hyperbola do not intersect.

36. d. The equation of the normal to the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ at $(2\sec\theta, \tan\theta)$ is $2x\cos\theta+y\cot\theta=5$

Slope of the normal is $-2\sin\theta=-1$

$\Rightarrow \quad \sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$

Y-intercept of the normal $=\frac{5}{\cot\theta}=\frac{5}{\sqrt{3}}$

As it touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We have $\quad a^2+b^2=\frac{25}{9}$

37. b. Equation of normals is given by $ty=t^3x-ct^4+c=0$. It passes through $(ct', c/t')$. Hence,

$\frac{c}{t'}=t^3ct'-ct^4+c=0$

$t=t^3t^2-tt^4+t'$

$t^3t'=-1$

38. a. Equation of normal at any point $\left(ct, \frac{c}{t}\right)$ is

$ct^4-xt^3+ty-c=0$

$\Rightarrow \quad$ Slope of normal $=t^2$

Let P be (h, k)

$\Rightarrow \quad ct^4-ht^3+tk-c=0$

$\Rightarrow \quad \Sigma t_i=\frac{h}{c}$ and $\Sigma t_it_j=0$

$\Rightarrow \quad \Sigma t_i^2=(\Sigma t_i)^2$

$\Rightarrow \quad h^2=c^2\lambda$

Therefore, the required locus is $x^2=\lambda c^2$.

39. d. The given hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

$\Rightarrow \quad e=\frac{5}{4}$

Its foci are $(\pm 5, 0)$.

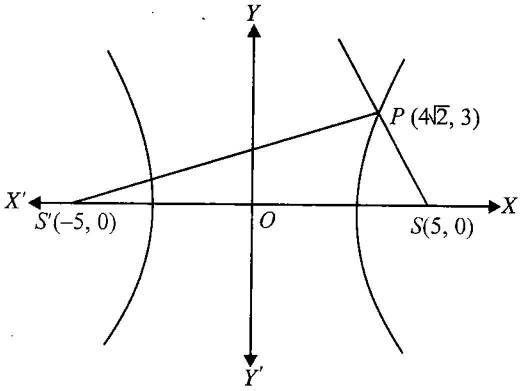


Fig. 5.55

The ray is incident at $P(4\sqrt{2}, 3)$.

The incident ray passes through $(5, 0)$; so the reflected ray must pass through $(-5, 0)$.

Its equation is $\quad \frac{y-0}{x+5}=\frac{3}{4\sqrt{2}+5}$

or $\quad 3x-y(4\sqrt{2}+5)+15=0$

40. b.

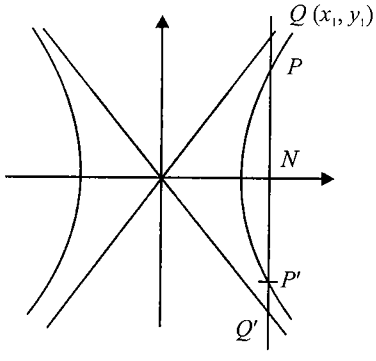


Fig. 5.56

$$NP=\frac{4}{5}\sqrt{x_1^2-25}$$

Q is on $y=\frac{4}{5}x$

$$NQ=\frac{4}{5}x_1$$

$$PQ=NQ-NP=\frac{4}{5}(x_1-\sqrt{x_1^2-25})$$

$$P'Q=\frac{4}{5}(x_1+\sqrt{x_1^2-25}$$

$$PQ\cdot P'Q=16$$

41. b. Let the asymptotes be $2x + 3y + \lambda_1 = 0$ and $x + 2y + \lambda_2 = 0$

It will pass through centre (1, 2). Hence,

$$\Rightarrow \quad \lambda_1 = -8, \lambda_2 = -5$$

The equation of the hyperbola is

$(2x + 3y - 8)(x + 2y - 5) + \lambda = 0$

It passes through (2, 4), therefore

$(4 + 12 - 8)(2 + 8 - 5) + \lambda = 0 \Rightarrow \lambda = -40$

Hence, equation of hyperbola is

$(2x + 3y - 8)(x + 2y - 5) = 40$

42. c. Let P be (h, k) be any point. The chord of contact of P w.r.t. the hyperbola is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \qquad \text{(i)}$$

The chord of contact of P w.r.t. the auxiliary circle is

$$hx + ky = a^2 - b^2 \qquad \text{(ii)}$$

Now, $\quad \frac{h}{a^2} \times \frac{b^2}{k} \times \left(-\frac{h}{k}\right) = -1$

$$\Rightarrow \quad \frac{h^2}{a^2} - \frac{k^2}{b^2} = 0$$

Therefore, P lies on one of the asymptotes.

43. c. Slopes of asymptotes are

$$m_1 = \frac{b_1}{a_1}, m_2 = \frac{b_2}{a_2}$$

According to the question,

$$m_1 m_2 = -1$$

$$\Rightarrow \quad a_1 a_2 + b_1 b_2 = 0$$

44. c. For equation $S + K = 0$ to represent a pair of lines,

$$\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & 1+k \end{vmatrix} = 0$$

$\Rightarrow 3(1 + k) - 1 - 2(2 + 2k + 2) - 2(2 + 6) = 0$

$$\Rightarrow \quad k = -22$$

45. a.

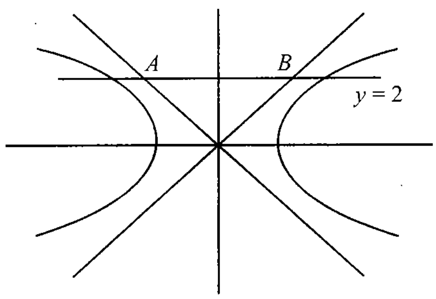


Fig. 5.57

For two distinct tangents on different branches the point should lie on the line $y = 2$ and between A and B (where A and B are the points on the asymptotes).

Equations of asymptotes are $4x = \pm 3y$

Solving with $\quad y = 2$, we have

$$x = \pm\frac{3}{2}$$

$$\therefore \quad -\frac{3}{2} < \alpha < \frac{3}{2}$$

46. d. Transverse axis is the equation of the angle bisector passing containing point (2, 3), which is given by

$$\frac{3x - 4y + 5}{5} = \frac{12x + 5y - 40}{13}$$

$$\Rightarrow \quad 21x + 77y = 265$$

47. d. Let $P(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The chord of contact of tangents from P to the hyperbola is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \text{(i)}$$

The equations of the asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0$$

and $\quad \frac{x}{a} + \frac{y}{b} = 0$

The points of intersection of (i) with the two asymptotes are given by

$$x_1 = \frac{2a}{\frac{x_1}{a} - \frac{y_1}{b}}, y_1 = \frac{2b}{\frac{x_1}{a} - \frac{y_1}{b}}$$

$$x_2 = \frac{2a}{\frac{x_1}{a} + \frac{y_1}{b}}, y_2 = \frac{2b}{\frac{x_1}{a} + \frac{y_1}{b}}$$

Area of the said triangle $= \frac{1}{2}(x_1 y_2 - x_2 y_1)$

$$= \frac{1}{2}\left|\left(-\frac{4ab \times 2}{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}}\right)\right| = 4ab$$

48. b. Since $c_1 c_2 (a_1 a_2 + b_1 b_2) < 0$, therefore origin lies in acute angle. $P(1, 2)$ lies in obtuse angle.

Acute angle between the asymptotes is $\frac{\pi}{3}$. Hence,

$$e = \sec\frac{\theta}{2} = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

49. a. Let the equation of asymptotes be

$$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0 \qquad \text{(i)}$$

This equation represents a pair of straight lines.

Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

We have

$$4\lambda + 25 - \frac{25}{2} - 8 - \lambda\frac{25}{4} = 0$$

$$\Rightarrow \quad -\frac{9\lambda}{4} + \frac{9}{2} = 0$$

$$\Rightarrow \quad \lambda = 2$$

Putting the value of λ in (i), we get

$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

This is the equation of the asymptotes.

50. a. The given hyperbola is

$$xy - hx - ky = 0 \quad \text{(i)}$$

The equation of asymptotes is given by

$$xy - hx - ky + c = 0 \quad \text{(ii)}$$

Equation (ii) gives a pair of straight lines. So,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & \frac{1}{2} & -\frac{h}{2} \\ \frac{1}{2} & 0 & -\frac{k}{2} \\ -\frac{h}{2} & -\frac{k}{2} & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{hk}{8} + \frac{hk}{8} - \frac{c}{4} = 0$$

$$\Rightarrow \quad c = hk$$

Hence, asymptotes are

$$xy - hx - ky + hk = 0$$

or $\quad (x - k)(y - h) = 0$

51. a. Any tangent to hyperbola forms a triangle with the asymptotes which has constant area ab.

Given

$$ab = a^2 \tan \lambda$$

$$\Rightarrow \quad \frac{b}{a} = \tan \lambda$$

$$\Rightarrow \quad e^2 - 1 = \tan^2 \lambda$$

$$\Rightarrow \quad e^2 = 1 + \tan^2 \lambda = \sec^2 \lambda$$

$$\Rightarrow \quad e = \sec \lambda$$

52. a. The asymptotes of a rectangular hyperbola are perpendicular to each other.

Given one asymptote,

$$x + y + c = 0$$

Let the other asymptote be

$$x - y + \lambda = 0$$

We also know that the asymptotes pass through centre of the hyperbola. Therefore, the line $2x - y = 0$ and the asymptotes must be concurrent.

Thus, we have

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & c \\ 1 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = -3c$$

53. d. The equation of rectangular hyperbola is

$$(x - 3)(y - 5) + \lambda = 0$$

which passes through (7, 8). Hence,

$$4 \times 3 + \lambda \Rightarrow \lambda = -12$$

$$\therefore \quad xy - 5x - 3y + 15 - 12 = 0$$

$$\Rightarrow \quad xy - 3y - 5x + 3 = 0$$

54. c. Foci of hyperbola lie on $y = x$. So, the major axis is $y = x$.

Major axis of hyperbola bisects the asymptote

$\Rightarrow$ equation of other asymptote is $x = 2y$

$\Rightarrow$ equation of hyperbola is $(y - 2x)(x - 2y) + k = 0$

Given that it passes through (3, 4) $\Rightarrow k = 10$

Hence, required equation is

$$2x^2 + 2y^2 - 5xy + 10 = 0$$

55. a. If (x_i, y_i) is the point of intersection of given curves, then

$$\frac{\sum_{j=1}^{4} x_i}{4} = \frac{1+1}{2} \text{ and } \frac{\sum_{j=1}^{4} y_i}{4} = 0$$

Now $$\frac{\sum_{i=1}^{3} x_i}{3} = \frac{4 - x_4}{3} \text{ and } \frac{\sum_{i=1}^{3} y_i}{3} = -\frac{y_4}{4}$$

Centroid $\left(\frac{\sum_{j=1}^{3} x_i}{3}, \frac{\sum_{i=1}^{3} y_i}{3}\right)$ lies on the line $y = 3x - 4$.

Hence,

$$\frac{-y_4}{3} = \frac{3(4 - x_4)}{3} - 4$$

$$\Rightarrow \quad y_4 = 3x_4$$

Therefore, the locus of D is $y = 3x$.

56. b.

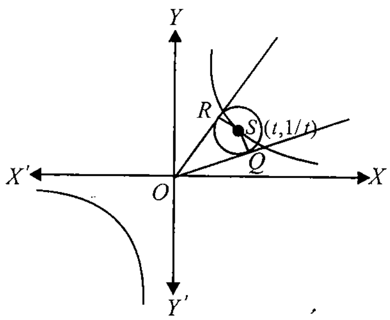


Fig. 5.58

Let S be a point on the rectangular hyperbola $\left[\text{say } \left(t, \frac{1}{t}\right)\right]$

Now, circumcircle of ΔOQR also passes through S.

Therefore, circumcentre is the midpoint of OS. Hence,

$$x = \frac{t}{2}, y = \frac{1}{2t}$$

So, the locus of the circum centre is $xy = \frac{1}{4}$

57. a. The points are such that one of the points is the orthocentre of the triangle formed by other three points. When the vertices of a triangle lie on a rectangular hyperbola the orthocentre also lies on the same hyperbola.

58. b.

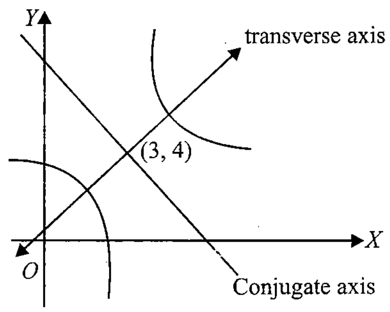


Fig. 5.59

$$(x-3)(y-4) = 5$$

The axes of the hyperbola are $x = 3$ and $y = 4$

Since the hyperbola is rectangular, axes are bisectors of these axes.

Hence, their slopes are ±1, out of which conjugate axis has slope $m = -1$ and passes through (3, 4).

Hence, its equation is

$$y - 4 = -1(x-3)$$

59. b.

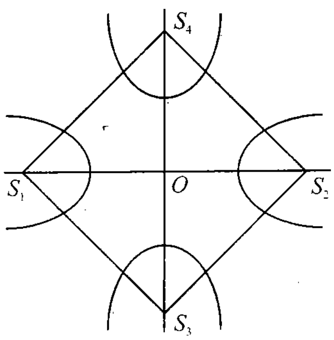


Fig. 5.60

Required area = 4 area $\Delta S_2OS_4 = 4 \times \frac{1}{2} ae \times 8 be_1$

$- 4 \times \frac{1}{2} \times 2 \times 3 \times ee_1$ (i)

$$b^2 = a^2(e^2 - 1) \Rightarrow e^2 = \frac{9}{4} + 1 = \frac{13}{4}$$

Also $\quad \frac{1}{e_1^2} = 1 - \frac{1}{e^2} = 1 - \frac{4}{13} = \frac{9}{13}$

$$e_1^2 = \frac{13}{9}$$

$$\text{Required area} = 12 \times \frac{\sqrt{13}}{2} \times \frac{\sqrt{13}}{3}$$

$$= 2 \times 13 = 26$$

60. a. For $\lambda = -3$, the equation becomes

$$x^2 + y^2 - 3xy = 0$$

which represents a pair of lines through origin.

61. d. $xy = c^2$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have

$$-x\frac{dx}{dy} + y = 0$$

$$\Rightarrow ydy - xdx = 0$$

Integrating, we have

$$x^2 - y^2 = k^2$$

where k is the parameter which represents family of hyperbolas.

62. d.

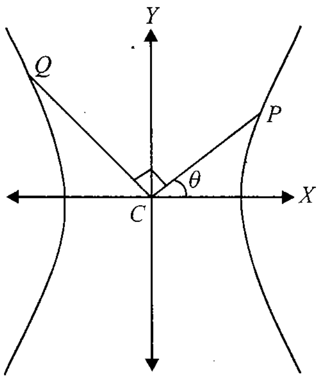


Fig. 5.61

Let $CP = r_1$ be inclined to transverse axis at an angle θ so that P is $(r_1 \cos\theta, r_1 \sin\theta)$ and P lies on the hyperbola. It gives

$$r_1^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$$

Replacing θ by 90° + θ, we have

$$r_2^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$$

$$\Rightarrow \quad \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} + \frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\cos^2\theta\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\sin^2\theta\times\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

63. a. The midpoint of the chord is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The equation of the chord in terms of its midpoint is

$$T = S_1$$

$$\Rightarrow x\left(\frac{y_1+y_2}{2}\right)+y\left(\frac{x_1+x_2}{2}\right)-c^2=2\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right)-c^2$$

$$\Rightarrow \quad x(y_1+y_2)+y(x_1+x_2)=(x_1+x_2)(y_1+y_2)$$

$$\Rightarrow \quad \frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$$

64. c. A rectangular hyperbola circumscribing a triangle also passes through its orthocentre.

If $\left(ct_i, \frac{c}{t_i}\right)$, where $i = 1, 2, 3$, are the vertices of the triangle then the orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$, where $t_1t_2t_3t_4 = 1$.

Hence, orthocentre is $\left(-ct_4, \frac{-c}{t_4}\right) = (-x_4, -y_4)$.

65. b.

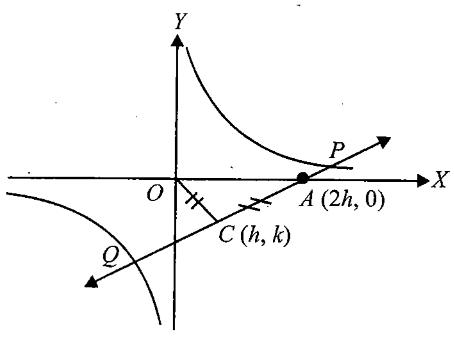


Fig. 5.62

Hyperbola is

$$xy = a^2$$

or $\quad 2xy - 2a^2 = 0$

Chord with a given middle point is given by

$$hy + kx - a^2 = 2hk - a^2$$

$$\Rightarrow \quad \frac{x}{h}+\frac{y}{k}=2$$

From the diagram ΔOCA is isosceles with $OC = CA$.

66. b.

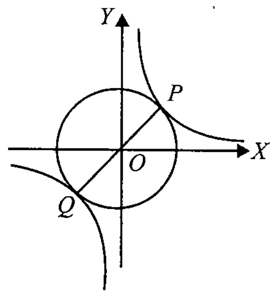


Fig. 5.63

From the diagram PQ = diameter of the circle = 2

67. b. $x - 2 = m$

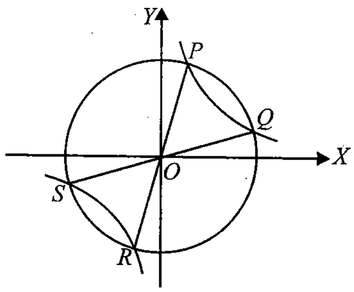


Fig. 5.64

$$y + 1 = \frac{4}{m}$$

$\therefore \quad (x-2)(y+1) = 4$

$\Rightarrow \quad XY = 4$, where $X = x - 2, Y = y + 1$

$$S \equiv (x-2)^2 + (y+1)^2 = 25$$

$\Rightarrow \quad X^2 + Y^2 = 25$

Curve 'C' and circle S both are concentric.

$\therefore \; OP^2 + OQ^2 + OR^2 + OS^2 = 4r^2 = 4 \times 25 = 100$

68. a.

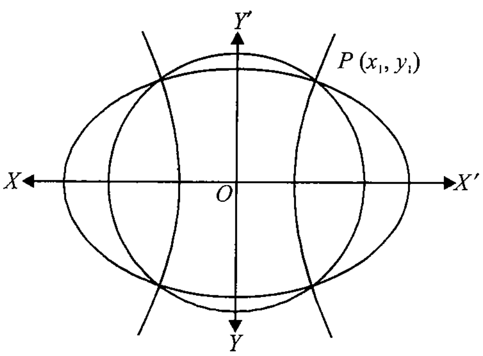


Fig. 5.65

Since ellipse and hyperbola intersect orthogonally, they are confocal.

Hence, $a = 2$ (equating foci).

Let point of intersection in the first quadrant be $P(x_1, y_1)$.

P lies on both the curves. Therefore,

$$4x_1^2 + 9y_1^2 = 36 \text{ and } 4x_1^2 - y_1^2 = 4$$

Adding these two results, we get

$$8(x_1^2 + y_1^2) = 40$$

$\Rightarrow$ $x_1^2 + y_1^2 = 5 \Rightarrow r = \sqrt{5}$

Hence, equation of the circle is

$$x^2 + y^2 = 5$$

69. b. The chord of contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

It meets the axes at the points $\left(\frac{a^2}{x_1}, 0\right)$ and $\left(0, \frac{b^2}{y_1}\right)$.

Area of the triangle is $\frac{1}{2}\frac{a^2}{x_1}\frac{b^2}{y_1} = k$ (constant)

$\Rightarrow$ $x_1y_1 = \frac{a^2b^2}{2k} = c^2$ (c is constant)

$\Rightarrow$ $xy = c^2$ is the required locus.

70. c.

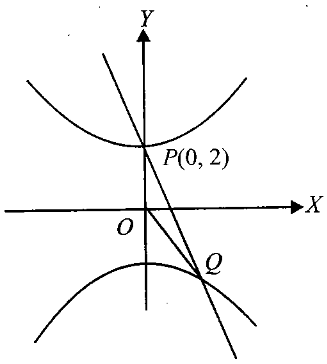


Fig. 5.66

Homogenizing the hyperbola using the straight line, we get pair of straight lines OP and OQ, which are given by

$$y^2 - x^2 = 4\left(\frac{\sqrt{3}x + y}{2}\right)^2$$

$\Rightarrow$ $y^2 - x^2 = 3x^2 + y^2 + 2\sqrt{3}xy$

$\Rightarrow$ $4x^2 + 2\sqrt{3}\,xy = 0$

$\Rightarrow$ $x = 0$ and $2x + \sqrt{3}\,y = 0$

Angle between the lines is $\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

71. a. Let the variable chord be

$$x\cos\alpha + y\sin\alpha = p \qquad \text{(i)}$$

Let this chord intersect the hyperbola at A and B. Then the combined equation of OA and OB is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

$$x^2\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) - y^2\left(\frac{1}{b^2} + \frac{\sin^2\alpha}{p^2}\right) - \frac{2\sin\alpha\cos\alpha}{p}xy = 0$$

This chord subtends a right angle at centre. Therefore,

Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2} - \frac{\cos^2\alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2\alpha}{p^2} = 0$$

$\Rightarrow$ $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2}$

$\Rightarrow$ $p^2 = \frac{a^2b^2}{b^2 - a^2}$

Hence, p is constant, i.e. it touches the fixed circle.

72. a. Equation of tangent at point $P(\alpha\cos\theta, \sin\theta)$ is

$$\frac{x}{\alpha}\cos\theta + \frac{y}{1}\sin\theta = 1 \qquad \text{(i)}$$

Let it cut the hyperbola at points P and Q.

Homogenizing the hyperbola $\alpha^2x^2 - y^2 = 1$ with the help of the above the equation, we get

$$\alpha^2x^2 - y^2 = \left(\frac{x}{\alpha}\cos\theta + y\sin\theta\right)^2$$

This is a pair of straight lines OP and OQ.

Given $\angle POQ = \frac{\pi}{2}$

$\Rightarrow$ Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - \sin^2\theta = 0$$

$$\Rightarrow \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - 1 + \cos^2\theta = 0$$

$\Rightarrow$ $\cos^2\theta = \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1}$

Now,

$$0 \le \cos^2\theta \le 1$$

$\Rightarrow$ $0 \le \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1} \le 1$

Solving, we get $\alpha^2 \in \left[\frac{\sqrt{5} + 1}{2}, 2\right]$

Multiple Correct Answers Type

1. a. We have,

$$\left|\sqrt{x^2 + (y-1)^2} - \sqrt{x^2 + (y+1)^2}\right| = K$$

which is equivalent to $|S_1P - S_2P|$ = constant, where $S_1 \equiv (0, 1)$, $S_2 \equiv (0, -1)$ and $P \equiv (x, y)$.

The above equation represents a hyperbola. So, we have

$$2a = K$$

and $$2ae = S_1S_2 = 2$$

where $2a$ is the transverse axis and e is the eccentricity.

Dividing, we have

$$e = \frac{2}{K}$$

Since, $e > 1$ for a hyperbola, therefore $K < 2$.

Also, K must be a positive quantity.

So, we have $K \in (0, 2)$.

2. a., b., c.

$3x^4 - 2(19y + 8)x^2 + [(19y)^2 + (10)^2 + (10)^2 + y^4 + y^4 + 8^2]$
$= 2(19 \times 10y + 10y^2 - 8y^2)$

$\Rightarrow 3x^4 - 2(19y + 8)x^2 + (19y - 10)^2 + (10 - y^2)^2 + (y^2 + 8)^2 = 0$

$\Rightarrow 3x^4 - 2(19y - 10 + 10 - y^2 + y^2 + 8)x^2 + (19y - 10)^2 + (10 - y^2)^2 + (y^2 + 8)^2 = 0$

$\Rightarrow [x^2 - (19y - 10)]^2 + [x^2 - (10 - y^2)]^2 + [x^2 - (y^2 + 8)]^2 = 0$

$\Rightarrow x^2 - 19y + 10 = 0,\ x^2 - 10 + y^2 = 0$ and $x^2 - y^2 - 8 = 0$

The three curves represented by the given equation are $x^2 = 19y - 10$ (parabola), $x^2 + y^2 = 10$ (circle) and $x^2 - y^2 = 8$ (hyperbola).

3. a., c. $|PS_1 - PS_2| = 2a$

$$2a = K$$

$$\Rightarrow 2a = \sqrt{(24 - 0)^2 + (7 - 0)^2} - \sqrt{12^2 + 5^2} = 12$$

$$\therefore \quad a = 6$$

$$2ae = \sqrt{(24 - 5)^2 + (12 - 7)^2}$$

$$= \sqrt{386}$$

$$\therefore \quad e = \frac{\sqrt{386}}{12}$$

$$LR = \frac{2b^2}{a} = \frac{2a^2(e^2 - 1)}{a}$$

$$= 2 \times 6\left(\frac{386}{144} - 1\right) = \frac{121}{6}$$

4. a., d.,

Let A (5, 12) and B (24, 7) be two fixed points.

So, $|OA - OB| = 12$ and $|OA + OB| = 38$.

If the conic is an ellipse, then

$$e = \frac{\sqrt{386}}{38} \quad (\because 2ea = \sqrt{386} \text{ and } a = 19)$$

If the conic is a hyperbola, then

$$e = \frac{\sqrt{386}}{12} \quad (\because 2ae = \sqrt{386} \text{ and } a = 6)$$

5. b. Locus of point of intersection of perpendicular tangents is director circle for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Equation of director circle is $x^2 + y^2 = a^2 - b^2$ which is real if $a > b$.

6. a., b., c., d.

Given hyperbola can be written as

$$\frac{(x - 1)^2}{16} - \frac{(y - 1)^2}{9} = 1$$

$$\Rightarrow \quad \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

(where $X = x - 1$, $Y = y - 1$)

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

Directrices are

$$X = \pm\frac{a}{e}$$

$$\Rightarrow \quad x - 1 = \pm\frac{16}{5} \Rightarrow x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

Length of latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$

The foci are given by

$$X = \pm ae,\ Y = 0$$

$\Rightarrow (6, 1),\ (-4, 1)$

7. a., b.,d.

For the ellipse,

$$a = 5 \text{ and } e = \sqrt{\frac{25 - 9}{25}} = \frac{4}{5}$$

$$\therefore \quad ae = 4$$

Hence, the foci are (–4, 0) and (4, 0).

For the hyperbola,

$$ae = 4,\ e = 2$$

$$\therefore \quad a = 2$$

$$b^2 = 4(4 - 1) = 12$$

$$\Rightarrow \quad b = \sqrt{12}$$

8. a., b., c., d.

Solving $xy = c^2$ and $x^2 + y^2 = a^2$, we have

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$$

$$\Rightarrow \quad \Sigma x_i = 0 \text{ and } x_1x_2x_3x_4 = c^4$$

Similarly, if we eliminate y, then $\Sigma y_i = 0$ and $y_1y_2y_3y_4 = c^4$

9. b., d.

$$\frac{dx}{dy} = \frac{3y}{2x}$$

$$\Rightarrow \quad \int 2x\,dx = \int 3y\,dy$$

$$\Rightarrow \quad x^2 = \frac{3y^2}{2} + c$$

or $\frac{x^2}{3}-\frac{y^2}{2}=\frac{c}{3}$

If c is positive, then

$$e=\sqrt{1+\frac{2}{3}}=\sqrt{\frac{5}{3}}$$

If c is negative, then

$$e=\sqrt{1+\frac{3}{2}}=\sqrt{\frac{5}{2}}$$

10. a., d.

Circle with points $P(2t_1, 2/t_1)$ and $Q(2t_2, 2/t_2)$ as diameter is given by

$$(x-2t_1)(x-2t_2)+\left(y-\frac{2}{t_1}\right)\left(y-\frac{2}{t_2}\right)=1 \quad \text{(i)}$$

Also, slope of PQ is given by

$$-\frac{1}{t_1t_2}=1\Rightarrow t_1t_2=-1$$

Hence, from (1), circle is

$(x^2+y^2-8)-(t_1+t_2)(x-y)=0$

which is of the form $S+\lambda L=0$.

Hence, circles pass through the points of intersection of the circle $x^2+y^2-8=0$ and the line $x=y$.

The points of intersection are (2, 2) and (–2, –2).

11. b., c., d.

$(x-\alpha)^2+(\gamma-\beta)^2=k(lx+my+n)^2$

$$\Rightarrow \sqrt{(x-\alpha)^2+(y-\beta)^2}=\sqrt{k}\sqrt{l^2+m^2}\frac{(lx+my+n)}{\sqrt{l^2+m^2}}$$

$$\Rightarrow \frac{PS}{PM}=\sqrt{k}\sqrt{l^2+m^2}$$

where point $P(x, y)$ is any point on the curve.

Fixed point $S(\alpha, \beta)$ is focus and fixed line $lx+my+n=0$ is directrix.

If $k(l^2+m^2)=1$, P lies on parabola.

If $k(l^2+m^2)<1$, P lies on ellipse.

If $k(l^2+m^2)>1$, P lies on hyperbola.

If $k=0$, P lies on a point circle.

12. a., b.

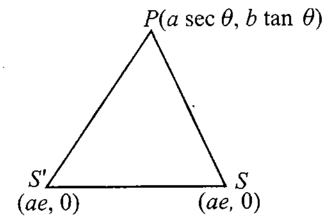


Fig. 5.67

Let (h, k) be excentre, then

$$h=\frac{a(ae\sec\theta+a)-ae(ae\sec\theta-a)-2ae(a\sec\theta)}{2ae(\sec\theta-1)}$$

$h=-a\Rightarrow x=-a$ (for $a\sec\theta>0$)

Similarly, $x=a$ for $a\sec\theta<0$

$\Rightarrow$ locus is $x^2=a^2$

Again let (h, k) be excentre opposite $\angle S'$,

$$\therefore h=\frac{2a^2e\sec\theta+a^2e^2\sec\theta+a^2e+a^2e^2\sec\theta-a^2e}{2a+2ae}$$

$$\Rightarrow h=ae\sec\theta,\ k=\frac{2aeb\tan\theta}{2a+2ae}$$

$\Rightarrow$ locus is hyperbola.

13. b., d. Differentiating $xy=1$ w.r.t. x, we have

$$\frac{dy}{dx}=-\frac{1}{x^2}<0$$

Hence, the slope of normal at any point $P(x_1, y_1)$ is

$$x_1^2>0$$

Therefore, slope of the normal must always be positive.

Hence, possible lines are (b) and (d).

14. c., d.

Equations of asymptotes are $4y-3x=0$ and $4y+3x=0$

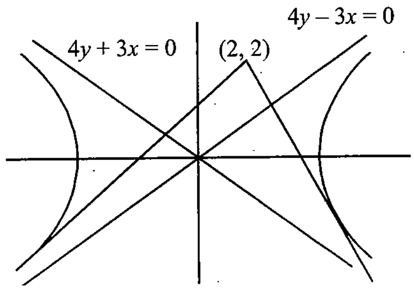
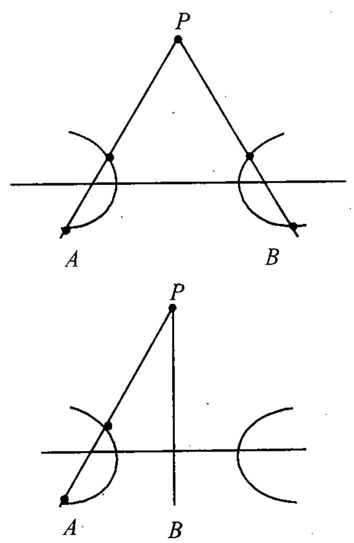


Fig. 5.68

As point (2, 2) lies above the asymptote.

Hence, points of contact of the tangents are in III and IV quadrants.

15. b., c., d.



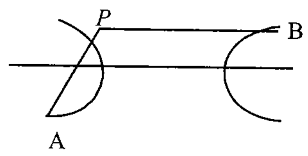


Fig. 5.69

Different cases have been shown in the above diagrams. Therefore, number of points can be 2, 3 or 4.

16. a., b., c.

Locus of point of intersection of perpendicular tangents is director circle $x^2 + y^2 = a^2 - b^2$.

Now,

$$e^2 = 1 + \frac{b^2}{a^2}$$

If $a^2 > b^2$, then there are infinite (or more than 1) points on the circle $\Rightarrow e^2 < 2 \Rightarrow e < \sqrt{2}$.

If $a^2 < b^2$, there does not exist any point on the plane $\Rightarrow e^2 > 2 \Rightarrow e > \sqrt{2}$.

If $a^2 = b^2$, there is exactly one point (centre of the hyperbola) $\Rightarrow e = \sqrt{2}$.

17. a., b., c., d.

Normal at point P (2 sec θ, 2 tan θ) is

$$\frac{2x}{\sec\theta} + \frac{2y}{\tan\theta} = 8$$

It meets the axes at points G (4 sec θ, 0) and g (0, 4 tan θ). Then

$$PG = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Pg = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$PC = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Gg = \sqrt{16\sec^2\theta + 16\tan^2\theta}$$

$$= 2\sqrt{4\sec^2\theta + \tan^2\theta} = 2\,PC$$

Reasoning Type

1. b. Statement 1 is correct, see the theory.

Also statement 2 is true as asymptotes are perpendicular, they are bisectors of transverse and conjugate axes of hyperbola.

But statement 2 does not explain statement 1, as in hyperbolas other than rectangular hyperbolas asymptotes are not bisectors of transverse and conjugate axes.

2. a. Tangent to hyperbola having slope m is

$$y = mx \pm \sqrt{4m^2 - 16}$$

which is real line if

$4m^2 - 16 > 0 \Rightarrow m^2 > 4 \Rightarrow m \in (-\infty, -2) \cup (2, \infty)$

Hence, statement 2 is correct.

Also statement 1 is correct and statement 2 is correct explanation of statement 1.

3. a. Let P be the position of the gun and Q be the position of the target. Let u be the velocity of sound, v be the velocity of bullet and R be the position of the man. Then, we have

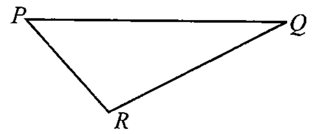


Fig. 5.70

$$\frac{PR}{u} = \frac{QR}{u} + \frac{PQ}{v}$$

$$\Rightarrow \quad \frac{PR}{u} - \frac{QR}{u} = \frac{PQ}{v}$$

$$\Rightarrow \quad PR - QR = \frac{u}{v} PQ = \text{constant}$$

and $\quad \frac{u}{v} PQ < PQ$

Therefore, locus of R is a hyperbola.

4. a. For ellipse $\frac{x^2}{27/12} - \frac{y^2}{27/4} = 1$

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$$a = 5$$

The foci are (±3, 0).

For hyperbola $\frac{x^2}{27/12} - \frac{y^2}{27/4} = 1$

$$e = \sqrt{1 + \frac{12}{4}} = 2$$

$$a = \frac{3}{2}$$

The foci are (±3, 0).

Therefore, the two conics are confocal. Hence, curves are orthogonal.

5. a. See the theory.

6. d. Statement 2 is true. See the theory.

For the points (2, 2), (4, 1) and (6, 2/3), $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$, respectively.

For the point (1/4, 16), $t_4 = \frac{1}{8}$.

Now $\quad t_1 t_2 t_3 t_4 = \frac{3}{4} \neq 1.$

Hence, statement 1 is false.

7. d.

Statement 1 is false as points in region A lie below the asymptote

$$y = \frac{b}{a}x \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} > 0$$

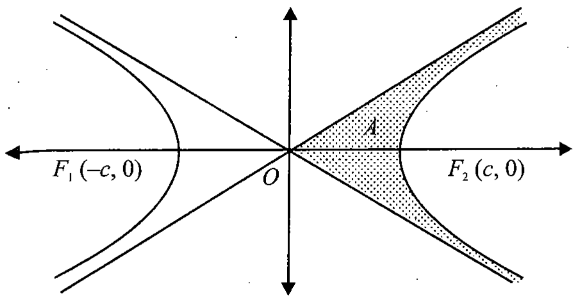


Fig. 5.71

Statement 2 is true (standard result). Indeed for points in region A

$$0 < \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$$

8. **b.** Given hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

Now line having slope $m = 3$ is tangent to the hyperbola. So, its equation is

$$y = 3x \pm \sqrt{3(3)^2 - 2}$$

or $\quad y = 3x \pm 5$

Hence, statement 1 is correct.

Also statement 2 is correct, but information is not enough to get the equations of tangents.

9. **d.** The locus of point of intersection of two mutually perpendicular tangents drawn on to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is its director circle whose equation is $x^2 + y^2 = a^2 - b^2$.

For $\quad \frac{x^2}{9} - \frac{y^2}{16} = 1, x^2 + y^2 = 9 - 16$

So director circle does not exist.

10. **b.** Chord of contact of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. point $P(x_1, y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ (i)

Equation (i) can be written as

$$\frac{x(-x_1)}{a^2} - \frac{y(-y_1)}{b^2} = -1$$

which is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

at point $(-x_1, -y_1)$.

Obviously, points (x_1, y_1) and $(-x_1, -y_1)$ lie on the different branches of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Hence, statement 1 is correct.

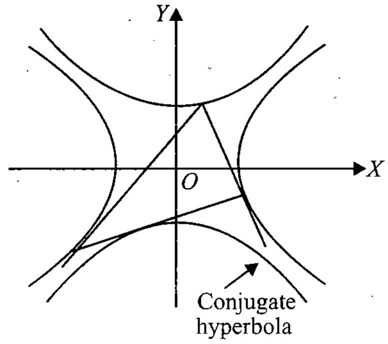


Fig. 5.72

Statement 2 is also correct but does not explain statement I.

11. **d.** We have

$$\sqrt{(\lambda - 3)^2 + 16} - 4 = 1 \Rightarrow \lambda = 0 \text{ or } 6$$

12. **a.** Given that

$$\frac{r_2}{r_3} = k \text{ (constant)}$$

$$\frac{\frac{\Delta}{s-b}}{\frac{\Delta}{s-c}} = k$$

where Δ is area of triangle and s is semi-perimeter

$$\Rightarrow \quad \frac{s-c}{s-b} = k$$

$$\Rightarrow \quad \frac{b-c}{2s-(b+c)} = \frac{k-1}{k+1}$$

$\Rightarrow b - c = a\left(\frac{k-1}{k+1}\right)$ (constant) (as $BC = a$ is given)

Therefore, the locus of vertex A is a hyperbola.

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. b.

Sol.

Let the curve be $y = f(x)$.

Now tangent at point P to the curve is

$$Y - y = m(X - x)$$

It meets y-axis when

$$X = 0 \Rightarrow Y = y - mx$$

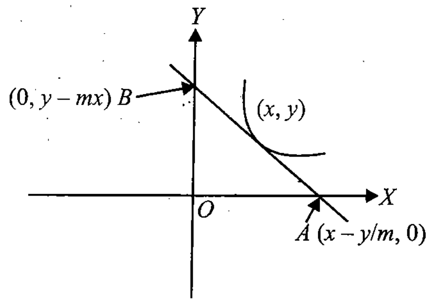


Fig. 5.73

and x-axis when

$$Y = 0 \Rightarrow X = x - \frac{y}{m}$$

Given that P is midpoint of AB. Hence,

$$x - \frac{y}{m} = 2x$$

$$\Rightarrow \quad \frac{y}{m} = -x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \quad \log_e xy = c$$

$$\Rightarrow \quad xy = c$$

As the curve passes through (2, 4), so

$$xy = 8$$

Solving with $\quad y = x$, we get

$$x = 2\sqrt{2}$$

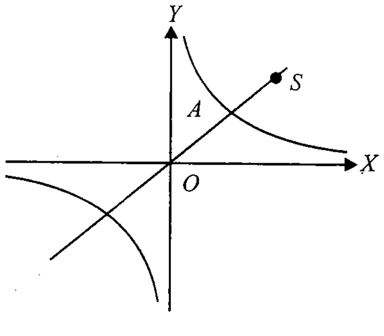


Fig. 5.74

$$\therefore \quad OA = \sqrt{8+8} = 4$$

$$\Rightarrow \quad OS = 4\sqrt{2}$$

Hence, coordinates of S are (4, 4) or (−4, −4).

Directrix is at distance $4/\sqrt{2}$ from origin.

Hence, its equations $x + y = \pm 4$.

For Problems 4–6

4. d., 5. b., 6. c.

Sol. 4. d. Centre ≡ (1, 2)

Radius of auxiliary circle $= a = \sqrt{(2-1)^2 + (5-2)^2}$

$$= \sqrt{10}$$

$$2ae = \sqrt{8^2 + 8^2} = 8\sqrt{2} \Rightarrow e = \frac{4}{\sqrt{5}}$$

$$b^2 = a^2e^2 - a^2 = 32 - 10 = 22$$

$$\Rightarrow \quad 2b = 2\sqrt{22}$$

5. b. Directrix is perpendicular to the transverse axis. Let it be $x + y = k$.

Its distance from centre $\frac{a}{e}$

$$\Rightarrow \quad \frac{|1+2-k|}{\sqrt{2}} = \frac{5}{2\sqrt{2}} \Rightarrow k = 3 + \frac{5}{2} = \frac{11}{2}$$

6. c. The tangent is

$$y - 5 = -\frac{5-2}{6-5}(x-2)$$

$$\Rightarrow \quad 3x + y = 11$$

The hyperbola is

$$(x-5)^2 + (y-6)^2 = \frac{16}{5} \times \frac{(2x+2y-11)^2}{8}$$

Solving, we get

$$(x-5)^2 + (5-3x)^2 = \frac{2}{5}(2x + 22 - 6x - 11)^2$$

$$\Rightarrow \quad 5[10x^2 - 40x + 50] = 2(11-4x)^2$$

$$\Rightarrow \quad 9x^2 - 12x + 4 = 0$$

$$\Rightarrow \quad (3x-2)^2 = 0 \Rightarrow x = \frac{2}{3}, y = 9$$

For Problems 7–9

7. c., 8. d., 9. b.

Sol. 7. c. $2a = 3$

Distance between the foci (1, 2) and (5, 5) is 5.

$$\therefore \quad 2ae = 5$$

$$\therefore \quad e = \frac{5}{3}$$

Now if e' is eccentricity of the corresponding conjugate hyperbola, then

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow \quad e' = \frac{5}{4}$$

8. d. Director circle is given by

$$(x-h)^2 + (y-k)^2 = a^2 - b^2$$

where (h, k) is centre, i.e. the

midpoint of foci $\left(\frac{1+5}{2}, \frac{2+5}{2}\right) \equiv \left(3, \frac{7}{2}\right)$.

$$Y - y = m(X - x)$$

$$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2 - 1\right) = 4$$

Therefore, the director circle is

$$(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{9}{4} - 4$$

$$\Rightarrow \quad (x-3)^2 + \left(\frac{y-7}{2}\right)^2 = -\frac{7}{4}$$

This does not represent any real point.

9. b. Slope of transverse axis is $\frac{3}{4}$.

Therefore, the angle of rotation is

$$\theta = \tan^{-1}\frac{3}{4}.$$

For Problems 10–12

10. a., 11. c., 12. b.

Sol. 10. a. Equation of tangent in parametric form is given by

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = \pm 3\sqrt{2}$$

$$\Rightarrow \quad A \equiv (4, -2),\ B \equiv (-2, 4)$$

Equations of asymptotes (OA and OB) are given by

$$y + 2 = \frac{-2}{4}(x-4) \Rightarrow 2y + x = 0$$

and

$$y - 4 = \frac{4}{-2}(x+2) \Rightarrow 2x + y = 0$$

Hence, the combined equation of asymptotes is

$$(2x+y)(x+2y) = 0$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 5xy = 0$$

11. c. $m_{OA} = -\frac{1}{2}$, $m_{OB} = -2$

$$\tan\theta = \left|\frac{-1/2+2}{1+1}\right| = \frac{3}{4}$$

$$\Rightarrow \quad \sin\theta = \frac{3}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{3}{5}\right)$$

12. b. Let the equation of the hyperbola be

$$2x^2 + 2y^2 + 5xy + \lambda = 0$$

It passes through (1, 1). Therefore,

$$2 + 2 + 5 + \lambda = 0$$

$$\Rightarrow \quad \lambda = -9$$

So, the hyperbola is

$$2x^2 + 2y^2 + 5xy = 9$$

Equation of the tangent at $\left(-1, \frac{7}{2}\right)$ is given by

$$2x(-1) + 2y\left(\frac{7}{2}\right) + 5\,\frac{x(7/2) + (-1)y}{2} = 9$$

$$\Rightarrow \quad 3x + 2y = 4$$

For Problems 13–15

13. b., 14. c., 15 d.

13. b. Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$

Normal at this point is $y - 4/t = t^2(x - 4t)$.

If the normal passes through $P(h, k)$, then $k - 4/t = t^2(h - 4t)$

$$\Rightarrow \quad 4t^4 - t^3h + tk - 4 = 0$$

This equation has roots t_1, t_2, t_3, t_4 which are parameters of the four feet of normals on the hyperbola. Therefore,

$$\Sigma t_1 = \frac{h}{4}$$

$$\Sigma t_1t_2 = 0$$

$$\Sigma t_1t_2t_3 = -\frac{k}{4}$$

and

$$t_1t_2t_3t_4 = -1$$

$$\therefore \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{4}$$

$$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = k$$

According to the question,

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{16} = k$$

Hence, the locus of (h, k) is

$$x^2 = 16y$$

14. c. $x^2 = 16y$

Equation of tangent of P is

$$x\,4t = \frac{16(y + t^2)}{2}$$

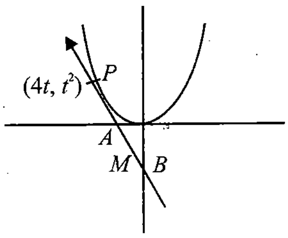


Fig. 5.75

$$\Rightarrow \quad 4tx = 8y + 8t^2$$

$$\Rightarrow \quad tx = 2y + 2t^2$$

$$A = (2t, 0),\ B = (0, -t^2)$$

$M(h, k)$ is the middle point of AB.

$$h = t,\ k = -\frac{t^2}{2} \Rightarrow 2k = -h^2$$

Therefore, the locus of $M(h, k)$ is $x^2 + 2y = 0$.

15. d. $\tan 30° = \frac{4t_1}{t_1^2} = \frac{4}{t_1}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{4}{t_1} \Rightarrow t_1 = 4\sqrt{3}$$

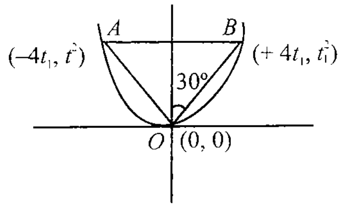


Fig. 5.76

$$AB = 8t_1 = 32\sqrt{3}$$

Area of $\Delta OAB = \frac{\sqrt{3}}{4} \times 32\sqrt{3} \times 32\sqrt{3}$

$$= 768\sqrt{3}\text{ sq. units}$$

For Problems 16–18

16. b., 17. c., 18. d.

Sol. 16. b. Perpendicular tangents intersect at the centre of rectangular hyperbola. Hence, centre of hyperbola is (1, 1) and equation of asymptotes are $x - 1 = 0$ and $y - 1 = 0$.

17. c. Let the equation of hyperbola is $xy - x - y + 1 + \lambda = 0$. It passes through (3, 2) hence $\lambda = -2$.

So the equation of hyperbola is

$$xy = x + y + 1$$

18. d. From the centre of hyperbola we can draw two real tangents to the rectangular hyperbola.

Matrix-Match Type

1. a → p; b → p, q; c → q; d → s.

We have

$$A = ae_E \text{ and } a = Ae_H$$

$$\Rightarrow \quad e_E\, e_H = 1 \Rightarrow e_E + e_H > 2$$

$$B^2 = A^2\left(e_H^2 - 1\right) = a^2\left(1 - e_E^2\right)$$

$$= b^2$$

$$\Rightarrow \quad \frac{b}{B} = 1$$

Also the angle between the asymptotes is

$$2\tan^{-1}\frac{B}{A} = \frac{2\pi}{3}$$

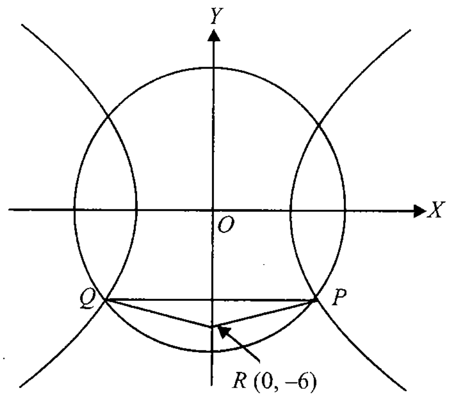
Also, $\quad \dfrac{B}{A} = \sqrt{3} \Rightarrow \dfrac{b}{ae_E} = \sqrt{3} \Rightarrow e_E^2 = \dfrac{1}{4}$

Solving $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2 e_E^2} - \dfrac{y^2}{b^2} = 1$

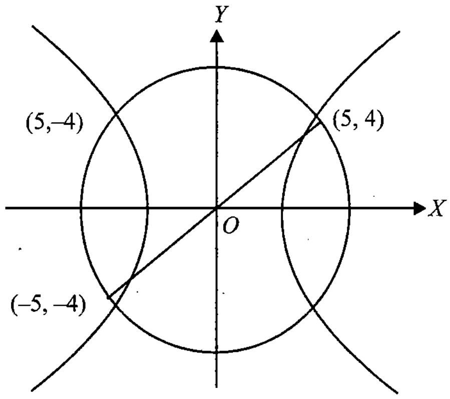
we get

$$x^2 = \frac{2a^2 e_E^2}{b^2(1 - e_E^2)} = 4$$

2. a → p, q, r, s ; b → q, r ; c → p ; d → p, s.



(a)



(b)

Fig. 5.77

a. Obviously all the points in column II are common to the hyperbola and circle.

b. Chord of contact of hyperbola w.r.t. $\left(0, -\frac{9}{4}\right)$ is $0(x) - \left(-\frac{9}{4}\right)y = 9$ or $y = 4$

Solving this with hyperbola we have

$$x^2 - 16 = 9 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

Hence, points of contact are (±5, 4).

c. Obviously the required point is (–5, –4).

d. Let the points on the hyperbola be $P(h, k)$ and $Q(-h, k)$.

Then area of triangle is $\frac{1}{2}|2h||-6-k| = 10$

$$\Rightarrow \quad |h||6 + k| = 10 \tag{i}$$

Also points P and Q lie on the hyperbola. Hence,

$$h^2 - k^2 = 9 \tag{ii}$$

Obviously points (±5, –4) satisfy both Eqs. (i) and (2).

3. a → p; b → s, c → r; d → p.

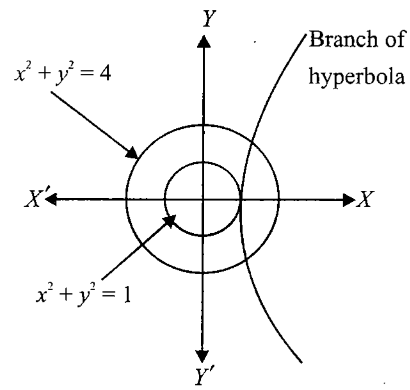


Fig. 5.78

Locus of point P satisfying $PA - PB = 2$ is a branch of the hyperbola $x^2 - y^2/3 = 1$.

For $r = 2$ the circle and the branch of the hyperbola intersect at two points. For $r = 1$ there is no point of intersection.

If m be the slope of the common tangent, then

$$m^2 - 3 = r^2(1 + m^2)$$

$$\Rightarrow \quad m^2 = \frac{r^2 + 3}{1 - r^2}$$

Hence, there are no common tangents for $r > 1$ and two common tangents for $r < 1$.

4. **a** $\to$ **r**; **b** $\to$ **p**; **c** $\to$ **s**; **d** $\to$ **q**.

a. $\text{Im}(z^2) = 3$

$\Rightarrow \text{Im}((x + iy)^2) = 3$

$\Rightarrow 2xy = 3$, which is a rectangular hyperbola having eccentricity $\sqrt{2}$

b.

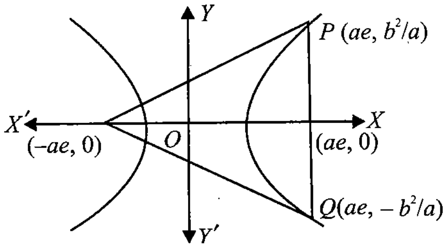


Fig. 5.79

$$\tan 30° = \frac{b^2/a}{2ae}$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}e = e^2 - 1$$

$$\Rightarrow \quad \sqrt{3}e^2 - 2e - \sqrt{3} = 0$$

$$\Rightarrow \quad e = \frac{2 \pm \sqrt{4 + 12}}{2\sqrt{3}} = \frac{2 \pm 4}{2\sqrt{3}}$$

$$\Rightarrow \quad e = \frac{3}{\sqrt{3}} = \sqrt{3}$$

c. Eccentricity of the hyperbola $= \frac{AB}{PA - PB} = \frac{6}{4} = \frac{3}{2}$

If eccentricity of conjugate hyperbola is e', then $\frac{1}{\left(\frac{3}{2}\right)^2} + \frac{1}{e'^2} = 1$

$$\Rightarrow \quad e' = \frac{3}{\sqrt{5}}$$

d. Angle between the asymptotes is $\tan^{-1}\left|\frac{2ab}{a^2 - b^2}\right| = \frac{\pi}{3}$

$$\Rightarrow \quad \left|\frac{2\frac{a}{b}}{\frac{a^2}{b^2} - 1}\right| = \sqrt{3}$$

$\Rightarrow \frac{2\sqrt{e'^2 - 1}}{|e'^2 - 2|} = \sqrt{3}$ (where e' is eccentricity of conjugate hyperbola)

$$\Rightarrow \quad e' = 2$$

5. **a** $\to$ **p., s.**; **b** $\to$ **q, r**; **c** $\to$ **r**; **d** $\to$ **p, s.**

a. We must have

$$e_1 < 1 < e_2 \Rightarrow f(1) < 0 \Rightarrow 1 - a + 2 < 0 \Rightarrow a > 3$$

b. We must have both the roots greater than 1.

i. $D > 0$ or $a^2 - 4 > 0$ or $a \in (-\infty, -2) \cup (2, \infty)$

ii. $1 \cdot f(1) > 0$ or $1 - a + 2 > 0$ or $a < 3$

iii. $\frac{a}{2} >= 1 \Rightarrow a > 2$

From Eqs. (i), (ii) and (iii) we have $a \in (2, 3)$.

c. We must have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

$$\Rightarrow \quad \frac{(e_1 + e_2)^2 - 2e_1e_2}{e_1^2e_2^2} = 1$$

$$\Rightarrow \quad \frac{a^2 - 4}{4} = 1$$

$$\Rightarrow \quad a = \pm 2\sqrt{2}$$

d. We must have

$$e_2 < \sqrt{2} < e_1$$

$$\Rightarrow \quad f(\sqrt{2}) < 0$$

$$\Rightarrow \quad 2 - a\sqrt{2} + 2 < 0$$

$$\Rightarrow \quad a > 2\sqrt{2}$$

Integer Type

1. **(5)** For given equation of hyperbola foci are $S(3, 2)$ and $S'(-1, -1)$,

Using definition of hyperbola $|SP - S'P| = 2a$,

We have $SS' = 5$ and $2a = 1$

Hence eccentricity is $\frac{SS'}{2a} = 5$.

2. **(5)** $e^2 = \frac{b^2}{a^2} + 1 \Rightarrow \frac{b^2}{a^2} = e^2 - 1 = 24$

Now $y = mx + c$ is tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then we must have $a^2m^2 - b^2 \geq 0$

or $m^2 \geq b^2/a^2$ or $m^2 \geq 24$ then least positive integral value of m is 5.

3. **(4)** We have $y = Ax^2$, $y^2 + 3 = x^2 + 4y$; $A > 0$

Now $y^2 - 4y = x^2 - 3$

$$\Rightarrow \quad (y - 2)^2 = x^2 + 1$$

$$\Rightarrow \quad (y - 2)^2 - x^2 = 1$$

If $x = 0$, $y - 2 = 1$ or $-1 \Rightarrow y = 3$ or 1

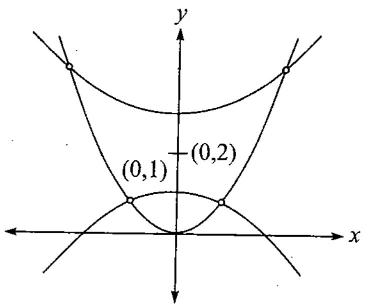


Fig. 5.80

Hence the two graphs of $y = Ax^2$ $(A > 0)$ and the hyperbola $(y-2)^2 - x^2 = 1$ are as shown which intersects in 4 points.

4. **(7)** Given hyperbola is

$$3x^2 - 2y^2 = 6 \text{ or } \frac{x^2}{2} - \frac{y^2}{3} = 1$$

Tangents from the point (α, β)

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

or $(y - mx)^2 = a^2m^2 - b^2$

or $(\beta - m\alpha)^2 = 2m^2 - 3$ ($\because a^2 = 2$ and $b^2 = 3$)

or $m^2\alpha^2 + \beta^2 - 2m\alpha\beta - 2m^2 + 3 = 0$

$m^2(\alpha^2 - 2) - 2\alpha\beta m + \beta^2 + 3 = 0$

$$m_1 . m_2 = \frac{\beta^2 + 3}{\alpha^2 - 3} = 2 = \tan\theta \cdot \tan\phi$$

$\therefore$ $\beta^2 + 3 = 2(\alpha^2 - 2)$

or $2\alpha^2 - \beta^2 = 7$

5. **(3)** Since tangent drawn from the point $A(a, 2)$ are perpendicular then A must lie on the director circle $x^2 + y^2 = 7$. Putting $y = 2$ we get the value of $x^2 = a^2 = 3$

6. **(3)** $(a, 2)$ lies on director circle $x^2 + y^2 = 7$.

$\therefore$ $a^2 = 3$

7. **(8)** Hyperbola is $x^2 - 9y^2 = 9$ or $\frac{x^2}{9} - \frac{y^2}{1} = 1$

Equation of tangent is $y = mx \pm \sqrt{a^2m^2 - b^2}$ (1)

It passes through (3, 2)

$\Rightarrow$ $2 = 3m \pm \sqrt{9m^2 - 1}$

or $4 + 9m^2 - 12m = 9m^2 - 1$

Solving we get values of m as $m_1 = \frac{5}{12}$ and $m_2 = \infty$

Equation of tangent (1) for $m_1 = \frac{5}{12}$

$$y = \frac{5}{12}x \pm \sqrt{9\left(\frac{5}{12}\right)^2 - 1}$$

or $y = \frac{5}{12}x \pm \frac{3}{4}$

on taking (–)ve sign point $P(3, 2)$ does not satisfy the equation of tangent therefore rejecting (–)ve sign. Hence equation of tangent is $y = \frac{5x}{12} + \frac{3}{4}$ (2)

now equation of tangent (1) for $m_2 = \infty$ is $x \pm 3 = 0$

rejecting (+) sign (since taking (+) sign point $P(3, 2)$ does not satisfy this equation.)

Hence equation of tangent is $x - 3 = 0$ (3)

Now equation of chord of contact w.r.t. point $P(3, 2)$ is $T = 0$

or $3x - 18y = 9$

or $x - 6y = 3$ (4)

Solving (2) and (4); $x = -5, y = -\frac{4}{3}$

Solving (3) and (4); $x = 3, y = 0$

Now vertices of triangle are (3, 2), (3, 0), (–5, –4/3)

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -\frac{4}{3} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \left| 3\left(\frac{4}{3}\right) - 2(3+5) + 1(-4) \right|$$

$$= \frac{1}{2} \,|4 - 16 - 4|$$

$= 8$ sq. units

8. **(2)** Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate

$$\therefore \frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\therefore 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

line passing through the points $(e, 0)$ and $(0, e')$

$e'x + ey - ee' = 0$

It is tangent to the circle $x^2 + y^2 = r^2$

$$\therefore \frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$$\therefore r^2 = \frac{e^2e'^2}{e^2 + e'^2} = 4$$

$\therefore$ $r = 2$

9. **(3)** Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

then $2a = ae$, i.e. $e = 2$

$$\therefore \frac{b^2}{a^2} = e^2 - 1 = 3$$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2} = 3$

10. (5) Equation of tangents to hyperbola having slope m are

$y = mx \pm \sqrt{9m^2 - 49}$

Distance between tangents is 2

$\Rightarrow \quad \dfrac{2\sqrt{9m^2 - 49}}{\sqrt{1+m^2}} = 2$

$\Rightarrow \quad 9m^2 - 49 = 1 + m^2$

$\Rightarrow 8m^2 = 50 \Rightarrow m = \pm\dfrac{5}{2}$

11. (4) Equation of hyperbola $(x-3)(y-2) = c^2$

or $xy - 2x - 3y + 6 = c^2$

It passes through (4, 6), then

$4 \times 6 - 2 \times 4 - 3 \times 6 + 6 = c^2$

$\Rightarrow \quad c = 2$

$\therefore \quad$ Latus rectum $= 2\sqrt{2}c = 2\sqrt{2} \times 2 = 4\sqrt{2}$

12. (6) Equation of hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{18} = 1$

or $9x^2 - 8y^2 - 144 = 0$

Homogenization of this equation using

$\dfrac{x\cos\alpha + y\sin\alpha}{p} = 1$

we have $9x^2 - 8y^2 - 144\left(\dfrac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$

Since these lines are perpendicular to each other

$\therefore 9p^2 - 8p^2 - 144(\cos^2\alpha + \sin^2\alpha) = 0$

$p^2 = 144$ or $p = \pm 12$

$\therefore \quad$ radius of the circle = 12

$\therefore \quad$ diameter of the circle = 24

13. (8) The point $P\left(\dfrac{\pi}{6}\right)$ is $\left(a\sec\dfrac{\pi}{6}, b\tan\dfrac{\pi}{6}\right)$ or $P\left(\dfrac{2a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$

$\therefore$ Equation of tangent at P is $\dfrac{x}{\frac{\sqrt{3}a}{2}} - \dfrac{y}{\sqrt{3}b} = 1$

$\therefore$ Area of the triangle $= \dfrac{1}{2} \times \dfrac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$

$\therefore \quad \dfrac{b}{a} = 4$

$\therefore \quad e^2 = 1 + \dfrac{b^2}{a^2} = 17$

14. (4) Eccentricity of the hyperbola $x^2 - y^2\sec^2\theta = 5$ is

$e_1 = \sqrt{\dfrac{1+\sec^2\theta}{\sec^2\theta}} = \sqrt{1+\cos^2\theta}$

Eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$ is

$e_2 = \sqrt{\dfrac{\sec^2\theta - 1}{\sec^2\theta}} = |\sin\theta|$

Given $e_1 = \sqrt{3}\, e_2$

$\Rightarrow \quad 1 + \cos^2\theta = 3\sin^2\theta$

$\Rightarrow \quad \cos\theta = \pm\dfrac{1}{\sqrt{2}}$

$\therefore$ least positive value of θ is $\dfrac{\pi}{4}$

$\therefore p = 4$

15. (4) Let the point be (h, k). Then equation of the chord of contact is $hx + ky = 4$

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore x\left(\dfrac{4-hx}{k}\right) = 1$ has two equal roots

i.e. $hx^2 - 4x + k = 0$ has equal roots

$\therefore hk = 4$

$\therefore$ locus of (h, k) is $xy = 4$

i.e. $c^2 = 4$

Archives

Subjective Type

1. Consider any point $P(3\sec\theta, 2\tan\theta)$ on the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$.

Then, equation of chord of contact to the circle $x^2 + y^2 = 9$ w.r.t. point P is

$(3\sec\theta)x + (2\tan\theta)y = 9 \qquad \text{(i)}$

If (h, k) be the midpoint of chord of contact, then equation of chord of contact will be

$hx + ky - 9 = h^2 + k^2 - 9$ (from $T = S_1$)

or $\quad hx + ky = h^2 + k^2 \qquad \text{(ii)}$

But Eqs. (i) and (ii) represent the same straight line. Hence,

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2}$$

$$\Rightarrow \quad \sec\theta = \frac{3h}{h^2+k^2}, \ \tan\theta = \frac{9k}{2(h^2+k^2)}$$

Eliminating θ, we have

$$\frac{9h^2}{(h^2+k^2)^2} - \frac{81k^2}{4(h^2+k^2)^2} = 1$$

$$\Rightarrow \quad 4h^2 - 9k^2 = \tfrac{4}{9}(h^2+k^2)^2$$

$$\Rightarrow \quad \frac{h^2}{9} - \frac{k^2}{4} = \left(\frac{h^2+k^2}{9}\right)^2$$

Therefore, the locus of (h, k) is

$$\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$$

Objective Type

Fill in the blanks

1.

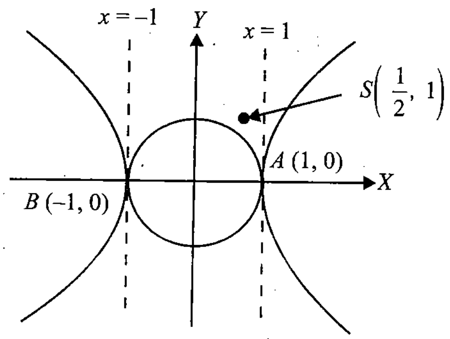


Fig. 5.81

For the circle $x^2 + y^2 = 1$ and rectangular hyperbola $x^2 - y^2 = 1$, one common tangent is evidently $x = 1$, the other being $x = -1$. The required standard form of the ellipse with focus at $S\left(\frac{1}{2}, 1\right)$ and directrix $x = 1$ is

$$\left(x-\frac{1}{2}\right)^2+(y-1)^2=\left(\frac{1}{2}\right)^2(1-x)^2$$

$$\Rightarrow \quad \frac{3x^2}{4}-\frac{x}{2}+(y-1)^2=0$$

$$\Rightarrow \quad \frac{3}{4}\left(x-\frac{1}{3}\right)^2+(y-1)^2=\frac{1}{12}$$

$$\Rightarrow \quad 9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$$

Multiple choice questions with one correct answer

1. d. Given that

$$\frac{x^2}{1-r}-\frac{y^2}{1+r}=1$$

As $\quad r > 1$, so $1 - r < 0$ and $1 + r > 0$

Let $\quad 1 - r = -a^2,\ 1 + r = b^2$.

Then we get

$$\frac{x^2}{-a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

2. c.

$x^2 + 2y^2 \le 1$ represents interior region of circle, where on taking any two points the midpoint of that segment will also lie inside that circle.

Max $\{|x|, |y|\} \le 1 \Rightarrow |x| \le 1, |y| \le 1 \Rightarrow -1 \le x \le 1$ and $-1 \le y \le 1$

which represents the interior region of a square with its sides $x = \pm 1$ and $y = \pm 1$ in which for any two points, their midpoint also lies inside the region.

$x^2 - y^2 \le 1$ represents the exterior region of hyperbola in which we take two points (4, 3) and (4, −3). Then their midpoint (4, 0) does not lie in the same region (as shown in the figure).

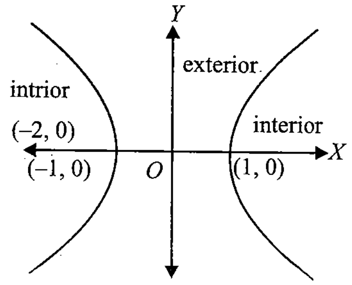


Fig. 5.82

$y^2 \le x$ represents interior region of parabola in which for any two points, their midpoint also lies inside the region.

3. c. We have

$$2x^2 + 3y^2 - 8x - 18y + 35 = k$$

$$\Rightarrow 2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$$

$$\Rightarrow \quad 2(x-2)^2 + 3(y-3)^2 = k$$

For $\quad k = 0$, we get

$$2(x-2)^2 + 3(y-3)^2 = 0$$

which represents the point (2, 3).

4. c. We have

$$2x^2 + 3y^2 - 18y + 35 = k$$

$$\Rightarrow 2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$$

$$\Rightarrow \quad 2(x-2)^2 + 3(y-3)^2 = k$$

For $\quad k = 0$, we get

$$2(x-2)^2 + 3(y-3)^2 = 0$$

which represents the point (2, 3).

5. d. Normals at $p(\theta)$, $Q(\phi)$ are

$$ax\cos\theta + by\cot\theta = a^2 + b^2$$

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

where $\varphi = \frac{\pi}{2} - \theta$ and these pass through (h, k). Therefore,

$$ah\cos\theta + bk\cot\theta = a^2 + b^2$$

and

$$ah\sin\theta + bk\tan\theta = a^2 + b^2$$

Eliminating h, we have

$$bk(\cot\theta\sin\theta - \tan\theta\cos\theta) = (a^2 + b^2)(\sin\theta - \cos\theta)$$

$$\Rightarrow \quad k = -\left(\frac{a^2+b^2}{b}\right)$$

6. b. Let a pair of tangents be drawn from point (x_1, y_1) to hyperbola

$$x^2 - y^2 = 9$$

Then chord of contact will be

$$xx_1 - yy_1 = 9 \quad \text{(i)}$$

But the given chord of contact is

$$x = 9 \quad \text{(ii)}$$

As Eqs. (i) and (ii) represent the same line, these equations should be identical and hence

$$\frac{x_1}{1} = -\frac{y_1}{0} = \frac{9}{9} \Rightarrow x_1 = 1, y_1 = 0$$

Therefore, the equation of pair of tangents drawn from (1, 0) to $x^2 - y^2 = 9$ is

$$(x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x.1 - y.0 - 9)^2$$

(using $SS_1 = T^2$)

$$\Rightarrow \quad (x^2 - y^2 - 9)(-8) = (x - 9)^2$$

$$\Rightarrow \quad -8x^2 + 8y^2 + 72 = x^2 - 18x + 81$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

7. b. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$

$$a^2 = \cos^2\alpha$$

$$\therefore \quad a^2 e^2 = 1$$

Hence, the foci are $(\pm ae, 0) = (\pm 1, 0)$, which are independent of α.

8. d. Equation of tangent to hyperbola $x^2 - 2y^2 = 4$ at any point (x_1, y_1) is $xx_1 - 2yy_1 = 4$.

Comparing with $2x + \sqrt{6}\,y = 2$ or $4x + 2\sqrt{6}\,y = 4$, we have

$$x_1 = 4 \text{ and } -2y_1 = 2\sqrt{6}$$

$\Rightarrow (4, -\sqrt{6})$ is the required point of contact

9. a. The length of transverse axis is $2\sin\theta = 2a$

$$\Rightarrow \quad a = \sin\theta$$

Also, for ellipse

$$3x^2 + 4y^2 = 12$$

or

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$a^2 = 4, b^2 = 3$$

$$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

Hence, the focus of ellipse is $\left(2 \times \frac{1}{2}, 0\right) = (1, 0)$.

As hyperbola is confocal with ellipse, focus of hyperbola = (1, 0). Now,

$$ae = 1 \Rightarrow \sin\theta \times e = 1$$

$$\Rightarrow \quad e = \operatorname{cosec}\theta$$

$\therefore b^2 = a^2(e^2 - 1) = \sin^2\theta(\operatorname{cosec}^2\theta - 1) = \cos^2\theta$

Therefore, the equation of hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$$

or

$$x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$$

10. b. The given hyperbola is

$$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$$

$$\Rightarrow \quad a = 2, b = \sqrt{2}, e = \sqrt{\frac{3}{2}}$$

Therefore, the required area $= \frac{1}{2}a(e-1) \times \frac{b^2}{a}$

$$= \frac{1}{2}\frac{(\sqrt{3}-\sqrt{2})\times 2}{\sqrt{2}}$$

$$= \frac{(\sqrt{3}-\sqrt{2})}{\sqrt{2}}$$

$$= \left(\sqrt{\frac{3}{2}} - 1\right)$$

11. b. $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

$$\Rightarrow ax^2 + by^2 + c = 0$$

or

$$x^2 - 5xy + 6y^2 = 0$$

$$\Rightarrow x^2 + y^2 = \left(-\frac{c}{a}\right) \text{ iff } a = b, x - 2y = 0 \text{ and } x - 3y = 0$$

Hence, the given equation represents two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a.

12. b. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xb^2}{ya^2}$$

$\Rightarrow$ slope of normal at (6, 3) is $\frac{-a^2}{2b^2}$

Equation of normal is $(y - 3) = \frac{-a^2}{2b^2}(x - 6)$

It passes through the point (9, 0)

$$\Rightarrow \quad \frac{a^2}{2b^2} = 1 \Rightarrow e = \sqrt{\frac{3}{2}}$$

Multiple choice questions with one or more than one correct answer

1. a, c. For the given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$$

Hence, the eccentricity of hyperbola $=\frac{5}{3}$.

Let the hyperbola be

$$\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$$

Then

$$B^2=A^2\left(\frac{25}{9}-1\right)=\frac{16}{9}A^2$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{A^2}-\frac{9y^2}{16A^2}=1$$

As it passes through (3, 0), we get $A^2=9$, $B^2=16$.

The equation is

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

Focus of hyperbola is $(\pm ae, 0)\equiv(\pm 5, 0)$.

Vertex of hyperbola is (3, 0).

2. a, b Ellipse and hyperbola will be confocal. So,

$$\left(\pm a\times\frac{1}{\sqrt{2}}, 0\right)\equiv(\pm 1, 0)$$

$\Rightarrow$ $a=\sqrt{2}$ and $e=\frac{1}{\sqrt{2}}$

$\Rightarrow$ $b^2=a^2(1-e^2)\Rightarrow b^2=1$

Therefore, the equation of ellipse is

$$\frac{x^2}{2}+\frac{y^2}{1}=1$$

3. b, d

For ellipse $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$,

$\Rightarrow$ $1^2=2^2(1-e^2)\Rightarrow e=\frac{\sqrt{3}}{2}$

$\therefore$ eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$

$\Rightarrow$ $b^2=a^2\left(\frac{4}{3}-1\right)\Rightarrow 3b^2=a^2$

One of the foci of the ellipse is $(\sqrt{3}, 0)$

Hyperbola passes through $(\sqrt{3}, 0)$

$\Rightarrow$ $\frac{3}{a^2}=1\Rightarrow a^2=3$ and $b^2=1$

$\therefore$ Equation of hyperbola is $x^2-3y^2=3$

Focus of hyperbola is $(ae, 0)\equiv\left(\sqrt{3}\times\frac{2}{\sqrt{3}}, 0\right)\equiv(2, 0)$

Comprehension type

1. b.

A tangent to $\frac{x^2}{9}-\frac{y^2}{4}=1$ is $y=mx+\sqrt{9m^2-4}, m>0$

It is tangent to $x^2+y^2-8x=0$

$$\therefore \quad \frac{4m+\sqrt{9m^2-4}}{\sqrt{1+m^2}}=4$$

$\Rightarrow$ $495m^4+104m^2-400=0$

$\Rightarrow$ $m^2=\frac{4}{5}$ or $m=\frac{2}{\sqrt{5}}$

$\Rightarrow$ the tangent is $y=\frac{2}{\sqrt{5}}m+\frac{4}{\sqrt{5}}$

$\Rightarrow$ $2x-\sqrt{5}y+4=0$.

2. a.

A point on hyperbola is $(3\sec\theta, 2\tan\theta)$

It lies on the circle, so $9\sec^2\theta+4\tan^2\theta-24\sec\theta=0$

$\Rightarrow$ $13\sec^2\theta-24\sec\theta-4=0\Rightarrow \sec\theta=2, -\frac{2}{13}$

$\therefore$ $\sec\theta=2\Rightarrow\tan\theta=\sqrt{3}$.

The point of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$

$\therefore$ The circle with AB as diameter is

$(x-6)^2+y^2=(2\sqrt{3})^2\Rightarrow x^2+y^2-12x+24=0$

Match the following

1. a → p; b → s, t; c → r; d → q, s.

p. Line $hx+ky=1$ touches the circle $x^2+y^2=4$. Hence,

$$\frac{1}{\sqrt{h^2+k^2}}=2$$

$\Rightarrow$ locus is $x^2+y^2=\left(\frac{1}{2}\right)^2$ which is a circle

q. If $|z-z_1|-|z-z_2|=k$ where $k<|z_1-z_2|$, the locus is a hyperbola.

r. Let $t=\tan\alpha$

$\Rightarrow$ $x=\sqrt{3}\cos 2\alpha$ and $y=\sin 2\alpha$

$\Rightarrow$ $\cos 2\alpha=\frac{x}{\sqrt{3}}$ and $\sin 2\alpha=y$

$\Rightarrow \frac{x^2}{3}+y^2=\sin^2 2\alpha+\cos^2 2\alpha=1$ which is an ellipse.

s. If eccentricity is $[1,\infty)$, then the conic can be a parabola (if $e=1$) and a hyperbola if $e\in(1,\infty)$.

t. Let $z=x+iy; x, y\in R$

$\Rightarrow$ $(x+1)^2-y^2=x^2+y^2+1$

$\Rightarrow$ $y^2=x$; which is a parabola.

Integer type

1. (2)

Substituting $\left(\frac{a}{e}, 0\right)$ in $y = -2x + 1$

$\therefore\ 0 = -\frac{2a}{e} + 1$

$\therefore\ \frac{2a}{e} = 1$

$\therefore\ a = \frac{e}{2}$

Also, $1 = \sqrt{a^2 m^2 - b^2}$

$\Rightarrow\ 1 = a^2m^2 - b^2$

$\Rightarrow\ 1 = 4a^2 - b^2$

$\Rightarrow\ 1 = \frac{4e^2}{4} - b^2$

$\Rightarrow\ b^2 = e^2 - 1$

Also, $b^2 = a^2 (e^2 - 1)$

$\therefore\ a = 1, e = 2$

Appendix

Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. c. $L = \sqrt{4+12} = 4$

$\Rightarrow \quad p^2 + q^2 = 16$ and $(p-2)^2 + (q - 2\sqrt{3})^2 = 16$

$\Rightarrow \quad p + \sqrt{3}q = 4$

Now from options, only (4, 0) satisfies the equation.

2. c. Distance $= \sqrt{a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2}$

$= a\sqrt{\sin^2\alpha + \cos^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

$= a\sqrt{2\{1-\cos(\alpha-\beta)\}}$

$= 2a\sin\left(\frac{\alpha-\beta}{2}\right)$

3. In ΔABC, $A \equiv (-3, 0)$; $B \equiv (4, -1)$, and $C \equiv (5, 2)$

$$BC = \sqrt{(5-4)^2 + (2+1)^2} = \sqrt{1+9} = \sqrt{10}$$

and area of $\Delta ABC = \frac{1}{2}[-3(-1-2) + 4(2-0) + (0+1)]$

$= 11$

Therefore, altitude $AL = \frac{2\Delta ABC}{BC} = \frac{2\times 11}{\sqrt{10}} = \frac{22}{\sqrt{10}}$

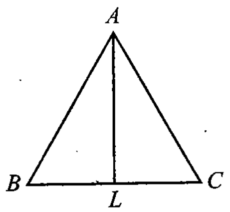


Fig. S–1.1

4. Midpoints of the diagonals must be the same. Therefore,

$$\frac{x-2}{2} = \frac{-3+3}{2}$$

$\Rightarrow \quad x = 2$

and $\quad \frac{-1+3}{2} = \frac{-2+y}{2}$

$\Rightarrow \quad y = 4$

5. Let the vertices of triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$\Rightarrow \quad \frac{x_1+x_2}{2} = 2, \frac{x_2+x_3}{2} = -1, \frac{x_3+x_1}{2} = 4$

Solving, we get $\quad x_1 = 7, x_2 = -3$ and $x_3 = 1$

Similarly, y_1, y_2, and y_3 can be found.

6. c. $a = \sqrt{(8+2)^2 + (-2-2)^2} = \sqrt{116}$

$b = \sqrt{(-4-8)^2 + (-3+2)^2} = \sqrt{145}$

$c = \sqrt{(-4+2)^2 + (-3-2)^2} = \sqrt{29}$

$\Rightarrow a^2 + c^2 = b$

7. d. The given points are collinear as the point $[(kc + la)/(k + l), (kd + lb)/(k + l)]$ divides the points (a, b) and (c, d) in the ratio of $k : L$.

8. d. $l_1 = \sqrt{(2a)^2 + (2b)^2} = 2\sqrt{a^2+b^2}$

$l_2 = \sqrt{(a^2-a)^2 + b^2(a-1)^2} = (a-1)\sqrt{a^2+b^2}$ (if $a > 1$)

$l_3 = \sqrt{(a^2+a)^2 + b^2(a+1)^2} = (a+1)\sqrt{a^2+b^2}$

Now $l_1 + l_2 = l_3$. Hence, points are collinear.

Also when $0 < a < 1$.

$l_2 = (1-a)\sqrt{a^2+b^2}$,

and hence $\quad l_1 = l_2 + l_3$.

In that case also points are collinear.

9. b. Since circumcenter $P(x, y)$ is equidistant from the vertices of the triangle $A(0, 0)$, $B(-2, -2)$, $C(-4, -8)$

we have $\quad AP = CP$ and $AP = BP$

or $\quad x^2 + y^2 = (x+4)^2 + (y+8)^2$

$\Rightarrow \quad 8x + 16y + 80 = 0 \qquad$ (i)

and $\quad x^2 + y^2 = (x+2)^2 + (y+2)^2$

$\Rightarrow \quad 4x + 4y + 8 = 0 \qquad$ (ii)

Solving (i) and (ii), we get $y = -8$ and $x = 6$.

10. The required area

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 7 & 21 \\ 7 & -3 \\ 12 & 2 \\ 0 & -3 \\ 1 & 1 \end{vmatrix}$$

$= \frac{1}{2}\{(21 - 21 + 14 - 36 + 0) - (7 + 147 - 36 + 0 - 3)\}$

$= 127/2$ sq. units

11. As we know that the centroid of the triangle ABC and that of the triangle formed by joining the middle points of the sides of triangle ABC are same.

Therefore, the required centroid is

$$\left(\frac{4+4-2}{3}, \frac{5-3+3}{3}\right) = \left(2, \frac{5}{3}\right)$$

12.

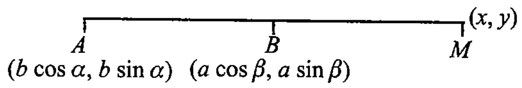


Fig. S–1.2

$$\frac{AM}{BM} = \frac{b}{a}$$

$$\Rightarrow M\left[\frac{ab\cos\alpha - ab\cos\beta}{(a-b)}, \frac{ab\sin\alpha - ab\sin\beta}{(a-b)}\right] \equiv M(x, y)$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$$

$$= \frac{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= -\tan\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad x + \tan\left(\frac{\alpha+\beta}{2}\right)y = 0$$

13. Let (x, y) be the required point. Therefore,

$$\frac{1}{2}\begin{vmatrix} x & y \\ 1 & 5 \\ 3 & -7 \\ x & y \end{vmatrix} = \pm 21$$

$$\Rightarrow 5x - y - 7 - 15 + 3y + 7x = \pm 42$$

$$\Rightarrow 12x + 2y = 64 \text{ or } 12x + 2y = -20$$

$$\Rightarrow \quad 6x + y = 32 \text{ or } 6x + y = -10$$

14. Let $A \equiv (4, -8)$, $B \equiv (-9, 7)$, and $G \equiv (1, 4)$.

Let $C(x, y)$ be the third vertex of ΔABC. Then,

$$1 = (4 - 9 + x)/3 \text{ or } x = 8$$

and $$4 = (-8 + 7 + y)/3 \text{ or } y = 13$$

Hence, $C \equiv (8, 13)$

Now area of ΔABC

$$= \frac{1}{2}\left\|\begin{matrix} 4 & -8 & 1 \\ -9 & 7 & 1 \\ 8 & 13 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}|[4(7 - 13) + 8(-9 - 8) + 1(-117 - 56)]|$$

$= 166.5$ sq. units

15. a. Let A (4, 0), B (– 1, – 1), and C (3, 5) be the given points. Then, we get

$$|AB| = \sqrt{(-1-4)^2 + (-1-0)^2}$$

$$= \sqrt{25+1} = \sqrt{26}$$

$$|BC| = \sqrt{(3+1)^2 + (5+1)^2}$$

$$= \sqrt{16+36} = \sqrt{52}$$

and $$|CA| = \sqrt{(4-3)^2 + (0-5)^2}$$

$$= \sqrt{1+25} = \sqrt{26}$$

Clearly, $|AB| = |CA|$

⇒ Triangle is isosceles.

And $BC^2 = AB^2 + CA^2$ $[\because 52 = 26 + 26]$

⇒ Triangle is right angled.

Exercise 1.2

1. Let the point be (h, k). Therefore,

$$(h - a)^2 + (k - 0)^2 = h^2$$

$$\Rightarrow \quad h^2 + a^2 - 2ah + k^2 = h^2$$

Hence, locus is $y^2 - 2ax + a^2 = 0$.

2. Let the point be (x, y). Then,

$$(x - a)^2 + y^2 - (x + a)^2 - y^2 = 2k^2$$

$$\Rightarrow \quad -4ax - 2k^2 = 0$$

$$\Rightarrow \quad 2ax + k^2 = 0$$

The required equation to the line of the point P.

3. Let (h, k) be the centroid of the triangle, then

$$\Rightarrow \quad h = \frac{\cos\alpha + \sin\alpha + 1}{3}$$

and $$k = \frac{\sin\alpha - \cos\alpha + 2}{3}$$

$$\Rightarrow \quad 3h - 1 = \cos\alpha + \sin\alpha$$

and $$3k - 2 = \sin\alpha - \cos\alpha$$

Squaring and adding, we get $(3h - 1)^2 + (3k - 2)^2 = 2$

$$\Rightarrow 9(h^2 + k^2) - 6h - 12k + 3 = 0$$

$$\Rightarrow 3(h^2 + k^2) - 2h - 4k + 1 = 0$$

Therefore, locus of centroid is $3(x^2 + y^2) - 2x - 4y + 1 = 0$.

4. Let C be (α, β)

The centroid is

$$\left(\frac{2 - 2 + \alpha}{3}, \frac{-3 + 1 + \beta}{3}\right), \text{ i.e., } \left(\frac{\alpha}{3}, \frac{\beta - 2}{3}\right)$$

This lies on $2x + 3y = 1$, therefore, we get

$$2\left(\frac{\alpha}{3}\right) + 3\left(\frac{\beta - 2}{3}\right) = 1$$

$$\Rightarrow \quad 2\alpha + 3\beta = 9$$

Hence, the locus of (α, β) is $2x + 3y = 9$.

5. Let Q be the point (X, Y) and P the point (x, y); the coordinates of Q satisfy the equation $2x + 3y + 4 = 0$, so that $2X + 3Y + 4 = 0$.

Applying the section formula for OQ, O being $(0, 0)$, we get

$$x = \frac{0+3X}{1+3}, y = \frac{0+3Y}{1+3}$$

from which we get $X = \frac{4}{3}x$, $Y = \frac{4}{3}y$

Substituting these values, then the locus of P is

$$\frac{8}{3}x + 4y + 4 = 0$$

$$\Rightarrow \quad 2x + 3y + 3 = 0$$

6.

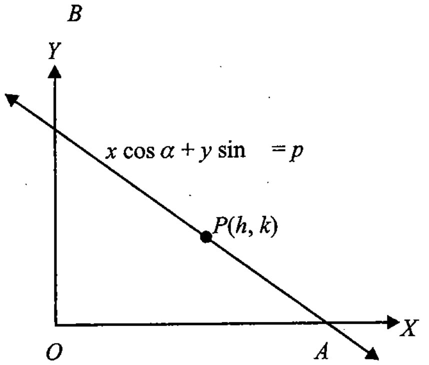


Fig. S–1.3

Equation of the variable line is

$$x\cos\alpha + y\sin\alpha = p \qquad \text{(i)}$$

Here p is a constant and α is the parameter (variable).

Let line Eq. (i) cut x- and y-axes at A and B, respectively, then

Putting $y = 0$ in Eq. (i), we get $A \equiv (p\sec\alpha, 0)$

Putting $x = 0$ in Eq. (i), we get $B \equiv (0, p\operatorname{cosec}\alpha)$

AB is the portion of the Eq. (i) intercepted between the axes.

Let $P(h, k)$ be the midpoint of AB. We have to find the locus of point. $P(h, k)$, For this, we will have to eliminate α and find a relation in h and k. Therefore,

$$h = \frac{p\sec\alpha + 0}{2} = \frac{p}{2}\sec\alpha \qquad \text{(ii)}$$

and

$$k = \frac{0 + p\operatorname{cosec}\alpha}{2} = \frac{p}{2}\operatorname{cosec}\alpha \qquad \text{(iii)}$$

From Eq. (ii), we get

$$\cos\alpha = \frac{p}{2h} \qquad \text{(iv)}$$

From Eq. (iii), we get

$$\sin\alpha = \frac{p}{2k} \qquad \text{(v)}$$

Squaring and adding Eqs. (iv) and (v), we get

$$\cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \text{ or } \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}$$

Hence, locus of point $P(h, k)$ is

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$$

7. Let (h, k) be the point of intersection of $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$. Then,

$$h\cos\alpha + k\sin\alpha = a \qquad \text{(i)}$$

$$h\sin\alpha - k\cos\alpha = b \qquad \text{(ii)}$$

Squaring and adding Eqs. (i) and (ii), we get

$$(h\cos\alpha + k\sin\alpha)^2 + (h\sin\alpha - k\cos\alpha)^2 = a^2 + b^2$$

$$\Rightarrow \quad h^2 + k^2 = a^2 + b^2$$

Hence, locus of (h, k) is

$$x^2 + y^2 = a^2 + b^2$$

Exercise 1.3

1.

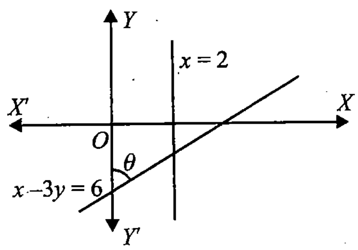


Fig. S-1.4

$$\theta = 90° - \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(3)$$

2. Midpoint of $AB = E\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$ and midpoint of $CD = F\left(\frac{a'-a}{2}, \frac{b-b'}{2}\right)$.

Hence, equation of line EF is

$$y - \frac{b+b'}{2} = \frac{b-b'-b-b'}{a'-a-a-a'}\left(x - \frac{a+a'}{2}\right)$$

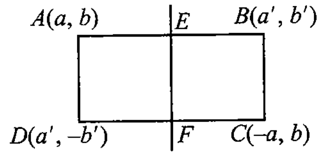


Fig. S–1.5

On simplification, we get

$$2ay - 2b'x = ab - a'b'$$

3. Required equation of median is

$$y + 8 = \frac{-\frac{3}{2} + 8}{-2 - 5}(x - 5)$$

$$\Rightarrow \quad 13x + 14y + 47 = 0$$

4. The given line is

$$bx - ay = ab$$

Obviously, it cuts x-axis at $(a, 0)$.

The equation of line perpendicular to Eq. (i) is

$$ax + by = k$$

but it passes through $(a, 0)$

$\Rightarrow \quad k = a^2$

Hence, the required equation of line is

$$ax + by = a^2$$

i.e., $\quad \frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

5. Slope of $DE = \frac{7-3}{5-1} = 1$

$\Rightarrow \quad$ Slope of $AB = 1$

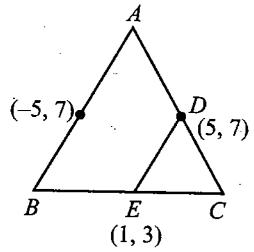


Fig. S–1.6

Hence equation of AB is

$$y - 7 = (x + 5)$$

$\Rightarrow \quad x - y + 12 = 0$

6. Angle between both the lines is

$$\tan^{-1} m \pm \tan^{-1} m = \tan^{-1} \frac{2m}{1-m^2} \text{ or } \tan^{-1} 0$$

Therefore, the equations of the lines are

$$y = 0, y = \frac{2mx}{1-m^2}$$

7. The midpoint of $P(-2, 6)$ and $Q(4, 2)$ is

$\left(\frac{-2+4}{2}, \frac{6+2}{2}\right)$, i.e., $(1, 4)$

and the gradient of line $PQ = \frac{2-6}{4+2} = \frac{-2}{3}$

Therefore, the slope of $L = \frac{3}{2}$.

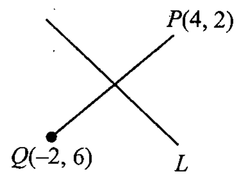


Fig. S–1.7

Hence, the equation of line which passes through point $(1, 4)$ is

$$y - 4 = \frac{3}{2}(x - 1)$$

$\Rightarrow \quad 3x - 2y + 5 = 0$

8. Slope of given lines are $-1/(a-1)$ and $-2/a^2$.

Since the lines are perpendicular, therefore, we get

$$\left(\frac{-1}{a-1}\right)\left(-\frac{2}{a^2}\right) = -1$$

$\Rightarrow \quad \frac{2}{(a-1)a^2} = -1$

$\Rightarrow \quad 2 + a^2(a-1) = 0$

$\Rightarrow \quad a^3 - a^2 + 2 = 0$

$\Rightarrow \quad a = -1$

9. $x + 2|y| = 1$ and $x = 0$ gives

$$|y| = \frac{1}{2}$$

$\therefore \quad y = \pm\frac{1}{2}$

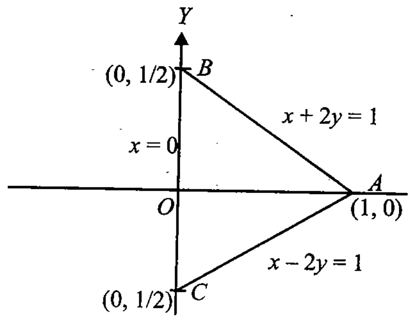


Fig. S–1.8

The required area = area of ΔABC (where AB is $x + 2y = 1$ and AC is $x - 2y = 1$)

$$\text{Required area} = \frac{1}{2}(BC)(OA) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

10. Point of intersection of $x - 2y = 1$ and $x + 3y = 2$ is $(7/5, 1/5)$.

Any line parallel to $3x + 4y = 0$ is $3x + 4y + K = 0$.

As this line passes through $(7/5, 1/5)$, we get

$$\frac{21}{5} + \frac{4}{5} + K = 0$$

$\Rightarrow \quad K = -5$

Therefore, the required line is

$$3x + 4y - 5 = 0$$

11. $\sqrt{3}x + y = 0$ makes an angle of $120°$ with OX, where as $\sqrt{3}x - y = 0$ makes an angle of $60°$ with OX. Therefore, the required line is

$$y - 2 = 0$$

12. Let 'a' and 'b' be the intercepts on the axes. Then the equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ (i)

Since Eq. (i) passes through (4, 3), we get

$$\frac{4}{a}+\frac{3}{b}=1 \quad \text{(ii)}$$

Also given that $a+b=-1$ (iii)

From Eq. (iii), we get

$$b=-1-a \quad \text{(iv)}$$

Putting in Eq. (ii), we get

$$\frac{4}{a}+\frac{3}{-1-a}=1$$

$\Rightarrow \quad -4-4a+3a=-a-a^2$

$\Rightarrow \quad a^2=4$

$\Rightarrow \quad a=\pm 2$

When $a=2$, then from Eq. (iv), $b=-1-2=-3$

and when $a=-2, b=-1+2=1$

Therefore, the line is

$$\frac{x}{2}+\frac{y}{-3}=1 \text{ and } \frac{x}{-2}+\frac{y}{1}=1$$

13. Let the line be

$$\frac{x}{a}+\frac{y}{b}=1$$

This line meets x-axis at $A(a, 0)$ and y-axis at $B(0, b)$. Therefore, we get

$$(3, 4)=\left(\frac{a}{2}, \frac{b}{2}\right)$$

$\Rightarrow \quad \frac{a}{2}=3, \frac{b}{2}=4$

$\Rightarrow \quad a=6, b=8.$

Therefore, the required line is

$$\frac{x}{6}+\frac{y}{8}=1$$

$\Rightarrow \quad 4x+3y=24$

14. $m_{AC}=\frac{6-2}{5-3}=2$

$\Rightarrow \quad m_{BD}=-\frac{1}{2}$

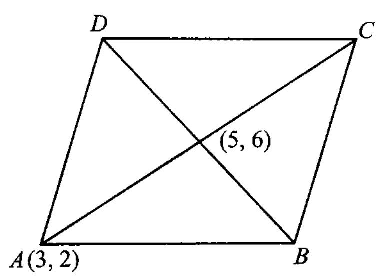


Fig. S–1.9

Thus, the equation of BD is

$$(y-6)=-\frac{1}{2}(x-5),$$

i.e., $\quad 2y+x-17=0$

15. Let $P(3, -4)$ be the foot of the perpendicular from the origin O on the required line. Then the slope of OP

$$=\frac{-4-0}{3-0}=-\frac{4}{3},$$

and therefore the slope of the required line is 3/4.

Hence, its equation is

$$y+4=\frac{3}{4}(x-3)$$

$\Rightarrow \quad 3x-4y=25$

16. The given form is $3x+3y+7=0$.

$\Rightarrow \quad \frac{3}{\sqrt{3^2+3^2}}x+\frac{3}{\sqrt{3^2+3^2}}y+\frac{7}{\sqrt{3^2+3^2}}=0$

$\Rightarrow \quad \frac{3}{3\sqrt{2}}x+\frac{3}{3\sqrt{2}}y=\frac{-7}{3\sqrt{2}},$

$\therefore p=\left|\frac{-7}{3\sqrt{2}}\right|=\frac{7}{3\sqrt{2}}$

17. Given line is $x+\sqrt{3}\,y+3\sqrt{3}=0$. Therefore, we get

$$y=\left(-\frac{1}{\sqrt{3}}\right)x-3$$

Therefore, slope of Eq. (i) $=-1/\sqrt{3}$.

Let the slope of the required line be m.

Also the angle between these lines is 60°

$\Rightarrow \quad \tan 60°=\left|\frac{m-(-1/\sqrt{3})}{1+m(1/\sqrt{3})}\right|$

$\Rightarrow \quad \sqrt{3}=\left|\frac{\sqrt{3}\,m+1}{\sqrt{3}-m}\right|$

$\Rightarrow \quad \frac{\sqrt{3}m+1}{\sqrt{3}-m}=\sqrt{3}$

$$\frac{\sqrt{3}m+1}{\sqrt{3}-m}=\sqrt{3}$$

$\Rightarrow \quad m=\frac{1}{\sqrt{3}}$

Using $y=mx+c$, the equation of the required line is

$$y=\frac{1}{\sqrt{3}}x+0,$$

i.e., $x-\sqrt{3}\,y=0$ (as the line passes through the origin, $c=0$)

$$\frac{\sqrt{3}m-1}{\sqrt{3}-m}=-\sqrt{3}$$

$\Rightarrow \quad \sqrt{3}\,m+1=-3+\sqrt{3}m$

$\Rightarrow$ m is not defined.

Therefore, the slope of the required line is not defined. Thus, the required line is a vertical line. This line passes through the origin.

Therefore, the equation of the required line is $x=0$.

18. Any point A on the first line is $(t, 5t - 4)$. Any point B on the second line is $\left[r, \frac{(3r-4)}{4}\right]$. Hence,

$$1 = \frac{2r+t}{3}$$

and $$5 = \frac{\frac{3r-4}{2} + 5t - 4}{3}$$

$\Rightarrow$ $$2r + t = 3 \text{ and } 3r + 10t = 42$$

On solving, we get $t = \frac{75}{17}$.

Hence A is $\left(\frac{75}{77}, \frac{304}{17}\right)$.

19.

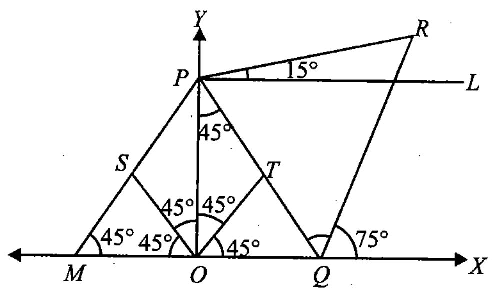


Fig. S-1.10

Equation of line OT: (0, 0) and slope of $OT = \tan 45° = 1$. Therefore, equation of line OT will be

$$y - 0 = 1(x - 0) \text{ or } x - y = 0$$

Equation of line OS: (0, 0) and $\angle SOX = 135°$. Therefore, the slope of line $OS = \tan 135° = -1$.

Equation of line OS will be

$$y - 0 = (-1)(x - 0)$$

or $$y = -x \text{ or } x + y = 0$$

Equation of line SP: Given that $OT = 2\sqrt{2}$. Therefore,

$$OP = OT \sec 45° = 2\sqrt{2}\sqrt{2} = 4$$

$\therefore$ $P \equiv (0, 4)$. Also the slope of $SP = \tan 45° = 1$

Therefore, equation of line SP will be

$$y - 4 = 1(x - 0) \text{ or } x - y + 4 = 0$$

Equation of QR: Given that $OQ = OT \sec 45° = 2\sqrt{2}\sqrt{2} = 4$. Therefore, $Q \equiv (4, 0)$. Also slope of line $QR = \tan 75° = 2 + \sqrt{3}$.

Therefore, equation of line QR will be

$$y - 0 = (2 + \sqrt{3})(x - 4)$$

or $(2 + \sqrt{3})x - y - 8 - 4\sqrt{3} = 0$

Equation of PR: $P \equiv (0, 4)$

Slope of line $PR = \tan 15° = 2 - \sqrt{3}$

Therefore, equation of line PR is

$$y - 4 = (2 - \sqrt{3})(x - 0)$$

or $$y - 4 = (2 - \sqrt{3})x$$

or $(2 - \sqrt{3})x - y + 4 = 0$

Equation of PQ: $P \equiv (0, 4)$ and $Q \equiv (4, 0)$

Therefore, equation of line PQ will be

$$\frac{x}{4} + \frac{y}{4} = 1$$

or $$x + y = 4$$

20. Let m_1 and m_2 be the slopes of the straight lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$. Then,

$$m_1 = -\frac{1}{-2} = \frac{1}{2} \text{ and } m_2 = -\frac{3}{1} = -3$$

Let $$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \pm\left(\frac{\frac{1}{2} + 3}{1 - \frac{3}{2}}\right) = \pm 7$$

$\Rightarrow$ $$\theta = \tan^{-1}(\pm 7)$$

Thus, the acute angle between the lines is $\tan^{-1}(7)$ and the obtuse angle is $\pi - \tan^{-1}(7)$.

21. Let $A \equiv (a, 0)$, $B \equiv (0, b)$, $A' \equiv (a', 0)$, $B' \equiv (0, b')$

Equation of $A'B$ is

$$\frac{x}{a'} + \frac{y}{b} = 1 \qquad \text{(i)}$$

and the equation of AB' is

$$\frac{x}{a} + \frac{y}{b'} = 1 \qquad \text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get,

$$x\left(\frac{1}{a} - \frac{1}{a'}\right) + y\left(\frac{1}{b'} - \frac{1}{b}\right) = 0$$

$\Rightarrow$ $$\frac{x(a' - a)}{aa'} + \frac{y(b - b')}{bb'} = 0$$

$\Rightarrow$ $$\frac{x}{aa'} + \frac{y}{bb'} = 0 \qquad [\text{using } a' - a = b - b']$$

Exercise 1.4

1. $P(2, -1)$ goes 2 units along $x + y = 1$ up to A and 5 units along $x - 2y = 4$ up to B. Then,

$$\text{slope of } PA = -1 = \tan 135°$$

$$\text{slope of } PB = 1/2 = \tan\theta$$

$\Rightarrow$ $$\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$$

The coordinates of B, i.e., $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ are $(2\sqrt{5} + 2, \sqrt{5} - 1)$.

The coordinates of A, i.e., $(x_1 + r\cos 135°, y_1 + r\sin 135°)$ are $(2 - \sqrt{2}, \sqrt{2} - 1)$.

2. Since $m = 3/4$, then $\cos\theta = 4/5$ and $\sin\theta = 3/5$.

Any point on the line through A has the coordinates $(2 + 4r/5, 3 + 3r/5)$.

If this point is also the point of intersection, P, then these coordinates satisfy the equation of the given line. Hence,

$$5\left(2+\frac{4}{5}r\right)+7\left(3+\frac{3}{5}r\right)+40=0$$

or $$r\left(4+\frac{21}{5}\right)+71=0$$

or $$r=-\frac{355}{41}$$

Thus, the distance between A and P is 355/41 units, the vector $\overrightarrow{AP}$ being in the negative direction of the line.

3.

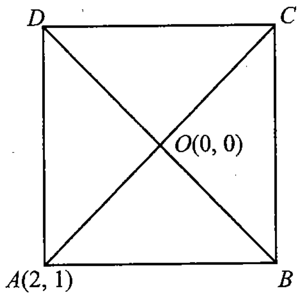


Fig. S–1.11

Let $ABCD$ be the square whose centre is O.

Given $A \equiv (2, 1)$ and $O \equiv (0, 0)$.

Now $AO = \sqrt{5}$

and slope of $AO = \frac{1-0}{2-0} = \frac{1}{2} = \tan\theta$ (say)

$\Rightarrow \cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

Now coordinates of the points on AC which are at a distance $\sqrt{5}$ from O will be

$(0 \pm \sqrt{5}\cos\theta, 0 \pm \sqrt{5}\sin\theta)$

i.e., $(\pm 2, \pm 1)$ or $(2, 1)$ and $(-2, -1)$

But $A \equiv (2, 1)$, therefore $C \equiv (-2, -1)$.

Again $BD \perp AC$

$\Rightarrow$ slope of $BD = -2 = \tan\alpha$ (say)

$\Rightarrow \cos\alpha = -\frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$

Since B and D are on BD at a distance $\sqrt{5}$ from O, therefore their coordinates (in some order) will be

$(0 \pm \sqrt{5}\cos\alpha, 0 \pm \sqrt{5}\sin\alpha)$.

i.e., $(0 \mp 1, 0 \pm 2)$ or $(-1, 2)$ and $(1, -2)$

Exercise 1.5

1. Since the perpendicular distance between the given lines is $\sqrt{2}$. Therefore, the required line is a straight line perpendicular to the given parallel lines and passes through $(-5, 4)$.

Any line perpendicular to given lines is

$$x - y + k = 0$$

This line passes through $(-5, 4)$, therefore

$$-5 - 4 + k = 0$$

$$\therefore \quad k = 9$$

Hence, the required line is

$$x - y + 9 = 0$$

2. Lines $3x + 4y + 2 = 0$ and $3x + 4y + 5 = 0$ are on the same side of the origin.

The distance between these lines is

$$d_1 = \left|\frac{2-5}{\sqrt{3^2+4^2}}\right| = \frac{3}{5}$$

Lines $3x + 4y + 2 = 0$ and $3x + 4y - 5 = 0$ are on the opposite sides of the origin.

The distance between these lines is

$$d_2 = \left|\frac{2+5}{\sqrt{3^2+4^2}}\right| = \frac{7}{5}$$

Thus, $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ in the ratio $d_1 : d_2$, i.e., 3 : 7.

3. Equation of any line parallel to $3x - 4y - 5 = 0$ is

$$3x - 4y + \lambda = 0 \quad \text{(i)}$$

Distance of (i) from $3x - 4y - 5 = 0$ is 1 unit.

$$\Rightarrow \frac{|5+\lambda|}{5} = 1$$

$$\Rightarrow \lambda = -10 \text{ or } 0$$

$\Rightarrow$ Required line is $2x - 4y - 10 = 0$ or $3x - 4y = 0$

These are the equations of the required lines.

4. Here, $p = \left|\frac{-k}{\sqrt{\sec^2\alpha + \text{cosec}^2\alpha}}\right|$, $p' = \left|\frac{-k\cos 2\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$

Hence, $$4p^2 + p'^2 = \frac{4k^2}{\sec^2\alpha + \text{cosec}^2\alpha} + \frac{k^2(\cos^2\alpha - \sin^2\alpha)^2}{1}$$

$= 4k^2\sin^2\alpha\cos^2\alpha + k^2(\cos^4\alpha + \sin^4\alpha) - 2k^2\cos^2\alpha \times \sin^2\alpha$

$= k^2(\sin^2\alpha + \cos^2\alpha)^2 = k^2$

Exercise 1.6

1. **a.** $L_1(8, -9) = 2(8) + 3(-9) - 4 = -15$

   $L_2(8, -9) = 6(8) + 9(-9) + 8 = -25$

   Hence, the point lies on the same side of the lines.

2. **d.** $L(-1, -1) = 3(-1) - 8(-1) - 7 < 0$

   $L(3, 7) = 3 \times 3 - 8 \times 7 - 7 < 0$

   Hence, $(-1, -1)$ and $(3, 7)$ lie on the same side of the line.

3. $L = 2x + 3y - 6$

   $$\Rightarrow \quad L(\alpha, 2 + \alpha) = 5\alpha$$

   $$L\left(\frac{3}{2}\alpha, \alpha^2\right) = 3\alpha + 3\alpha^2 - 6$$

   For the given condition, we get

   $$5\alpha(3\alpha + 3\alpha^2 - 6) < 0 \text{ or } \alpha(\alpha + 2)(\alpha - 1) < 0$$

   $$\Rightarrow \quad \alpha \in (-\infty, -2) \cup (0, 1)$$

Exercise 1.7

1. $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$

   or $a(x + y + 3) + b(-2x + 3y + 4) = 0$

   which represents a family of straight lines passing through the point of intersection of $x + y + 3 = 0$ and $-2x + 3y + 4 = 0$, i.e., $(-1, -2)$.

2. a, b, c are in H.P., then

   $$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \qquad \text{(i)}$$

   Given line is

   $$\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0 \qquad \text{(ii)}$$

   Subtracting Eq. (ii) from Eq. (i), we get

   $$\frac{1}{a}(x - 1) + \frac{1}{b}(y + 2) = 0$$

   Since $\quad a \neq 0, b \neq 0$, we get

   $$x - 1 = 0 \text{ and } y + 2 = 20$$

   $$\Rightarrow \quad x = 1$$

   and $\quad y = -2$

3. **b.** Let the equation of the variable line be

   $$ax + by + c = 0$$

   Then according to the given condition, we get

   $$\frac{2a + c}{\sqrt{a^2 + b^2}} + \frac{2b + c}{\sqrt{a^2 + b^2}} + \frac{-2a - 2b + c}{\sqrt{a^2 + b^2}} = 0$$

   $$\Rightarrow \quad c = 0$$

   which shows that the line passes through $(0, 0)$ for all values of a, b.

4. Lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ are concurrent at $(1, -1)$ and lines $x - y + 1 + \lambda_2(2x - y - 2) = 0$ are concurrent at $(3, 4)$.

   Thus, the equation of line common to both families is

   $$y - 4 = \frac{-1 - 4}{1 - 3}(x - 3)$$

   i.e., $\quad 5x - 2y - 7 = 0$

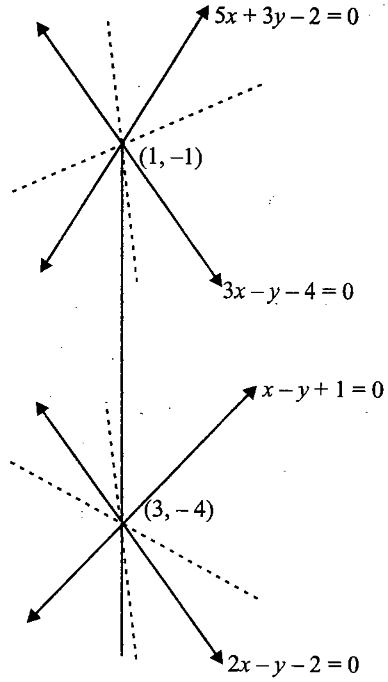


Fig. S–1.12

Exercise 1.8

1. The given pair is $(2x + y)(x - y) = 0$

   So, the required pair is $(2x + y + k)(x - y + k') = 0$, where $2x + y + k = 0$ and $x - y + k' = 0$ pass through $(1, 0)$.

   $$\therefore \quad k = -2, k' = -1$$

   Therefore, the required pair is

   $(2x + y - 2)(x - y - 1) = 0$

2. If m, $2m$ are the slopes, then

   $$m + 2m = -\frac{2/h}{1/b} = -\frac{2b}{h}$$

   and $\quad m \times 2m = \frac{1/a}{1/b} = \frac{b}{a}$

   Eliminating m, we get

   $$1\left(-\frac{2b}{3h}\right)^2 = \frac{b}{a}$$

   $$\Rightarrow \quad \frac{ab}{h^2} = \frac{9}{8}$$

3. Let

$$y = mx \quad \text{(i)}$$

be a line through the origin making an angle of 60° with the line

$$Ax + By + C = 0 \quad \text{(ii)}$$

Then, we have

$$\tan 60° = \pm \frac{m-(-A/B)}{1+m(-A/B)}$$

From Eq. (i) we have, $m = y/x$. Substituting this value of m in the above result,

i.e., $3(B - Am)^2 = (mB + A)^2$, we have

$$(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0 \quad \text{(iii)}$$

These are the straight lines passing through the origin and making angles of 60° with Eq. (ii), i.e., forming an equilateral triangle with Eq. (ii).

Now, OL = length of perpendicular to $Ax + By + C = 0$ from (0, 0)

$$= \frac{C}{\sqrt{(A^2+B^2)}}$$

So, $$\text{area} = \frac{C^2}{\sqrt{3(A^2+B^2)}}$$

4. Let $ax^2 + 2hxy + by^2 = b(y \pm x)(y - mx)$

Taking + ve sign, we get

$$ax^2 + 2hxy + by^2 = b(y + x)(y - mx)$$

Equating coefficient of x^2, we get

$$-bm = a$$

$$\Rightarrow \quad m = -\frac{a}{b}$$

Equating coefficient of xy, we get

$$b - bm = 2h$$

5. Given equation is

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$$

Writing the Eq. (i) as a quadratic equation in x, we have

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

$$\therefore x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - 4 \times 2(3y^2+7y+4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^2+4y+4}}{4} = \frac{-(5y+6)+(y+2)}{4}$$

$$\therefore x = \frac{-5y-6+y+2}{4}, \frac{-5y-6-y-2}{4}$$

or $\quad 4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$

or $\quad x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence, Eq. (i) represents a pair of straight lines whose equations are

$$x + y + 1 = 0 \quad \text{(i)}$$

and $$2x + 3y + 4 = 0 \quad \text{(ii)}$$

Solving these two equations, the required point of intersection is (1, – 2).

6. $y - 1 = m(x - 2)$

$$\Rightarrow \quad y - mx = 1 - 2m$$

or $$\frac{y-mx}{1-2m} = 1$$

Homogenizing the given pair of straight lines, we get

$$(4x^2 + y^2) - \frac{(x-4y)(y-mx)}{1-2m} - \frac{2(y-mx)^2}{(1-2m)^2} = 0$$

$$\Rightarrow (1-2m)^2(4x^2+y^2) - (x-4y)(y-mx)(1-2m) - 2(y-mx)^2 = 0$$

Equating coefficient of xy to 0, we get

$$4m - (1 + 4m)(1 - 2m) = 0 \text{ or } 8m^2 + 2m - 1 = 0$$

or $\quad 8m^2 + 4m - 2m - 1 = 0$

$\Rightarrow \quad 4m(2m + 1) - (2m + 1) = 0$

$\Rightarrow \quad m = -1/2$ or $m = 1/4$

Therefore, lines are $y - 1 = -\frac{1}{2}(x - 2)$

and $y - 1 = \frac{1}{4}(x - 2)$

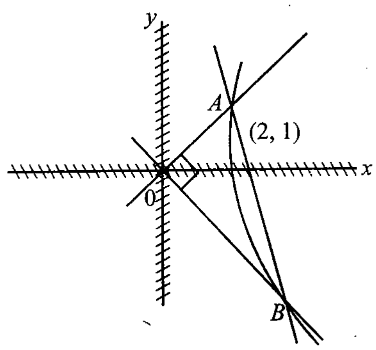


Fig. S–1.13

7. Since lines are parallel

$$\therefore \quad h^2 - ab = 0$$

$$\Rightarrow \quad \frac{(\lambda+\mu)^2}{4} - \lambda\mu = 0$$

$$\Rightarrow \quad (\lambda - \mu)^2 = 0$$

$$\Rightarrow \quad \lambda = \mu$$

8. Bisector of the angle between the positive directions of the axes is $y = x$.

Since it is one of the lines of the given pair of lines $ax^2 + 2hxy + by^2 = 0$, we have

$$x^2(a + 2h + b) = 0 \text{ or } a + b = -2h.$$

9. The given equation of pair of straight lines can be rewritten as

$$(\sqrt{3}x - y)(x - \sqrt{3}y) = 0$$

Their separate equations are

$$y = \sqrt{3}x \text{ and } y = \frac{1}{\sqrt{3}}x$$

or $\quad y = \tan 60° \, x$ and $y = \tan 30° \, x$

After rotation, the separate equations are

$$y = \tan 90° \, x \text{ and } y = \tan 60° \, x$$

or $\quad x = 0$ and $y = \sqrt{3}x$

Therefore, the combined equation in the new position is

$$x(\sqrt{3}x - y) = 0 \text{ or } \sqrt{3}x^2 - xy = 0$$

10. The given equation will represent a pair of real and distinct lines if $h^2 > ab$,

i.e., $\quad \left(-\frac{k}{2}\right) > (2)(2)$ or $k^2 > 16$

or $\quad (k-4)(k+4) > 0$

i.e. $\quad k \in (-\infty, -4) \cup (4, \infty)$.

11. Let $\phi(x, y) = 6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.

Differentiating with respect to x treating y as constant, we get

$$\frac{d\phi}{dx} = 12x + 5y + 13$$

Differentiating with respect to x treating y as constant, we get

$$\frac{d\phi}{dy} = 5x - 42y + 38$$

Solving equations $12x + 5y + 13 = 0$ and $5x - 42y + 38 = 0$, we get

$$x = -\frac{32}{23} \text{ and } y = \frac{17}{23}$$

Therefore, the point of intersection is (– 32/23, 17/32).

12. Let ϕ be the angle between the lines represented by

$$x^2 + 2xy \sec\theta + y^2 = 0 \qquad \text{(i)}$$

Here, $\quad a = 1, b = 1, h = \sec\theta$

Hence,

$$\tan\phi = \frac{2\sqrt{(\sec^2\theta - 1 \times 1)}}{1+1}$$

$$\Rightarrow \quad \tan\phi = \frac{2\sqrt{(\sec^2\theta - 1)}}{2} = \tan\theta$$

$$\therefore \quad \phi = \theta$$

Hence, the angle between the lines represented by Eq. (i) is θ.

Chapter 2

Exercise 2.1

1. Here the centre of circle (3, –1) must lie on the line $x + 2by + 7 = 0$.

Therefore, $3 - 2b + 7 = 0 \Rightarrow b = 5$.

2. Let a triangle has its three vertices as (0, 0), (a, 0), (0, b).

We have the moving point (h, k) such that $h^2 + k^2 + (h - a)^2 + k^2 + h^2 + (k - b)^2 = c$

$$\Rightarrow 3h^2 + 3k^2 - 2ah - 2bk + a^2 + b^2 = c$$

Therefore, locus is $3x^2 + 3y^2 - 2ax - 2by + a^2 + b^2 = c$ which is circle.

3. Here, radius $\sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{1-\lambda}{2}\right)^2 - 5} \le 5$

$$\Rightarrow 2\lambda^2 - 2\lambda - 119 \le 0$$

$$\Rightarrow \frac{1-\sqrt{239}}{2} \le \lambda \le \frac{1+\sqrt{239}}{2} \Rightarrow -7.2 \le \lambda \le 8.2 \text{ (approx.)}$$

$$\therefore \quad \lambda = -7, -6, \ldots, 8$$

4. Radius = perpendicular distance from (1, –3) to $3x - 4y - 5 = 0$, i.e., $\left|\frac{3 + 12 - 5}{\sqrt{5^2}}\right| = 2$.

5. Centre of the given circle is (1, 2). Let (α, β) be the other end.

$$\therefore \quad \frac{\alpha + 3}{2} = 1; \; \frac{\beta + 2}{2} = 2$$

$$\Rightarrow \quad \alpha = -1, \beta = 2$$

$\Rightarrow$ Other end is (–1, 2).

6. Let (h, k) be the centroid, then $h = \frac{a\cos t + b\sin t + 1}{3}$

and $\quad k = \frac{a\sin t - b\cos t + 0}{3}$

$$\Rightarrow \quad 3h - 1 = a\cos t + b\sin t \qquad \text{(i)}$$

$$\text{and} \quad 3k = a\sin t - b\cos t \qquad \text{(ii)}$$

Squaring and adding Eqs. (i) and (ii), $(3h - 1)^2 + (3k)^2 = a^2 + b^2$.

Hence, the locus of (h, k) is $(3x - 1)^2 + (3y)^2 = a^2 + b^2$.

7. Let centre be (h, k), then $(h-3)^2 + (k+2)^2 = (h+2)^2 + k^2$
$\Rightarrow 10h - 4k - 9 = 0$

Also the centre lies on the given line, so $2h - k = 3$.

On solving $\quad k = -6, h = -\frac{3}{2}$

Radius is $(h-3)^2 + (k+2)^2 = \frac{145}{4}$

8. Extremities of diameter are (5, 7) and (1, 4) and radius is half of the distance between them

$$= \frac{1}{2}\sqrt{(5-1)^2+(7-4)^2} = \frac{5}{2}$$

9.

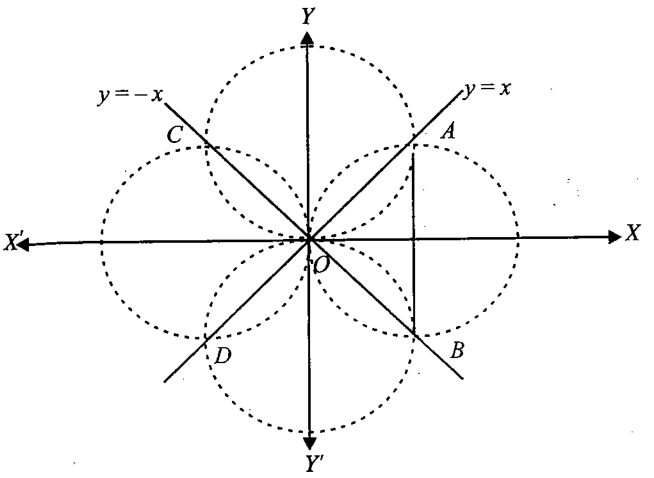


Fig. S-2.1

Coordinates of point A are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$,

and coordinates of point B are $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$.

Now from the geometry, A and B are end points of diameter of the circle.

Then, equation of circle is

$$\left(x-\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)+\left(y-\frac{a}{\sqrt{2}}\right)\left(y+\frac{a}{\sqrt{2}}\right)=0$$

or $\quad x^2+y^2-\sqrt{2}ax=0$

Similarly, circle with C and D as end points of diameter is

$$x^2+y^2+\sqrt{2}ax=0$$

With similar arguments, circles with A and C and B and D as end points of diameter are given by

$$x^2+y^2\pm\sqrt{2}ay=0$$

10. Centre (1, 2) and since circle touches x-axis, therefore, radius is equal to 2.

Hence, the equation is $(x-1)^2+(y-2)^2=2^2$

$\Rightarrow\ x^2+y^2-2x-4y+1=0$

11.

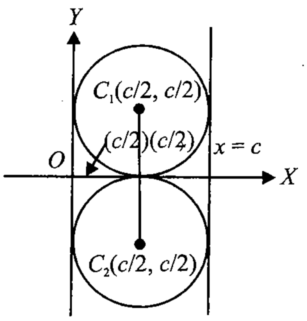


Fig. S–2.2

Since the circle touches both the coordinate axes and the line $x = c$, so there will be in all two such circles with centres C_1 and C_2 in 1st and 4th quadrants.

Hence, diameter of the circle = c,

Therefore, radius of the circle = $c/2$

and the coordinates of the centres are $(c/2, \pm c/2)$.

Hence, the equation of the two circles are $(x - c/2)^2 + (y \pm c/2)^2 = (c/2)^2$,

or $\ x^2+y^2-cx\pm cy+c^2/4=0$.

12.

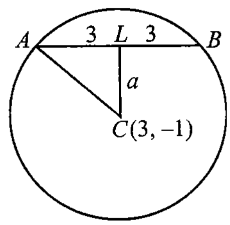


Fig. S–2.3

If the circle cuts off a chord AB of length 6 from the line $2x - 5y + 18 = 0$, then CL = length of ⊥ from centre $C(3, -1)$ on the line

$$= \frac{|2.3-5.(-1)+18|}{\sqrt{(4+25)}} = \sqrt{29}$$

But, $\qquad AL = BL = 3$

In right-angled triangle, CLA, $CA^2 = CL^2 + AL^2 = 29 + 9 = 38$.

Therefore, radius of the circle = $CA = \sqrt{38}$

Hence, the equation of the circle is $(x-3)^2+(y+1)^2 = \{\sqrt{38}\}^2$.

$\Rightarrow\ x^2+y^2-6x+2y-28=0$

13.

$$2\sqrt{g^2-c} = 2a \qquad \text{(i)}$$

$$2\sqrt{f^2-c} = 2b \qquad \text{(ii)}$$

On squaring Eqs. (i) and (ii) and then subtracting Eq. (ii) from Eq. (i), we get $g^2 - f^2 = a^2 - b^2$.

Hence, the locus is $x^2 - y^2 = a^2 - b^2$.

14. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (2, 0).

$\therefore \qquad 4+4g+c=0 \qquad$ (i)

Intercept on x-axis is $2\sqrt{g^2-c}=5$

$\therefore \qquad 4(g^2+4g+4)=25 \qquad$ [by Eq. (i)]

or $\quad (2g+9)(2g-1)=0 \Rightarrow g=-\frac{9}{2}, \frac{1}{2}$

Since centre $(-g, -f)$ lies in first quadrant, we choose

$g = -\frac{9}{2}$ so that $-g = \frac{9}{2}$ (positive)

$\therefore \qquad c = 14,$ [from Eq. (i)]

Value of t is variable.

15. Obviously, (3, 0) and (0, 4) are end points of diameter.

Then, equation is $(x-3)(x-0)+(y-0)(y-4)=0$ or

$$x^2+y^2-3x-4y=0$$

16. Given, equation of circle is $x^2+y^2-3x-4y+2=0$ and it cuts the x-axis.

$\therefore \quad x^2+0-3x+2=0$

$\Rightarrow \quad x^2-3x+2=0$

$\Rightarrow \quad (x-1)(x-2)=0$

$\Rightarrow \quad x=1,2$

Therefore, the points are (1, 0) and (2, 0).

17. Comparing the given equation with $x^2+y^2+2gx+2fy+c=0$, we get $g=5$

$\therefore$ Length of intercept on x-axis

$$=2\sqrt{g^2-c}$$

$$=2\sqrt{5^2-9}=8$$

18. Circle passing through points (1, 0), (0, 1) and (0, 0) is

$x(x-1)+y(y-1)=0$

It also passes through the point (2k, 3k)

$\Rightarrow 2k(2k-1)+3k(3k-1)=0$

$\Rightarrow \quad k=0$ or $k=\frac{5}{13}$

19. One end of the diameter is $P(1, 1)$.

Let centre be $Q(\alpha, \beta)$.

Now Q is midpoint of PR where R lies on the line $x+y=3$

Then, point R is $(2\alpha-1, 2\beta-1)$.

This point lies on the line, then $2\alpha-1+2\beta-1=3$

$\Rightarrow \quad 2\alpha+2\beta=5$

Hence, the locus is $2x+2y=5$.

20.

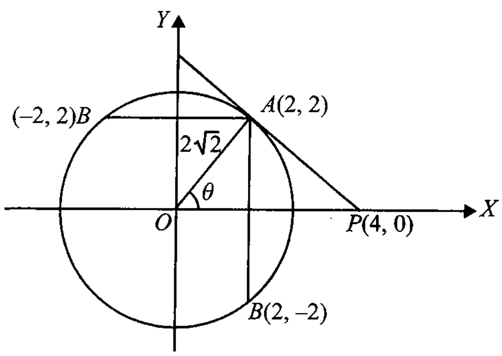


Fig. S–2.4

$$\cos\theta=\frac{2\sqrt{2}}{4}=\frac{1}{\sqrt{2}}$$

$$\theta=45^\circ$$

$\therefore \quad A \equiv (2, 2)$

Let $\quad B \equiv (x_1, y_1)$

Given $\quad AB=4$

$\therefore (x_1-2)^2+(y_1-2)^2=16$

$x_1^2+y_1^2-4x_1-4y_1=8$

Also $\quad x_1^2+y_1^2=8$

$\therefore \quad x_1+y_1=0$

$\therefore \quad 2x_1^2=8$

$x_1=\pm 2$

$\therefore \quad B \equiv (2,-2)$ or $(-2, 2)$

21.

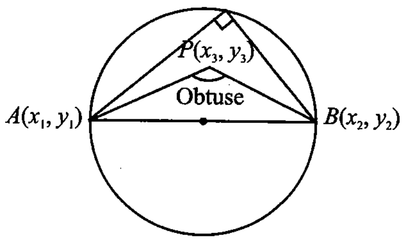


Fig. S–2.5

Equation of circle with AB as diameter, we get

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

since AB subtends obtuse angle at (x_3, y_3)

$\therefore$ P lies inside the circle.

$\therefore (x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)<0$

Exercise 2.2

1. Let the equation of the tangent be $\frac{x}{a}+\frac{y}{a}=1$,

i.e., $\qquad x+y=a \qquad$ (i)

$\therefore$ Length of perpendicular from the centre (–2, 2) on (i)

$=$ radius $=\sqrt{4+4-4}$

i.e., $\qquad \frac{|-2+2-a|}{\sqrt{1+1}}=2$

$\Rightarrow \qquad a=2\sqrt{2}$

Hence, the equation of the tangent is $x+y=2\sqrt{2}$.

2. According to the question,

$$\sqrt{(5)^2+(3)^2+2(5)+k(3)+17}=7$$

$\Rightarrow \quad 61 + 3k = 49 \Rightarrow k = -4$

3. Line $y = mx + c$ is tangent, if $c = \pm a\sqrt{1+m^2}$

Now $\quad lx + my + n = 0$

or $\quad y = -\frac{l}{m}x - \frac{n}{m}$ is tangent, if

$$-\frac{n}{m} = \pm a\sqrt{1+\left(\frac{l}{m}\right)^2}$$

or $\quad n^2 = a^2(m^2 + l^2)$

4. Equation of pair of tangent is given by $SS_1 = T^2$

Here $S = x^2 + y^2 + 20(x+y) + 20$, $S_1 = 20$

$$T = 10(x+y) + 20$$

$\therefore \quad SS_1 = T^2$

$\Rightarrow 20\{x^2 + y^2 + 20(x+y) + 20\} = 10^2(x+y+2)^2$

$\Rightarrow \quad 2x^2 + 2y^2 + 5xy = 0$

5. Normal passes through the centre of the circle.

Hence the equation of normal is $x - y = 0$.

6. Equation of line perpendicular to $5x + 12y + 8 = 0$ is $12x - 5y + k = 0$.

Now it is a tangent to the circle.

If radius of circle = Distance of line from centre of circle

$\Rightarrow \quad \sqrt{121+4-25} = \left|\frac{12(11) - 5(2) + k}{\sqrt{144+25}}\right|$

$$k = 8 \text{ or } -252.$$

Hence, equations of tangents are

$$12x - 5y + 8 = 0 \text{ and } 12x - 5y = 252$$

7. Clearly the point (1, 2) is the centre of the given circle and infinite tangents can only be drawn on a point circle.

Hence, radius should be 0.

$\therefore \quad \sqrt{1^2 + 2^2 - \lambda} = 0 \Rightarrow \lambda = 5$

8. We must have

Radius of given circle > Perpendicular distance from the centre of circle to the given line

$\Rightarrow \quad \sqrt{4+16+5} > \frac{|3(2) - 4(4) - m|}{\sqrt{9+16}}$

$\Rightarrow \quad |m + 10| < 25$

$\Rightarrow \quad -35 < m < 15$

9. The equation of tangents will be $y = mx$ or $y - mx = 0$

Then, applying condition for tangency,

$$\left|\frac{-5-4m}{\sqrt{1+m^2}}\right| = 5$$

$\Rightarrow 25 + 16m^2 + 40m = 25 + 25m^2$

$\Rightarrow \quad 9m^2 - 40m = 0$

$\Rightarrow \quad m = 0, \frac{40}{9}$

10. Let $P(x_1, y_1)$ be a point on $x^2 + y^2 = 4$.

Then, the equation of the tangent at P is $xx_1 + yy_1 = 4$ which meets the coordinates axes at $A\left(\frac{4}{x_1}, 0\right)$ and $B\left(0, \frac{4}{y_1}\right)$.

Let (h, k) be the midpoint of AB.

$\therefore \quad h = \frac{2}{x_1}, k = \frac{2}{y_1}$ i.e., $x_1 = \frac{2}{h}, y_1 = \frac{2}{k}$

But (x_1, y_1) lies on $x^2 + y^2 = 4$

$\Rightarrow \quad \frac{4}{h^2} + \frac{4}{k^2} = 1$

$\Rightarrow \quad 4(x^2 + y^2) = x^2y^2$

11. Let the point be (x_1, y_1)

According to question, $\frac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \frac{2}{3}$

Squaring both sides, $\frac{x_1^2 + y_1^2 + 4x_1 + 3}{x_1^2 + y_1^2 - 6x_1 + 5} = \frac{4}{9}$

$\Rightarrow \quad 9x_1 + 9y_1^2 + 36x_1 + 27 = 4x_1^2 + 4y_1^2 - 24x_1 + 20$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$

Hence, locus is $5x^2 + 5y^2 + 60x + 7 = 0$.

12. Let (x_1, y_1) be any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$

$\therefore \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1 = 0 \quad$ (i)

Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_2} = \sqrt{c_2 - c_1} \quad$ (using (i))

Exercise 2.3

1. Here the intersection point of chord and circle can be found by solving the equation of circle with the equation of given line. Therefore, the points of intersection are $(-4, -3)$ and $\left(\frac{24}{5}, \frac{7}{5}\right)$. Hence, the midpoint is

$$\left(\frac{-4+\frac{24}{5}}{2}, \frac{-3+\frac{7}{5}}{2}\right) = \left(\frac{2}{5}, -\frac{4}{5}\right).$$

2. **a.** An equation of the chord of contact of P with respect to the given circle is

$$2xh + 2yk - \frac{3}{2}(x+h) + \frac{5}{2}(y+k) - 7 = 0$$

$$\Rightarrow x\left(2h - \frac{3}{2}\right) + y\left(2k + \frac{5}{2}\right) - \frac{3}{2}h + \frac{5}{2}k - 7 = 0 \quad \text{(i)}$$

which should be same as the given line

$$9x + y - 18 = 0 \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$\frac{4h-3}{18} = \frac{4k+5}{2} = \frac{3h-5k+14}{36}$$

Comparing first two ratios, we get

$$h - 9k = 12 \quad \text{(iii)}$$

Comparing first and last ratios, we get

$$h + k = 4 \quad \text{(iv)}$$

Solving (iii) and (iv) for (h, k) we get

$$h = \frac{24}{5}, k = \frac{-4}{5}$$

Hence point P is $\left(\frac{24}{5}, \frac{-4}{5}\right)$

3. Equation of common chord will be

$$3x + 4y + 11 = 0 \quad \text{(i)}$$

Let the point of intersection of the tangents be (α, β).

$\therefore$ Equation of the chord of contact of the tangents drawn from (α, β) to first circle will be

$$x\alpha + y\beta = 9 \quad \text{(ii)}$$

$\because$ Equations (i) and (ii) are identical

$$\therefore \quad \frac{3}{\alpha} = \frac{4}{\beta} = -\frac{11}{9}$$

$$\therefore \quad (\alpha, \beta) = \left(-\frac{27}{11}, -\frac{36}{11}\right)$$

4. Chord of contact from origin $\equiv gx + fy + c = 0$

and from $(g, f) \equiv gx + fy + g(x+g) + f(y+f) + c = 0$

or $2gx + 2fy + g^2 + f^2 + c = 0$

$$\therefore \quad \text{Distance} = \frac{\left|\frac{g^2+f^2+c}{2} - c\right|}{\sqrt{g^2+f^2}} = \frac{|g^2+f^2-c|}{2\sqrt{g^2+f^2}}$$

5. Suppose point be (h, k). Equation of chord of contact is

$$hx + ky - a^2 = 0 \equiv lx + my + n = 0$$

or $$\frac{h}{l} = \frac{k}{m} = \frac{-a^2}{n}$$

or $$h = -\frac{a^2 l}{n}, k = \frac{-a^2 m}{n}$$

Exercise 2.4

1. **a.** Centres of circle $C_1(6, 6)$, $C_2(-3, -3)$

$$\therefore \quad C_1C_2 = \sqrt{(6+3)^2 + (6+3)^2} = 9\sqrt{2}$$

Radius of the circles $= \sqrt{36+36}, \sqrt{9+9} = 6\sqrt{2}, 3\sqrt{2}$

Since $\quad 9\sqrt{2} = 6\sqrt{2} + 3\sqrt{2}$

$\therefore$ Circles touch each other externally.

2. The equation of common chord PQ is $5ax + (c-d)$. $y + a + 1 = 0$ (i)

Also equation of PQ is $5x + by - a = 0$ (ii)

$$\therefore \quad \frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}$$

$$\Rightarrow \quad a = \frac{a+1}{-a}$$

$$\Rightarrow \quad a^2 + a + 1 = 0$$

$\Rightarrow$ No value of a. $[\because D < 0]$

3. **d.** Equation of circles is

$$S_1 = x^2 + y^2 + 2x - 3y + 6 = 0 \quad \text{(i)}$$

$$S_2 = x^2 + y^2 + x - 8y - 13 = 0 \quad \text{(ii)}$$

$\therefore$ Equation of common chord is

$$S_1 - S_2 = 0 \Rightarrow x + 5y + 19 = 0 \quad \text{(iii)}$$

and out of the four given points only point $(1, -4)$ satisfies it.

4. **b.** Centres and radii of the given circles are

Centres: $C_1(0, 1)$, $C_2(1, 0)$; Radii : $r_1 = 3, r_2 = 5$

Clearly, $\quad C_1C_2 = \sqrt{2} < (r_2 - r_1)$

Therefore, one circle lies entirely inside the other.

5. $(C_1C_2)^2 = r_1^2 + r_2^2$

$$\Rightarrow \quad 2a^2 = 18 \Rightarrow a = 3$$

6. Given circles are

$$2x^2 + 2y^2 - 3x + 6y + k = 0$$

or $$x^2 + y^2 - \frac{3}{2}x + 3y + \frac{k}{2} = 0 \quad \text{(i)}$$

and $x^2 + y^2 - 4x + 10y + 16 = 0$ (ii)

Circle (i) and (ii) cut orthogonally, then

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$2\left(-\frac{3}{4}\right)(-2) + 2\left(\frac{3}{2}\right)\cdot 5 = \frac{k}{2} + 16$$

$$3 + 15 = \frac{k}{2} + 16 \Rightarrow k = 4$$

7. Clearly, $r_1 - r_2 > C_1C_2$

$$r_1 = R, C_1(0, 0); r_2 = r; C_2(3, 4)$$

$$R - r > \sqrt{(3-0)^2 + (4-0)^2}$$

$$\Rightarrow \quad R - r > 5$$

8. Radical axes are

$$4x + 6y = 10 \text{ or } 2x + 3y = 5 \quad \text{(i)}$$

and $$2x + 2y = 4 \text{ or } x + y = 2 \quad \text{(ii)}$$

Point of intersection of (i) and (ii) is (1, 1).

9. Such circle is orthogonal to the given three circles. Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Then, according to the conditions given,

$$g + 2f = c + 3 \quad \text{(i)}$$

$$2g + 4f = c + 5 \quad \text{(ii)}$$

$$-7g - 8f = c - 9 \quad \text{(iii)}$$

$$\Rightarrow \quad g = \frac{2}{3}, f = \frac{2}{3}, c = -1$$

Therefore, the required equation is

$$3(x^2 + y^2) + 4(x + y) - 3 = 0$$

10.

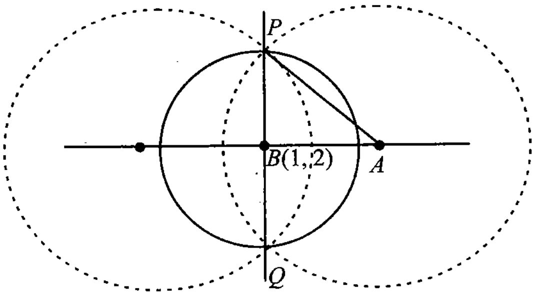


Fig. S–2.6

Clearly, diameter of 'C_1' will be the common chord.

Let the common chord be PQ and centre of C_2 be $A(h, k)$.

We have $AP = 5$, $PB = 3 \Rightarrow AB = 4$ units, where $B \equiv (1, 2)$.

Using parametric equation of line, we get

$$\frac{h-1}{-3/5} = \frac{k-2}{4/5} = \pm 4$$

$$\Rightarrow \quad h = -\frac{7}{5}, k = \frac{26}{5} \text{ or } h = \frac{17}{5}, k = -\frac{6}{5}$$

11.

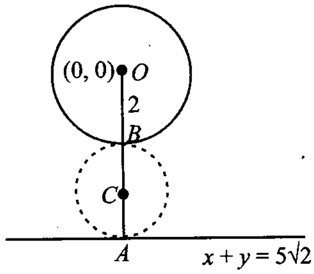


Fig. S–2.7

Here, OB = radius = 2

The distance of (0, 0) from $x + y = 5\sqrt{2}$ is 5.

$\therefore$ The radius of the smallest circle $= \frac{5-2}{2} = \frac{3}{2}$ and $OC = 2 + \frac{3}{2} = \frac{7}{2}$

The slope of $OA = 1 = \tan\theta$

$\therefore \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow C = (0 + OC\,.\cos\theta, 0 + OC\,.\sin\theta) = \left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}\right)$

12.

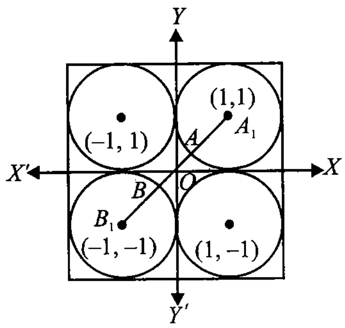


Fig. S–2.8

$$A_1B_1 = \sqrt{4+4} = 2\sqrt{2}$$

$$\Rightarrow \quad AB = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

Thus, equation of required circle is $x^2 + y^2 = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$.

13.

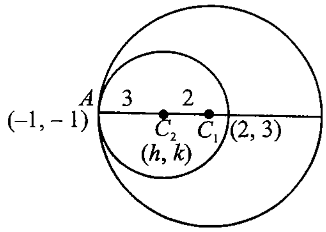


Fig. S–2.9

The given circle is $x^2 + y^2 - 4x - 6y - 12 = 0$, (i)

whose centre is $C_1(2, 3)$ and radius $r_1 = C_1A = 5$.

If $C_2(h, k)$ is the centre of the circle of radius 3 which touches the circle (i) internally at the point $A(-1, -1)$, then $C_2A = 3$.

and $C_1C_2 = C_1A - C_2A = 5 - 3 = 2$

Thus $C_2(h, k)$ divide C_1A in the ratio 2 : 3 internally,

$$\therefore \quad h = \frac{2(-1) + 3.2}{2+3} = \frac{4}{5}$$

$$\text{and} \quad k = \frac{2(-1) + 3.3}{2+3} = \frac{7}{5}$$

Hence, the equation of the required circle is $(x - 4/5)^2 + (y - 7/5)^2 = 3^2$ or $5x^2 + 5y^2 - 8x - 14y - 32 = 0$.

14. The two circles are $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$.

Centres: $C_1(2, 3)$ $C_2(-1, 1)$

radii: $r_1 = 4$ $\quad r_2 = 1$

We have, $C_1C_2 = 5 = r_1 + r_2$, circles touch externally therefore there are three common tangents to the given circles.

15. Let $A \equiv (0, 0)$ and $B \equiv (r_1 + r_2, 0)$ be the centres of the two given fixed circles.

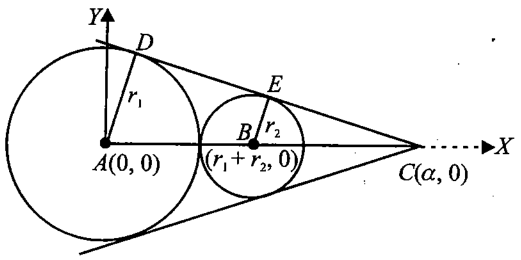


Fig. S–2.10

Let $C \equiv (\alpha, 0)$ be the point of intersection of direct common tangents.

$$\text{Now,} \quad \frac{r_2}{r_1} = \frac{\alpha - (r_1 + r_2)}{\alpha}$$

$$\Rightarrow \quad r_2\alpha = r_1\alpha - r_1^2 - r_1r_2$$

$$\Rightarrow \quad \alpha = \frac{r_1^2 + r_1r_2}{r_1 - r_2}$$

$$\therefore \quad \text{Locus of } C \text{ is } x = \frac{r_1^2 + r_1r_2}{r_1 - r_2} = a$$

which is a straight line.

16.

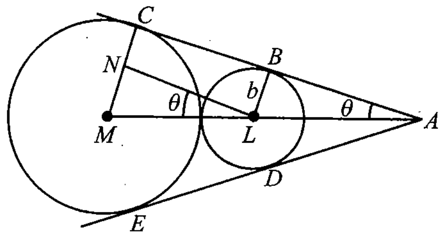


Fig. S–2.11

From Δ *MLN*

$$\sin\theta = \frac{a-b}{a+b}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{a-b}{a+b}\right)$$

∴ Angle between *AB* and *AD*

$$= 2\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$$

17. Given circles are

$$(x-1)^2 + (y-2)^2 = 1 \quad \text{(i)}$$

$$\text{and } (x-7)^2 + (y-10)^2 = 4 \quad \text{(ii)}$$

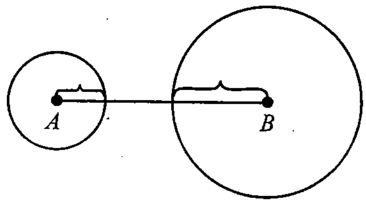


Fig. S–2.12

Let $\quad A \equiv (1, 2), B \equiv (7, 10), r_1 = 1, r_2 = 2$

$AB \equiv 10, r_1 + r_2 = 3$

$AB > r_1 + r_2$, hence the two circles are non-intersecting.

Radius of the two circles at time t are $1 + 0.3t$ and $2 + 0.4t$.

For the two circle to touch each other

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = (3 + 0.7t)^2, ((0.1)t + 1)^2$$

$$\Rightarrow \quad 3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$$

$$\Rightarrow \quad t = 10, t = 90 \qquad [\because t > 0]$$

The two circles will touch each other externally in 10 seconds and internally in 90 seconds.

Exercise 2.5

1.

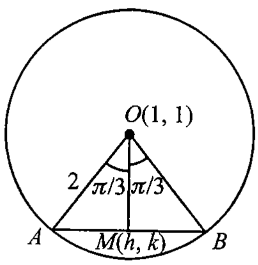


Fig. S–2.13

The coordinates of the centre and radius of the given circle are (1, 1) and 2, respectively. Let *AB* be the chord subtending an angle of $\frac{2\pi}{3}$ at the centre. Let *M* be the midpoint of *AB* and let its coordinates be (h, k).

In ΔOAM, $AM = OA.\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$\therefore$ $OM^2 = OA^2 - AM^2 = 4 - (\sqrt{3})^2 = 1$

But $OM^2 = (h-1)^2 + (k-1)^2$.

Therefore, $(h-1)^2 + (k-1)^2 = 1$

Hence, the locus of (h, k) is $(x-1)^2 + (y-1)^2 = 1$

or $x^2 + y^2 - 2x - 2y + 1 = 0$

2.

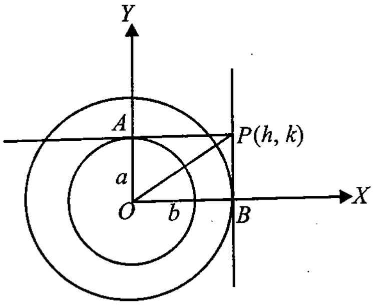


Fig. S–2.14

Let $OA = a, OB = b$

Since tangents at A and B meet at right angles in $P(h, k)$, $OAPB$ is a rectangle.

$\therefore$ $OP^2 = OB^2 + BP^2 = h^2 + k^2 = a^2 + b^2$

$\therefore$ Locus of P is

$$x^2 + y^2 = a^2 + b^2$$

which is concentric circle with given circles.

3.

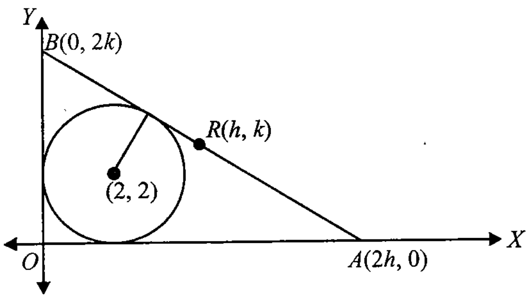


Fig. S–2.15

Let midpoint of AB is $R(h, k)$.

Then, coordinates of A and B are (2h, 0) and (0, 2k), respectively.

Equation of line AB is $\frac{x}{2h} + \frac{y}{2k} = 1$.

Since this line touches given circle, we have

$$\frac{\left|\frac{2}{2h} + \frac{2}{2k} - 1\right|}{\sqrt{\frac{1}{h^2} + \frac{1}{k^2}}} = 2$$

On simplifying, we get locus $x + y - xy + \sqrt{x^2 + y^2} = 0$.

4.

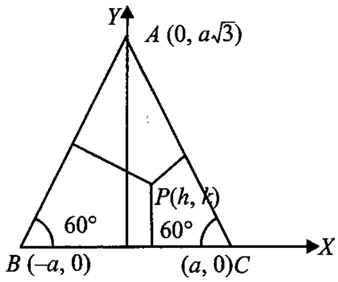


Fig. S–2.16

Taking midpoint of BC as origin, BC as x-axis and perpendicular to BC through O as y-axis, let C be $(a, 0)$, then B is $(-a, 0)$.

Since ΔABC is equilateral, A lies on y-axis.

As $\angle C = 60°$, A will be $(0, a\sqrt{3})$. Let P be (h, k).

Equation of AC and BC are $x\sqrt{3} + y - a\sqrt{3} = 0$ and $x\sqrt{3} - y + a\sqrt{3} = 0$, respectively.

According to the problem, $k^2 + \left(\frac{h\sqrt{3} - k + a\sqrt{3}}{2}\right)^2$

$$+\left(\frac{h\sqrt{3} - k + a\sqrt{3}}{2}\right)^2 = \lambda \text{ (say)}$$

$\Rightarrow 6h^2 + 6k^2 - 4a\sqrt{3}k + 6a^2 - 4\lambda = 0$

Hence, required locus of $P(h, k)$ is $6x^2 + 6y^2 - 4a\sqrt{3}y + 6a^2 - 4\lambda = 0$, which is a circle.

5.

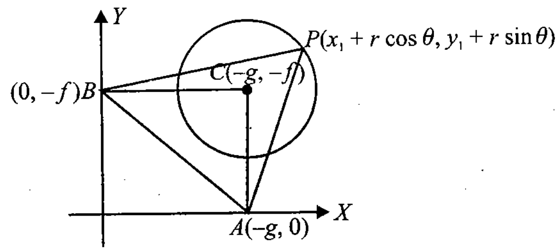


Fig. S–2.17

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Let the centroid of the ΔCAB be (h, k)

$3h = x_1 - g + r\cos\theta$

and $3k = y_1 f + r\sin\theta$

where $x_1 = -g; y_1 = -f$

$\therefore$ $3h = -2g + r\cos\theta$

$3k = -2f + r\sin\theta$

$\Rightarrow (3h + 2g)^2 + (3k + 2f)^2 = r^2$

$\Rightarrow (x + 2g/3)^2 + (y + 2f/3)^2 = r^2/9$

Hence, proved.

6.

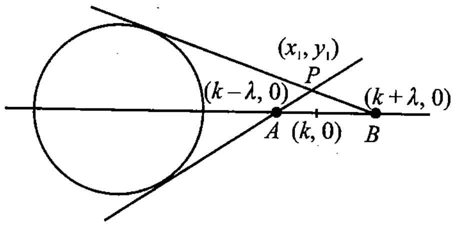


Fig. S–2.18

$$SS_1 = T^2$$

$\Rightarrow (x_1^2 + y_1^2 - a^2)\,[(k-\lambda)^2 - a^2] = [(k-\lambda)x_1 - a^2]^2$ (i)

and

$(x_1^2 + y_1^2 - a^2)\,[(k+\lambda)^2 - a^2] = [(k+\lambda)x_1 - a^2]^2$ (ii)

Equation (ii) – (i) gives

$$4(x_1^2 + y_1^2 - a^2)\,k\lambda = [(k+\lambda)x_1 - a^2 + (k-\lambda)x_1 - a^2] \times [(k+\lambda)x_1 - (k-\lambda)x_1]$$
$$= [2(kx_1 - a^2)]\,[2\lambda x_1]$$
$$= 4\lambda x_1\,[kx_1 - a^2]$$

$\Rightarrow \quad k(x_1^2 + y_1^2 - a^2) = kx_1^2 - a^2x_1$

$\Rightarrow \quad ky_1^2 = a^2(k - x_1)$

$\Rightarrow$ Hence, the locus is $ky^2 = a^2(k - x)$.

7. Let $A(-a, 0)$ and $B(a, 0)$ be two fixed points. Taking AB as x-axis and its right bisector as y-axis. Let the equation of the given line be

$x \cos\alpha + y \sin\alpha = p$ (i)

and line perpendicular to it and passing through $(-a, 0)$ is given by

$y \cos\alpha - x \sin\alpha = a \sin\alpha$ (ii)

Let AN and BM be the perpendiculars from A and B, then

$$AN \times BM = \frac{-a\cos\alpha + 0\sin\alpha - p}{\sqrt{\sin^2\alpha + \cos^2\alpha}} \times$$

$$\frac{a\cos\alpha + 0\sin\alpha - p}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \lambda^2(\text{contanstant})$$

$\Rightarrow \quad p^2 = \lambda^2 + a^2\cos^2\alpha$ (iii)

Eliminating p and α from Eqs. (i) and (ii), we get $x^2 + y^2 = \lambda^2 + a^2$ which is the required locus.

By changing a into $-a$, we get the same locus.

Hence, proved.

Chapter 3

Exercise 3.1

1.

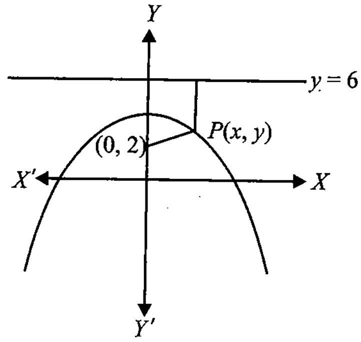


Fig. S–3.1

Since the focus and vertex of the parabola are on y-axis, therefore its directrix is parallel to x-axis and axis of the parabola is y-axis.

Let the equation of the directrix be $y = k$.

The directrix meets the axis of the parabola at $(0, k)$.

But vertex is the midpoint of the line segment joining the focus to the point where directrix meets axis of the parabola.

$$\therefore \quad \frac{k+2}{2} = 4 \Rightarrow k = 6$$

Thus, the equation of the directrix is $y = 6$.

Let (x, y) be a point on the parabola.

Then, by definition

$$(x-0)^2 + (y-2)^2 = (y-6)^2$$

$$\Rightarrow \quad x^2 + 8y = 32$$

2. The equation is

$$(x-0)^2 + (y-1)^2 = \left(\frac{x+2}{\sqrt{1}}\right)^2$$

or $\quad (y-1)^2 = 4(x+1)$

Clearly, $x = t^2 - 1$ and $y = 2t + 1$ satisfy it for all values of t.

3. Solving given parabola and circle, we have

$$x^2 + 4(x+3) + 4x = 0$$

$\Rightarrow \quad x^2 + 8x + 12 = 0$

$\Rightarrow \quad x = -2$ ($x = -6$ is not possible)

Since parabola and circle both are symmetrical about x-axis length of common chord is 4.

4. $\quad x^2 = 2(2x + y)$

or $\quad x^2 - 4x + 4 = 2(y + 2)$

or $\quad (x-2)^2 = 2(y+2)$.

So, the vertex is $(2, -2)$.

5. The given equation can be written as $(x-2)^2 = 3(y-2)$.
Shifting the origin at (2, 2), this equation reduces to

$$X^2 = 3Y, \text{ where } x = X+2, y = Y+2$$

The directrix of this parabola with reference to new axes is

$$Y = -a, \text{ where } a = \frac{3}{4}$$

$\Rightarrow \quad y - 2 = \frac{-3}{4}$

$\Rightarrow \quad y = \frac{5}{4}$

6. Shifting the origin at A, equation becomes

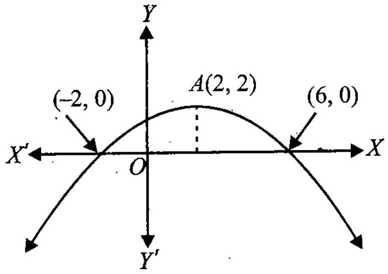


Fig. S–3.2

$$X^2 = -8Y$$

$\Rightarrow \quad (x-2)^2 = -8(y-2)$

7. $2a$ = perpendicular distance of the focus from the directrix

$= \left|-\frac{7}{\sqrt{71}}\right| = \frac{7}{\sqrt{17}}$

Therefore, latus rectum $= 4a = \frac{14}{\sqrt{17}}$

8.

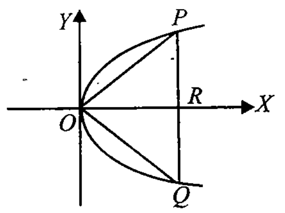


Fig. S–3.3

Let PQ be a double ordinate of length $8a$.

$\therefore \quad PR = RQ = 4a.$

Coordinates of P and Q are $(OR, 4a)$ and $(OR, -4a)$, respectively.

Since P lies on the parabola $y^2 = 4ax$,

$\therefore \quad (4a)^2 = 4a(OR)$

$\Rightarrow \quad OR = 4a$

Thus, the coordinates of P and Q are $(4a, 4a)$ and $(4a, -4a)$, respectively.

Now, $\quad m_1 = \text{slope of } OP = \frac{4a-0}{4a-0} = 1$

and $\quad m_2 = \text{slope of } OQ = \frac{-4a-0}{4a-0} = -1$

Clearly, $\quad m_1 m_2 = -1$

Thus, PQ subtends a right angle at the vertex of the parabola.

9.

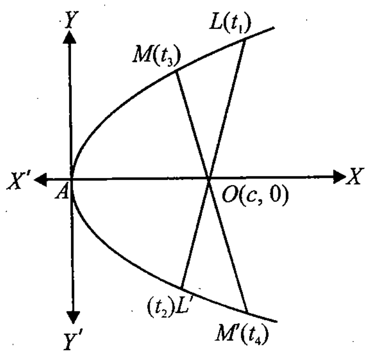


Fig. S–3.4

Equation of LOL': $2x - (t_1 + t_2)y + 2at_1t_2 = 0$

Since chord LOL' passes through $(c, 0)$

$\therefore \quad t_1t_2 = -\frac{c}{a}$

Similarly, for chord MOM',

$$t_3t_4 = -\frac{c}{a}$$

Now circle with LL' as diameter, the equation is

$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0$

or $x^2 + y^2 - a(t_1^2 + t_2^2)x - 2a(t_1 + t_2)y + c^2 - 4ac = 0 \quad$ (i)

(as $a^2t_1^2t_2^2 = c^2$ and $4a^2t_1t_2 = -4ac$)

Similarly, circle with MM' as diameter, the equation is

$x^2 + y^2 - a(t_3^2 + t_4^2)x - 2a(t_3 + t_4)y + c^2 - 4ac = 0 \quad$ (ii)

Radical axis of Eqs. (i) and (ii) is (i) – (ii)

or $a(t_1^2 + t_2^2 - t_3^2 - t_4^2)x - 2a(t_1 + t_2 - t_3 - t_4)y = 0$,

which passes through the origin (vertex of the parabola).

10.

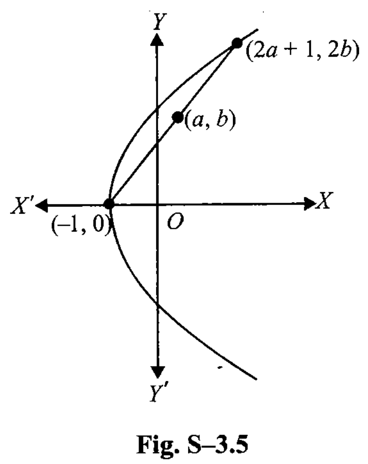


Fig. S–3.5

Since $((2a+1), 2b)$ lies on $y^2 = 4(x+1)$

$\therefore \quad 4b^2 = 4(2a+2)$

$b^2 = 2(a+1)$

11. The parabolas are $y^2 - x = 0$ and $y^2 + x = 0$. The point $(\lambda, -1)$ is an exterior point if

$1-\lambda > 0$ and $1+\lambda > 0$

$\Rightarrow \quad \lambda < 1$ and $\lambda > -1$

$\Rightarrow \quad -1 < \lambda < 1$

12. Let the fixed circle is $x^2 + y^2 = a^2$ and fixed line is $x = b$.

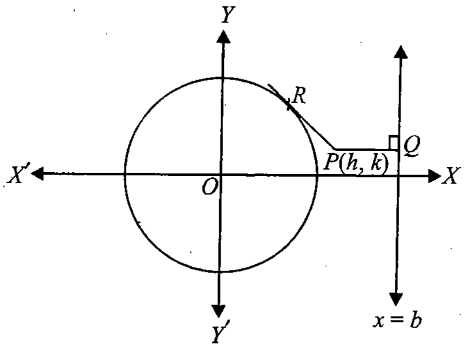


Fig. S–3.6

We have

$PQ = PR$

$\Rightarrow \quad b - h = \sqrt{h^2 + k^2 - a^2}$

$\Rightarrow \quad b^2 + h^2 - 2bh = h^2 + k^2 - a^2$

$\Rightarrow b^2 - 2bx = y^2 - a^2$ which is equation of parabola.

13

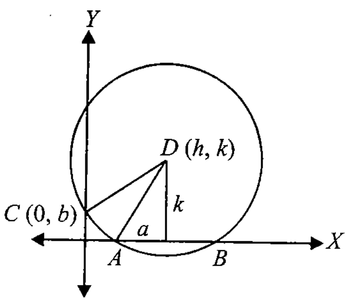


Fig. S–3.7

From the figure, $CD = AD$

$\Rightarrow \quad \sqrt{h^2 + (k-b)^2} = \sqrt{a^2 + k^2}$

$\Rightarrow h^2 + k^2 - 2bk + b^2 = a^2 + k^2$

$\Rightarrow x^2 - 2by + b^2 = a^2$, which is equation of parabola.

Exercise 3.2

1. $x^2 - 4x + 6y + 10 = 0$

$\Rightarrow \quad x^2 - 4x + 4 = -6 - 6y$

$\Rightarrow (x-2)^2 = -6(y+1)$ circle drawn on focal distance as diameter always touches the tangent drawn to parabola at vertex.

Thus, circle will touch the line $y + 1 = 0$.

2. Extremities of the latus rectum are (2, 4) and (2, –4).

Since any circle drawn with any focal chord as it's diameter touches the directrix, radius of the circle is $2a = 4$ ($\because a = 2$).

3. Length of focal chord making an angle 'α' with x-axis is $4a \operatorname{cosec}^2\alpha$. For $\alpha \in [0, \pi/4]$, its maximum length is $4a \times 2 = 8a$ units.

4. Latus rectum length of $y = ax^2 + bx + c$ is $1/a$. Now, semi-latus rectum is H.M. of SP and SQ.

Then, we have

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{1/2a}$$

$$\Rightarrow \quad 4a = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$\Rightarrow \quad a = \frac{5}{48}$$

5. Line joining $P(x_1, y_1) \equiv (at_1^2, 2at_1)$ and $Q(x_2, y_2) \equiv (at_2^2, 2at_2)$ is a focal chord

$\Rightarrow \quad t_1t_2 = -1$

Now

$$x_1x_2 + y_1y_2 = a^2t_1^4t_2^4 + 4a^2t_1t_2 = a^2 - 4a^2$$
$$= -3a^2$$

Exercise 3.3

1. The equation of any tangent to the parabola $y^2 = 4ax$ in terms of its slope m is

$$y = mx + \frac{a}{m}$$

Coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Therefore, the equation of tangent to $y^2 = ax$ is

$$y = mx + \frac{a}{4m}$$

and the coordinates of the point of contact are $\left(\frac{a}{4m^2}, \frac{a}{2m}\right)$.

It is given that $m = \tan 45° = 1$.

Therefore, the coordinates of the point of contact are $\left(\frac{a}{4}, \frac{a}{2}\right)$.

2.

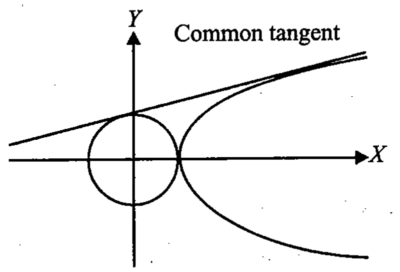


Fig. S-3.8

Equation of tangent to parabola $y^2 = 8ax$ having slope m is

$$y = mx + \frac{2a}{m} \quad \text{(i)}$$

Equation of tangent to circle $x^2 + y^2 = 2a^2$ having slope m is

$$y = mx \pm \sqrt{2}a\sqrt{1+m^2} \quad \text{(ii)}$$

Equations (i) and (ii) are identical

$$\Rightarrow \quad \frac{2a}{m} = \pm \sqrt{2}a\sqrt{1+m^2}$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2+2)(m^2-1) = 0$$

$$\Rightarrow \quad m^2 - 1 = 0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, required tangents are $x \pm y + 2a = 0$.

3. Given parabolas intersect at (16, 18) the slope of the tangent to parabola $y^2 = 4x$ at (16, 8) is given by

$$m_1 = \left(\frac{dy}{dx}\right)_{(16,8)} = \left(\frac{4}{2y}\right)_{(16,8)} = \frac{2}{8} = \frac{1}{4}$$

The slope of the tangent to parabola $x^2 = 32y$ at (16, 8) is given by

$$m_2 = \left(\frac{dy}{dx}\right)_{(16,8)} = \left(\frac{2x}{32}\right)_{(16,8)} = 1$$

$$\therefore \quad \tan\theta = \left|\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right| = \frac{3}{5}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

4. Here, $\quad S = y^2 - 4x = 0$

and $\quad S(0, -2) \equiv 0^2 - (-2) = 2 > 0.$

Thus, point (0, −2) lies outside parabola, hence two tangents can be drawn.

5. c. We know that the tangents to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet at $(at_1t_2, a(t_1+t_2))$.

Here, $a = 1$, $t_1 = 1$, $t_2 = 2$. So, they meet at (2, 3), which is on the line $y = 3$.

6. Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be two points on the parabola

$$y^2 = 4ax \quad \text{(i)}$$

Equations of tangents at P and Q on Eq. (i) are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$.

The point of intersection of these tangents is $T[at_1t_2, a(t_1+t_2)]$.

The coordinates of the focus S are $(a, 0)$.

$$\therefore (ST)^2 = (at_1t_2 - a)^2 + [a(t_1+t_2)]^2$$

$$= a^2(1+t_1^2)(1+t_2^2)$$

$$= (a+at_1^2)(a+at_2^2)$$

$$= SP \times SQ$$

7. Eliminating y, we have

$$a - x = x - x^2 \text{ or } x^2 - 2x + a = 0$$

Since the line touches the parabola, we must have equal roots.

$$\therefore \quad 4 - 4a = 0 \text{ or } a = 1$$

8. Equation of tangent to parabola having slope m is

$$y = mx + \frac{1}{m}$$

$\Rightarrow$ It passes through (h, k), therefore $m^2h - mk + 1 = 0$

$$\Rightarrow \quad m_1 + m_2 = \frac{k}{h};\ m_1m_2 = \frac{1}{h}$$

Given $\quad \theta_1 + \theta_2 = \frac{\pi}{4} \Rightarrow \tan(\theta_1 + \theta_2) = 1$

$$\Rightarrow \quad \frac{m_1+m_2}{1-m_1m_2} = 1 \Rightarrow \frac{k}{h} = 1 - \frac{1}{h}$$

$$\Rightarrow \quad y = x - 1$$

9. Equation of tangent $y = mx + \frac{1}{m}$ passing through (1, 3)

$$\Rightarrow \quad m^2 - 3m + 1 = 0$$

Hence, the roots are $m_1 = \frac{3+\sqrt{5}}{2}$, $m_2 = \frac{3-\sqrt{5}}{2}$.

$$\Rightarrow \quad \tan\theta = \left|\frac{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}}{1 + \frac{(3+\sqrt{5})(3-\sqrt{5})}{4}}\right| = \left|\frac{\sqrt{5}}{2}\right|$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) \text{ or } \pi - \tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$$

10. Any tangent to the parabola $y^2 = 8x$ is $y = mx + 2/m$, which is normal to the given circle.

Hence, tangents must pass through centre (−3, −4) of the circle.

Then, we have

$$-4 = -3m + \frac{2}{m}$$

$\Rightarrow \quad 3m^2 - 4m - 2 = 0$

$\Rightarrow \quad m = \dfrac{4 \pm \sqrt{40}}{6} = \dfrac{2 \pm \sqrt{10}}{3}$

11. The line $y = x - 1$ passes through (1, 0). That means it is a focal chord. Hence, the required angle is $\frac{\pi}{2}$.

12. We know that perpendicular tangents meet on the directric.

Given parabola is $y^2 + 4y - 6x - 2 = 0$

or $\quad (y + 2)^2 = 6(x + 1)$

Equation of directrix is $x + 1 = -6/4$ or $x = -5/2$ or $2x + 5 = 0$.

13. Diameter of triangle PAB = focal chord

$\Rightarrow$ minimum radius $= \dfrac{\text{length of latus ractum}}{2} = 8$

14. Origin (0, 0) lies on the directrix of given parabola which is $y = 0$. Then, angle between tangents is 90°.

15. Let P be (h, k). Any tangent $y = mx + a/m$

or $\quad k = mh + a/m$ or $m^2h - mk + a = 0$

Its roots are m_1 and $3m_1$ $\therefore$ $m_1 + 3m_1 = k/h$

$m_1 . 3m_1 = a/h$

Eliminating m_1,

$\Rightarrow \quad$ Locus is $3y^2 = 6x$

Exercise 3.4

1. We have

$$y - x\sqrt{2} + 4a\sqrt{2} = 0$$

or $\quad y = x\sqrt{2} - 4a\sqrt{2} \quad$ (i)

Comparing Eq. (i) with the equation $y = mx + c$, then

$$m = \sqrt{2},\ c = -4a\sqrt{2}$$

Since $-2am - am^2 = -2a\sqrt{2} - a(\sqrt{2})^3$

$$= -2a\sqrt{2} - 2a\sqrt{2} = -4a\sqrt{2}$$

$$= c$$

Hence, the given chord is normal to the parabola $y^2 = 4ax$.

The coordinates of the point are $(am^2, -2am) \equiv (2a, -2\sqrt{2}a)$.

2. The equation of a normal to the parabola $y^2 = 24x$ having slope m is $y = mx - 12m - 6m^3$.

It is parallel to $y = 2x + 3$, therefore, $m = 2$

Then equation of the parallel normal is

$$y = 2x - 24 - 48$$

or $\quad y = 2x - 72$

The distance between $y = 2x + 3$ and $y = 2x - 72$ is $\left|\dfrac{72 + 3}{\sqrt{4 + 1}}\right|$ $25\sqrt{5}$.

3. Slope m of the normal $x + y = 6$ is -1 and $a = 2$.

Normal to parabola $y^2 = 4ax$ at point $(am^2, -2am)$ is

$$y = mx - 2am - 2am - am^3$$

$\Rightarrow \quad y = -x + 4 + 2$ at (2, 4)

$\Rightarrow \quad x + y = 6$ is normal at (2, 4)

4. Normal at $P(at^2, 2at)$ is $y = -tx + 2at + at^3$.

It meets the axis $y = 0$ in $G(2a + at^2, 0)$.

If (x, y) be the midpoint of PG, then

$$2x = 2a + at^2 + at^2,\ 2y = 2at$$

Eliminating t, we have

$$x - a = at^2 = a(y/a)^2$$

or $y^2 = a(x - a)$ which is the required locus.

5.

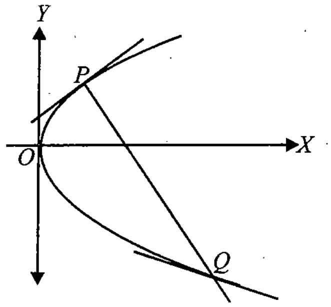


Fig. S–3.9

Slope of normal at point $P(t)$ is $-t$

or slope of line $PQ = -t$

Now, point Q has parameter $-t - \frac{2}{t}$.

Slope of tangent at point $Q = \dfrac{1}{-t - \frac{2}{t}} = \dfrac{-t}{t^2 + 2}$

Now, angle between normal and parabola at Q is equivalent to angle between normal and tangent at point

$$Q \Rightarrow \tan\theta = \left|\dfrac{-t + \dfrac{t}{t^2 + 2}}{1 + t\dfrac{t}{t^2 + 2}}\right| = \left|\dfrac{t}{2}\right|$$

Hence, $\quad \theta = \tan^{-1}\left|\dfrac{t}{2}\right|$

6.

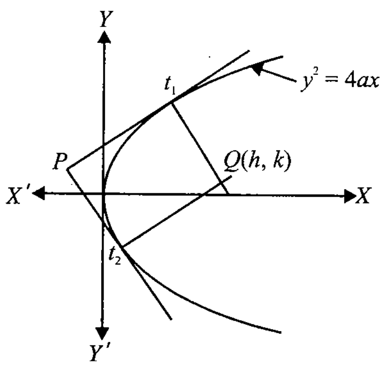


Fig. S–3.10

Let $P\left(at_1t_2, a(t_1+t_2)\right)$; P must satisfy $y^2 = a(x+b)$

Hence $a^2(t_1+t_2)^2 = a\left((at_1t_2)+b\right)$

$$a\left(t_1^2+t_2^2+t_1t_2\right) = b$$

Now coordinates of point of intersection of normals at t_1 and t_2 are

$$h = a\left(t_1^2+t_2^2+t_1t_2+2\right) \quad \text{(i)}$$

and $$k = -at_1t_2(t_1+t_2) \quad \text{(ii)}$$

From Eq. (i), $h = b + 2a$

$\Rightarrow$ $$x = b + 2a$$

7. If normal at $P(a\alpha^2, 2a\alpha)$ meets the parabola at $Q(a\beta^2, 2a\beta)$

$$\beta = -\alpha - \frac{2}{\alpha}$$

or $$2a(\beta+\alpha) = -\frac{4a}{\alpha},$$

$$2a\beta + 2a\alpha = -\frac{4a}{\alpha} = 3$$

$$4a = -3\alpha$$

$\Rightarrow$ $$2a = -1.5\alpha$$

8. For $y^2 = 4ax$, normal is

$$y = mx - 2am - am^3 \quad \text{(i)}$$

For $y^2 = 4c(x-b)$, normal is

$$y = m(x-b) - 2cm - cm^3 \quad \text{(ii)}$$

If two parabolas have common normal, then Eq. (i) and (ii) must be identical.

After comparing the coefficients we, get

$$m = \pm\sqrt{\frac{2(a-c)-b}{(c-a)}}$$

which is real if $-2 - \frac{b}{c-a} > 0$

$\Rightarrow$ $$\frac{b}{a-c} > 2$$

9. Normal having slope m is $y = mx - 2am - am^3$

or $$\frac{mx-y}{2am+am^3} = 1.$$

Make the parabola $y^2 = 4ax$ homogeneous and since the lines are at right angles, sum of the coefficients of x^2 and y^2 is zero.

$\Rightarrow$ $$m^2 = 2 \text{ or } m = \sqrt{2}$$

or $$\tan\theta = \sqrt{2}$$

or $$\theta = \tan^{-1}(\sqrt{2})$$

Exercise 3.5

1. Chord of contact of (h, k) is

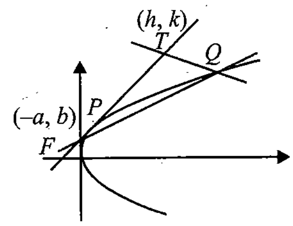


Fig. S–3.11

$$ky = 2a(x+h).$$

It passes through $(-a, b)$

$\therefore$ $$bk = 2a(-a+h)$$

$\Rightarrow$ Locus is $$by = 2a(x-a).$$

2. The chord of contact of parabola w.r.t. $(\alpha, 2)$ is

$$2y = 2(x+\alpha)$$

or $$x - y + \alpha = 0$$

Given $$\frac{|\alpha - 2 + \alpha|}{\sqrt{2}} = 4$$

$\Rightarrow$ $$|\alpha - 1| = 2\sqrt{2}$$

$\Rightarrow$ $$\alpha = 1 \pm 2\sqrt{2}.$$

$\Rightarrow$ $$\alpha = 1 - 2\sqrt{2}$$

(as for $\alpha = 1 + 2\sqrt{2}$, point lies inside paraboia)

3. Any chord PQ which get bisected at point $R(h, k)$ is $T = S_1$ or $ky - 2a(x+h) = k^2 - 4ah$.

Now given that this chord is focal chord, then it must passes through focus $S(a, 0)$.

Then $k(0) - 2a(a+h) = k^2 - 4ah$

$\Rightarrow$ $$k^2 = 2ah - 4a^2$$

$\Rightarrow$ $$y^2 = 2a(x-a)$$

4. Chord of contact of $P(0, \lambda)$ is $\lambda y = 2ax$.

$\Rightarrow$ $$2ax - \lambda y = 0$$

Line $\lambda x + 2ay = c$ perpendicular to it passes through $(0, \lambda)$

$\Rightarrow$ $$c = 2a\lambda$$

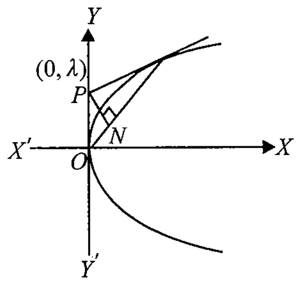


Fig. S–3.12

Hence equation of PN is

$$\lambda x + 2ay = 2a\lambda$$

or $\quad 2ay + \lambda(x - 2a) = 0.$

This passes through $(2a, 0)$.

Chapter 4

Exercise 4.1

1.

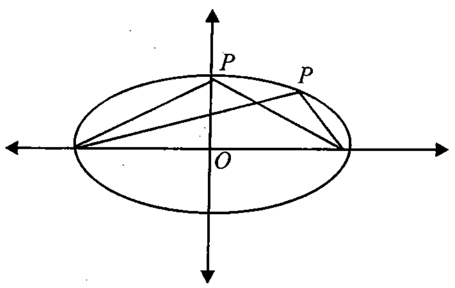


Fig. S–4.1

The maximum area corresponds to P when it is at either end of the minor axis and hence area for such a position of P is $\frac{1}{2}(2a)(b) = ab$.

2. Given $\frac{x}{3} = \cos t + \sin t$ and $\frac{y}{4} = \cos t - \sin t$.

Squaring and adding, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = (1 + \sin 2t) + (1 - \sin 2t)$$

or $$\frac{x^2}{9} + \frac{y^2}{16} = 2$$

which is the equation of the ellipse.

3. Let the equation of the semi-elliptical arch be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; (y > 0)$$

Length of the major axis $= 2a = 9 \Rightarrow a = \frac{9}{2}$

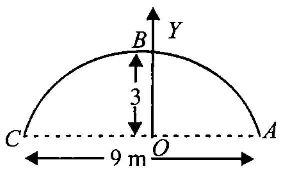


Fig. S–4.2

Length of the semi-minor axis $= b = 3$

So, the equation of the arc becomes $\frac{4x^2}{81} + \frac{y^2}{9} = 1$

If $x = 2$, then $y^2 = \frac{65}{9} \Rightarrow y = \frac{1}{3}\sqrt{65} = \frac{8}{3}$ approximately.

4.

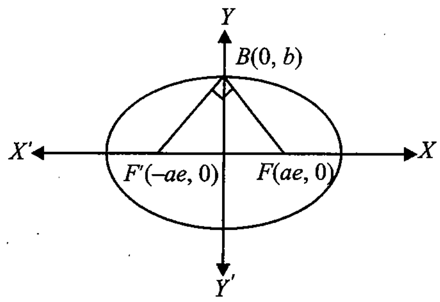


Fig. S–4.3

$$m_{BF} \cdot m_{BF'} = -1$$

$$\Rightarrow \quad \frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\Rightarrow \quad \frac{b^2}{a^2e^2} = 1$$

$$\Rightarrow \quad e^2 = 1 - \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow \quad e^2 = 1/2$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

5. Here $ae = 4$ and $e = \frac{4}{5}$. So, a = 5.

Now, $$b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad b^2 = 25\left(1 - \frac{16}{25}\right) = 9$$

Hence, the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

6.

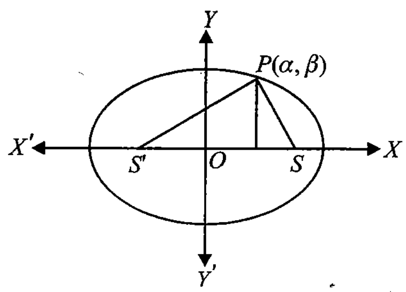


Fig. S–4.4

Since (α, β) lies on the ellipse $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$

$$\Rightarrow \quad \beta = b\sqrt{1 - \frac{\alpha^2}{a^2}}$$

Area of $\Delta SPS'$

$$= \frac{1}{2}\beta . SS'$$

$$= \frac{1}{2}\beta(2ae)$$

$$= bae\sqrt{1 - \frac{\alpha^2}{a^2}}$$

$$= be\sqrt{a^2 - \alpha^2}$$

7.

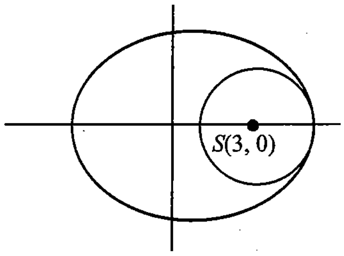


Fig. S–4.5

Given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \quad \text{(i)}$$

$$\Rightarrow \quad e^2 = 1 - \frac{16}{25} \Rightarrow e = \frac{3}{5}$$

Then foci are $(\pm ae, 0) \equiv (\pm 3, 0)$.

Now circle having centre (3, 0) is $(x - 3)^2 + y^2 = r^2$ (ii)

Eliminating y^2 from (i) and (ii), we get

$$\frac{x^2}{25} + \frac{r^2 - (x-3)^2}{16} = 1$$

or $16x^2 + 25r^2 - 25(x^2 - 6x + 9) = 400$

or $-9x^2 + 150x + 25r^2 - 625 = 0$

Since circle touches ellipse, above equation has equal roots. Hence, $D = 0 \Rightarrow 25500 + 36(25r^2 - 625) = 0$.

which is not possible.

Then circle will touch the ellipse at the end of the major axis (5, 0). Hence, radius is 2.

8. The given equation can be written as

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

which represents an ellipse whose centre is (– 1, – 2) and semi-major and minor axes are 5 and 3, respectively.

The eccentricity of the ellipse is given by

$$9 = (1 - e^2) \Rightarrow e = \frac{4}{5}$$

Shifting the origin at (– 1, – 2), the given equation reduces to

$$\frac{X^2}{9} + \frac{Y^2}{25} = 1 \quad \text{(i)}$$

where $x = X - 1, y = Y - 2$ (ii)

Coordinates of the foci of (i) are $(X = 0, Y = \pm be)$, where $b = 5$, $e = 4/5$, i.e., foci of (i) are $(X = 0, Y = \pm 4)$.

Therefore, the coordinates of the foci of the given ellipse are (– 1, 2) and (– 1, – 6).

9. Given ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Here $a^2 = 16, b^2 = 9 \Rightarrow a = 4, b = 3$

Sum of the focal distance from any point on the ellipse, $2a = 2(4) = 8$.

10. We have $a^2 = 16, b^2 = 9$. Then, $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{7}}{4}$.

Coordinates of focus S are $(\sqrt{7}, 0)$.

Therefore $CS = \sqrt{7}$.

Hence, CS: Major axis $= \sqrt{7} : 2a = \sqrt{7} : 8$.

11.

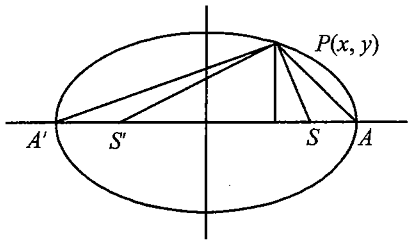


Fig. S–4.6

$$\frac{\Delta PSS'}{\Delta APA'} = \frac{\frac{1}{2}(SS')y}{\frac{1}{2}(AA')y}$$

$$= \frac{2ae}{2a} = e$$

12. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a < b)$

Since foci are $(0, \pm 1)$

$\Rightarrow \quad be = 1$ and $2a = 1$

[Since minor axis is of length = 1]

Also $a^2 = b^2(1 - e^2)$

$\Rightarrow \quad \frac{1}{4} = b^2 - b^2e^2 = b^2 - 1$

$\Rightarrow \quad b^2 = 1 + \frac{1}{4} = \frac{5}{4}$

Hence the equation of the ellipse is $\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1$ or $20x^2 + 4y^2 = 5$.

Exercise 4.2

1. $(3x - 12)^2 + (3y + 15)^2 = \frac{(3x + 4y + 5)^2}{25}$

$$\Rightarrow \sqrt{(x-4)^2 + (y+5)^2} = \frac{1}{3}\,\frac{|3x - 4y + 5|}{5}$$

Here, ratio of distances of variable point $P(x, y)$ from fixed point (focus) (4, 5) to that from fixed line (directrix

$\equiv 3x - 4y + 5 = 0$) is 1/3

Hence locus is ellipse and its eccentricity is 1/3.

Also length of latus rectum = 2(e)(distance of (4, 5) from the line $3x - 4y + 5 = 0$)

$$= 2 \times \frac{1}{3} \frac{|3 \times 4 - 4 \times 5 + 5|}{5} = \frac{2}{5}$$

2. The given equation is

$$\frac{\left(\frac{3x - 4y + 2}{5}\right)^2}{\frac{16}{25}} + \frac{\left(\frac{4x - 3y - 5}{5}\right)^2}{\frac{9}{25}} = 1$$

Hence $\quad a^2 = \frac{16}{25}$ and $b^2 = \frac{9}{25}$

$\Rightarrow$ Length of major axis $= 2a = \frac{8}{5}$ and length of minor axis $= 2b = \frac{6}{5}$

and $\quad e^2 = 1 - \frac{9}{16} \Rightarrow e = \frac{\sqrt{7}}{4}$

Exercise 4.3

1. The coordinates of any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is θ are $(a \cos \theta, b \sin \theta)$.

The coordinates of the end points of latus rectum are $\left(ae, \pm \frac{b^2}{a^2}\right)$.

$\therefore \quad a \cos \theta = ae$ and $b \sin \theta = \pm \frac{b^2}{a}$

$\Rightarrow \quad \tan \theta = \pm \frac{b}{ae}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\pm \frac{b}{ae}\right)$

Hence four points of latus rectum have eccentric angles $\tan^{-1}\left(\frac{b}{ae}\right), \pi - \tan^{-1}\left(\frac{b}{ae}\right), \pi + \tan^{-1}\left(\frac{b}{ae}\right), 2\pi - \tan^{-1}\left(\frac{b}{ae}\right)$ in I[st], II[nd], III[rd], IV[th] quadrants respectively.

2. Here $P(a \cos \alpha, b \sin \alpha)$, $Q(a \cos \beta, b \sin \beta)$, $S(ae, 0)$ are collinear, then

$$\begin{vmatrix} a \cos \alpha & b \sin \alpha & 1 \\ a \cos \beta & b \sin \beta & 1 \\ ae & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos \alpha & \sin \alpha & 1 \\ \cos \beta & \sin \beta & 1 \\ e & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad e(\sin \alpha - \sin \beta) + \sin (\beta - \alpha) = 0$

$\Rightarrow \quad 2e \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha - \beta}{2}$

$$\Rightarrow \quad e = \frac{\cos\left(\frac{\alpha - \beta}{2}\right)}{\cos\left(\frac{\alpha + \beta}{2}\right)}$$

$$= \frac{2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)}{2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right)}$$

$$= \frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)}$$

3.

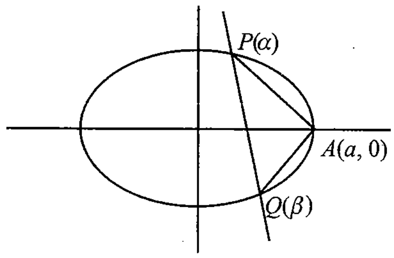


Fig. S–4.7

$$m_{AP} \times m_{QA} = -1$$

$\Rightarrow \quad \frac{b \sin \alpha}{a \cos \alpha - a} \times \frac{b \sin \beta}{a \cos \beta - a} = -1$

$\Rightarrow \quad \frac{4 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2}}{\left(2 \sin^2 \frac{\alpha}{2}\right)\left(2 \sin^2 \frac{\beta}{2}\right)} = \frac{-a^2}{b^2}$

$\Rightarrow \quad \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = -\frac{b^2}{a^2}$

Exercise 4.4

1. We know that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

Then comparing the given line $x \cos \alpha + y \sin \alpha = p$ with $y = mx + c$, we have $c = \frac{p}{\sin \alpha}$, $m = -\frac{\cos \alpha}{\sin \alpha}$.

So, the given line will be a tangent if

$$\frac{p^2}{\sin^2 \alpha} = a^2 \frac{\cos^2 \alpha}{\sin^2 \alpha} + b^2$$

$\Rightarrow \quad p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$

2. The equation of any tangent to the given ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If it touches $\quad x^2 + y^2 = r^2$.

Then, $\quad \sqrt{a^2m^2 + b^2} = r\sqrt{1 + m^2}$

$\Rightarrow \quad m^2(a^2 - r^2) = r^2 - b^2$

$\Rightarrow \quad m = \sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$

3. The equation of tangent to the given ellipse at point $P(a \cos \theta, b \sin \theta)$ is $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$.

Intercept of line on the axes are $\frac{a}{\cos\theta}$ and $\frac{b}{\sin\theta}$.

Given that $\frac{a}{\cos\theta} = \frac{b}{\sin\theta} = l \Rightarrow \cos\theta = \frac{a}{l}$ and $\sin\theta = \frac{b}{l}$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{a^2}{l^2} + \frac{b^2}{l^2}$

$\Rightarrow \quad l^2 = a^2 + b^2$

$\Rightarrow \quad l = \sqrt{a^2 + b^2}$

4.

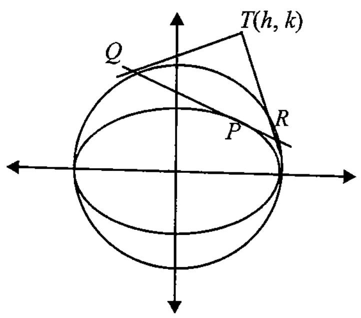


Fig. S–4.8

Equation of tangent to the ellipse at a given point is

$$x\left(\frac{1}{\sqrt{2}}\right) + 2y\left(\frac{1}{2}\right) = 1$$

or $\quad x + \sqrt{2}y = \sqrt{2} \quad$ (i)

Now QR is chord of contact of circle $x^2 + y^2 = 1$ with respect to point $T(h, k)$.

Then,

$$QR \equiv hx + ky = 1. \quad \text{(ii)}$$

Equations (i) and (ii) represent same straight line, then comparing ratio of coefficients, we have $\frac{h}{1} = \frac{k}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Hence, $(h, k) \equiv \left(\frac{1}{\sqrt{2}}, 1\right)$.

5. For a given ellipse, equation of tangent whose slope is m is $y = mx + \sqrt{18m^2 + 32}$.

For $m = -\frac{4}{3}$, tangent is

$$y = -\frac{4}{3}x \times \sqrt{18\left(\frac{16}{9}\right) + 32}$$

$$= -\frac{4}{3}x + 8$$

or $\quad 4x + 3y = 24$

This intersects major and minor axes at $A(6, 0)$ and $B(0, 8)$.

Then, area of ΔACB is $\frac{1}{2}(6)(8) = 24$.

6. Since the locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 + b^2.$$

Hence, the perpendicular tangents drawn to $\frac{x^2}{25} + \frac{y^2}{16}$ intersect on the curve $x^2 + y^2 = 25 + 16$, i.e., $x^2 + y^2 = 41$.

7. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse.

Equation of tangent at $P(\theta)$ is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{(i)}$$

Tangent meets the major axis at $T(a\sec\theta, 0)$.

Applying sine rule in ΔPST, we get

$$\frac{PS}{\sin(\pi - \alpha)} = \frac{ST}{\sin\beta}$$

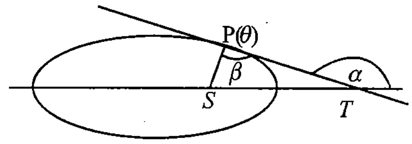


Fig. S–4.9

$$\frac{a(1 - e\cos\theta)}{\sin\alpha} = \frac{a(\sec\theta - e)}{\sin\beta}$$

$$\Rightarrow \quad \frac{1 - e\cos\theta}{\sin\alpha} = \frac{1 - e\cos\theta}{\cos\theta\sin\beta}$$

$$\Rightarrow \quad \cos\theta = \frac{\sin\alpha}{\sin\beta}$$

Slope of tangent is $-\frac{b}{a}\cot\theta = \tan\alpha$

$$\Rightarrow \quad \frac{b^2}{a^2} = \tan^2\alpha(\sec^2\theta - 1)$$

$$\Rightarrow \quad b^2/a^2 = \tan^2\alpha\left(\frac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha}\right)$$

$$= \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\alpha}$$

$$\Rightarrow \quad e = \sqrt{1 - \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\alpha}}$$

$$= \sqrt{\frac{1 - \sin^2\beta}{\cos^2\alpha}} = \frac{\cos\beta}{\cos\alpha}$$

Exercise 4.5

1. Let the line $lx + my + n = 0$ be normal to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

Now the equation of the normal at $(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2 \quad \text{(i)}$$

Comparing this line with the line $lx + my + n = 0$, we have

$$\frac{a}{l\cos\theta} = \frac{-b}{m\sin\theta} = \frac{a^2-b^2}{-n}$$

$$\Rightarrow \quad \cos\theta = \frac{-an}{l(a^2-b^2)}$$

and $$\sin\theta = \frac{bn}{m(a^2-b^2)}$$

Squaring and adding, we get

$$\Rightarrow \frac{a^2n^2}{l^2(a^2-b^2)^2} + \frac{b^2n^2}{m^2(a^2-b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} + \frac{b^2}{m} = \frac{(a^2-b^2)^2}{n^2}$$

2. The equation of the normal at (x_1, y_1) to the given ellipse is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

Here, $x_1 = ae$ and $y_1 = \frac{b^2}{a}$

So, the equation of the normal at positive end of the latus rectum is

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2e^2 \qquad [\because b^2 = a^2(1-e^2)]$$

$$\Rightarrow \quad \frac{ax}{e} - ay = a^2e^2 \Rightarrow x - ey - e^3a = 0$$

3.

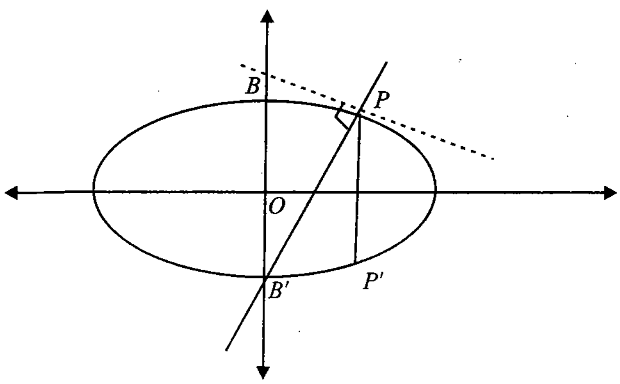


Fig. S–4.10

From the above question, equation of normal at the extremity of latus rectum is $x - ey = ae^3$.

This passes through the extremity of the minor axis, i.e., $B'(0, -b)$

Then $\quad 0 + eb - ae^3 = 0$

$$\Rightarrow \quad b = ae^2 \quad \Rightarrow b^2 = a^2e^4$$

$$\Rightarrow \quad a^2(1-e^2) = a^2e^4 \quad \Rightarrow e^4 + e^2 - 1 = 0$$

4. The equations of the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\frac{a^2x}{x_2} - \frac{b^2y}{y_2} = a^2 - b^2$$

$$\frac{a^2x}{x_3} - \frac{b^2y}{y_3} = a^2 - b^2, \text{ respectively}$$

These three lines will be concurrent, if

$$\begin{vmatrix} \frac{a^2}{x_1} & \frac{-b^2}{y_1} & a^2-b^2 \\ \frac{a^2}{x_2} & \frac{-b^2}{y_2} & a^2-b^2 \\ \frac{a^2}{x_3} & \frac{-b^2}{y_3} & a^2-b^2 \end{vmatrix} = 0$$

$$\Rightarrow -a^2b^2(a^2-b^2)\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} y_1 & x_1 & x_1y_1 \\ y_2 & x_2 & x_2y_2 \\ y_3 & x_3 & x_3y_3 \end{vmatrix} = 0$$

Exercise 4.6

1. The equations of the chords of contact of tangents drawn from (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{(i)}$$

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1 \quad \text{(ii)}$$

It is given that (i) and (ii) are at right angles.

$$\therefore \quad \frac{-b^2}{a^2}\frac{x_1}{y_1} \times \frac{-b^2}{a^2}\frac{x_2}{y_2} = -1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

2.

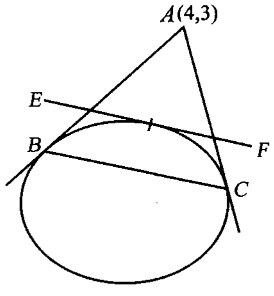


Fig. S–4.11

The equation of the chord of contact is $\frac{x}{4}+\frac{y}{3}=1$.

$\Rightarrow$ Slope of line EF is $\frac{-3}{4}$.

$\Rightarrow$ EF is $y=-\frac{3}{4}x+\sqrt{18}$.

$\Rightarrow$ Distance of (4, 3) from EF is $\frac{|24-4\sqrt{18}|}{5}$.

3. Equation of chord of ellipse which gets bisected at $P(h, k)$ is $\frac{hx}{4}+\frac{ky}{9}=\frac{h^2}{4}+\frac{k^2}{9}$. (i)

Its distance from origin (0, 0) is 2.

$$\Rightarrow \frac{\left|0+0-\left(\frac{h^2}{4}+\frac{k^2}{9}\right)\right|}{\sqrt{\frac{h^2}{16}+\frac{k^2}{81}}}=2$$

$\Rightarrow$ Locus is $4\left(\frac{x^2}{16}+\frac{y^2}{81}\right)=\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2$.

4.

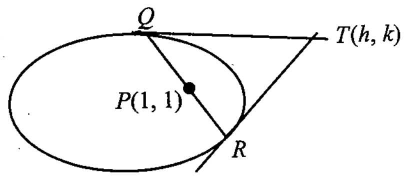


Fig. S–4.12

Equation of chord which gets bisected at point $P(1, 1)$ is

$$\frac{x^2}{16}+\frac{y^2}{9}-1=\frac{1}{16}+\frac{1}{9}-1$$

or $$\frac{x}{16}+\frac{y}{9}=\frac{25}{144}$$

or $$9x+16y=25 \quad \text{(i)}$$

Also line QR is chord of contact with respect to point $T(h, k)$.

Its equation is

$$\frac{hx}{16}+\frac{ky}{9}=1 \text{ or } 9hx+16ky=144 \quad \text{(ii)}$$

Equations (i) and (ii) represent the same straight line.

Hence, $$\frac{9h}{9}=\frac{16k}{16}=\frac{144}{25}$$

$$\Rightarrow (h, k)\equiv\left(\frac{144}{25}, \frac{144}{25}\right)$$

5.

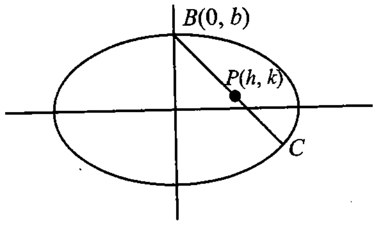


Fig. S–4.13

From the diagram, P is a midpoint of BC

$\Rightarrow$ Coordinates of C are $(2h, 2k-b)$

Now C lies on the ellipse, then

$$\frac{(2h)^2}{a^2}+\frac{(2k-b)^2}{b^2}=1$$

or $$\frac{4x^2}{a^2}+\frac{(2y-b)^2}{b^2}=1$$

which is the ellipse with centre at $(0, b/2)$.

Chapter 5

Exercise 5.1

1.

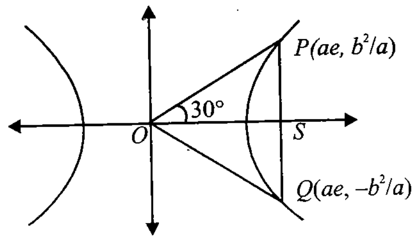


Fig. S–5.1

$$\tan 30^\circ=\frac{b^2/a}{ae}$$

$$\Rightarrow \frac{e}{\sqrt{3}}=e^2-1$$

$$\Rightarrow \sqrt{3}e^2-e-\sqrt{3}=0$$

$$\Rightarrow e=\frac{1\pm\sqrt{13}}{2\sqrt{3}}$$

$$\Rightarrow e=\frac{1+\sqrt{13}}{2\sqrt{3}}$$

2. Distance between the two directrix is $2a/e=10$ units

$$\Rightarrow a=5e$$

Now distance between the foci

$=2ae=10e^2=10(2)=20$ (as rectangular hyperbola, $e=\sqrt{2}$)

3.

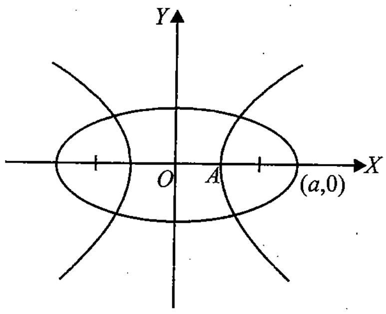


Fig. S–5.2

Let foci of ellipse are $(\pm ae_1, 0)$ and those of hyperbola are $(\pm Ae_2, 0)$.

According to question, we have

$$ae_1 = Ae_2 \quad \text{(i)}$$

Also it is given that the conjugate axis of hyperbola is equal to the minor axis of the ellipse. Therefore,

$$a^2(1 - e_1^2) = A^2(e_2^2 - 1) \quad \text{(ii)}$$

From (i) and (ii), we have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 2$$

4. Let equation of the lines be

$$y = m_1(x - a)$$

and

$$y = m_2(x + a)$$

$\therefore$ $m_1m_2 = p$

$\therefore$ $y^2 = m_1m_2(x^2 - a^2) = p(x^2 - a^2)$

Hence locus of the points of intersection is

$$y^2 = p(x^2 - a^2)$$

or

$$px^2 - y^2 = pa^2$$

which is a hyperbola.

5. Squaring and subtracting the given equations, we get

$$x^2 - y^2 = a^2$$

which is rectangular hyperbola.

6. Let $P \equiv (a \sec \theta, b \tan \theta)$

Then $N \equiv (a \sec \theta, 0)$

Since Q divides AP in the ratio $a^2 : b^2$

Therefore, coordinates of Q are $\frac{ab^2 + a^2 \sec \theta}{a^2 + b^2}, \frac{a^2 b \tan \theta}{a^2 + b^2}$.

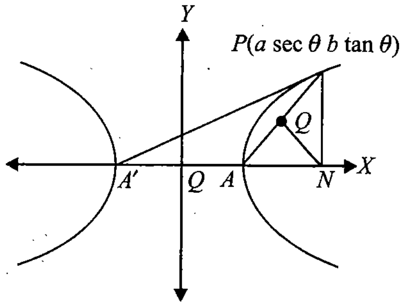


Fig. S–5.3

Slope of $A'P = \frac{b \tan \theta}{a(\sec \theta + 1)}$

Slope of $QN = \frac{a^2 b \tan \theta}{ab^2 + a^3 \sec \theta - ab^2 \sec \theta}$

$$= \frac{a^2 b \tan \theta}{ab^2(1 - \sec \theta)}$$

$\therefore$ Slope of $A'P \times$ slope of $QN = \frac{ab^2b^2 \tan^2 \theta}{-a^2b^2 \tan^2 \theta} = -1$.

$\therefore$ $QN \perp A'P$.

7. Taking AOB and COD as x- and y-axes and their point of intersection O as origin, clearly O is midpoint of AB and CD. Let A be $(a, 0)$ and C be $(0, c)$. Then B is $(-a, 0)$ and D is $(0, -c)$.

Let $P = (x, y)$.

Given that $PA \cdot PB = PC \cdot PD$

$\Rightarrow \sqrt{(x-a)^2 + y^2}\,\sqrt{(x+a)^2 + y^2}$

$$= \sqrt{x^2 + (y-c)^2}\,\sqrt{x^2 + (y+c)^2}$$

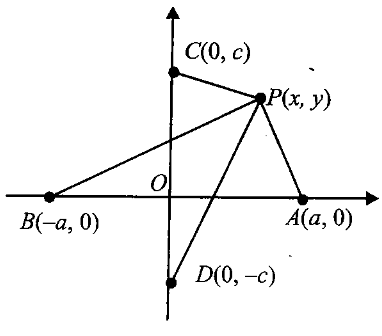


Fig. S–5.4

After simplification, we have

$$x^2 - y^2 = (a^2 - c^2)/2.$$

which is a rectangular hyperbola whose eccentricity is $\sqrt{2}$.

8. Let any point P on the hyperbola

$$x^2 - y^2 = a^2 \text{ be } (x_1, y_1).$$

Now $SP = ex_1 - a$ and $S'P = ex_1 + a$

$\Rightarrow$ $SP \times S'P = e^2x_1^2 - a^2$

$= 2x_1^2 - a^2$

$= 2x_1^2 - (x_1^2 - y_1^2)$

[$\because$ point (x_1, y_1) lies on the hyperbola]

$= x_1^2 + y_1^2$

$= CP^2$

9. The centre of the hyperbola is the midpoint of the line joining the two foci.

So the coordinate of the centre are $\left(\frac{8+0}{2}, \frac{3+3}{2}\right)$, i.e. (4, 3).

Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and let e be the eccentricity.

Then the equation of the hyperbola is

$$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1 \quad \text{(i)}$$

Now, distance between the foci = $2ae$

$\Rightarrow \sqrt{(8-0)^2 + (3-3)^2} = 2ae \Rightarrow ae = 4$

$\Rightarrow \quad a = 3 \left(\because e = \frac{4}{3}\right)$

Now, $\quad b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad b^2 = 9\left(-1 + \frac{16}{9}\right) = 7$

Thus, the equation of the hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1$$

Exercise 5.2

1. Equation of tangent is

$\sec\theta x - a\tan\theta y - ab = 0$

$\therefore \quad CN = \dfrac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$

Equation of normal P is

$ax\cos\theta + by\cot\theta = a^2 + b^2$

$\therefore \quad CM = \dfrac{a^2 + b^2}{\sqrt{a^2\cos^2\theta + b^2\cot^2\theta}}$

$A = \frac{1}{2} CM \times CN = \dfrac{ab(a^2+b^2)}{\sqrt{2a^2b^2 + b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta}}$

A is maximum when $b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta$ is minimum

Now, $b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta \ge 2a^2b^2$.

$A_{max} = \dfrac{ab(a^2+b^2)}{2ab} = \dfrac{a^2+b^2}{2}$, where $\theta = \sin^{-1}\frac{b}{a}$.

2. Given equation of hyperbola is

$$9x^2 - 16y^2 = 144$$

or

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Equation of tangent to hyperbola having slope m is

$$y = mx \pm \sqrt{16m^2 - 9}$$

If it touches the circle, then distance of this line from centre of the circle is radius of the circle. Hence,

$$\frac{\sqrt{16m^2-9}}{\sqrt{m^2+1}} = 3$$

$\Rightarrow \quad 9m^2 + 9 = 16m^2 - 9$

$\Rightarrow \quad 7m^2 = 18$

$\Rightarrow \quad m = \pm 3\sqrt{\frac{2}{7}}$

So, the equation of tangents is

$$y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$$

3. Let m be the slope of the tangent to $4x^2 - 9y^2 = 1$.

Then, m = (slope of the line $4y = 5x + 7$) = 5/4.

We have,

$$\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $\quad a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$

The equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

or

$$y = \frac{5}{4}x \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$$

$\Rightarrow \quad 24y - 30x = \pm\sqrt{161}$

4. The line $y = \alpha x + \beta$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ if } \beta^2 = a^2\alpha^2 - b^2$$

Hence, the locus of (α, β) is

$$y^2 = a^2x^2 - b^2$$

$\Rightarrow \quad a^2x^2 - y^2 = b^2$

$\Rightarrow \quad \dfrac{x^2}{b^2/a^2} - \dfrac{y^2}{b^2} = 1$

which is a hyperbola.

5. Equation of chord of contact of parabola w.r.t. $P(x_1, y_1)$ is given by

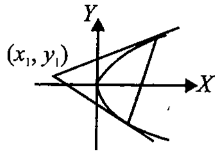


Fig. S–5.5

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

(i) touches the curve

$$x^2 - y^2 = c^2 \quad \text{(ii)}$$

Using the condition of tangency on

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

We get, $\left[\because \text{(i) is tangent to } \frac{x^2}{c^2} - \frac{y^2}{c^2} = 1\right]$

$$\frac{4a^2x_1^2}{y_1^2} = c^2\frac{4a^2}{y_1^2} - c^2$$

$$\Rightarrow \quad 4a^2x_1^2 = 4a^2c^2 - c^2y_1^2$$

$$\Rightarrow \quad 4a^2x_1^2 + c^2y_1^2 = 4a^2c^2$$

Hence locus is

$$\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1 \text{ which is an ellipse.}$$

6. Let $P \equiv (a \sec\theta, b\tan\theta)$

Since Q divides AP in the ratio $a^2 : b^2$

Then $N \in (a \sec\theta, 0)$

Therefore, coordinates of Q are $\left(\frac{ab^2 + a^2\sec\theta}{a^2 + b^2}, \frac{a^2b\tan\theta}{a^2 + b^2}\right)$.

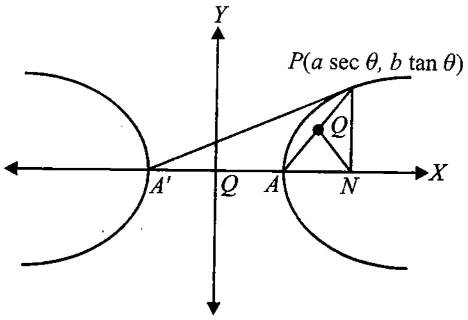


Fig. S–5.6

$$\text{Slope of } A'P = \frac{b\tan\theta}{a(\sec\theta + 1)}$$

$$\text{Slope of } QN = \frac{a^2b\tan\theta}{ab^2 + a^3\sec\theta - a^3\sec\theta - ab^2\sec\theta}$$

$$= \frac{a^2b\tan\theta}{ab^2(1 - \sec\theta)}$$

$$\therefore \text{Slope of } A'P \times \text{slope of } QN = \frac{a^2b^2\tan^2\theta}{-a^2b^2\tan^2\theta} = -1$$

$\therefore QN \perp A'P$

7. Tangent at $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

It meets $bx - ay = 0$ at Q. The point Q is given by

$$\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$$

It meets $bx + ay = 0$ at R. The point R is given by

$$\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$$

$$\therefore \quad CQ \cdot CR = \frac{\sqrt{a^2 + b^2}}{(\sec\theta - \tan\theta)}\frac{\sqrt{a^2 + b^2}}{(\sec\theta + \tan\theta)}$$

$$= a^2 + b^2$$

Exercise 5.3

1. Let the perpendicular line cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P(x_1, y_1)$ and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at point $Q(x_1, y_2)$.

Normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point P is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad \text{(i)}$$

Normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at Q is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_2} = a^2 + b^2 \quad \text{(ii)}$$

In Eqs. (i) and (ii), putting $y = 0$ we get

$$x = \frac{a^2 + b^2}{a^2}x_1$$

Hence both normals meet on x-axis.

2. The equation of normal at the point $Q(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

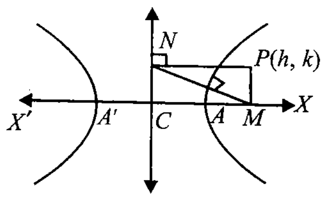


Fig. S–5.7

$$ax\cos\phi + by\cot\phi = a^2 + b^2 \quad \text{(i)}$$

The normal (i) meets the x-axis at

$M\left(\frac{a^2 + b^2}{a}\sec\phi, 0\right)$ and y-axis at $N\left(0, \frac{a^2 + b^2}{b}\tan\phi\right)$

Let point P be (h, k).

From the diagram,

$$h = \frac{a^2 + b^2}{a} \sec \phi$$

and

$$k = \frac{a^2 + b^2}{b} \tan \phi$$

Eliminating ϕ by using the relation $\sec^2\phi - \tan^2\phi = 1$, we have

$$\left(\frac{ah}{a^2 + b^2}\right)^2 - \left(\frac{bk}{a^2 + b^2}\right)^2 = 1$$

$\Rightarrow a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

3. Equation of the given hyperbola can be written as

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

Therefore, equation of the chord of this hyperbola in terms of the middle point (5, 3) is

$$\frac{5x}{16} - \frac{3y}{25} - 1 = \frac{5^2}{16} - \frac{9}{25} - 1$$

$$\Rightarrow \quad 125x - 48y = 481$$

4.

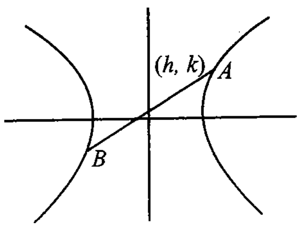


Fig. S–5.8

Equation of chord AB with $T = S_1$ is given by

$$\frac{hx}{3} - \frac{ky}{2} = \frac{h^2}{3} - \frac{k^2}{2}$$

Given that it has slope tan 45° = 1. Hence,

$$\frac{h}{3}\frac{2}{k} = 1$$

$$\Rightarrow \quad 2x = 3y, \quad \text{(as which is the required locus)}$$

5. Normal at a point $(a \sec \theta,\ a \tan \theta)$ is

$$x \cos \theta + y \cot \theta = 2a \qquad \text{(i)}$$

If $P(x_1, y_1)$ be the point of intersection of the tangents at the ends of normal chord (i), then (i) must be the chord of contact of $P(h, k)$ whose equation is given by

$$hx - ky = a^2 \qquad \text{(ii)}$$

Comparing (i) and (ii) and eliminating θ, we get

$$\frac{a^2}{4h^2} - \frac{a^2}{4k^2} = 1$$

Hence the locus is

$$\frac{1}{x^2} - \frac{1}{y^2} = \frac{4}{a^2}$$

6. **b.** Equation of chord joining α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\therefore \quad \alpha + \beta = 3\pi$$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) + \frac{y}{b} = 0$$

It passes through the centre (0, 0).

Exercise 5.4

1. Angle between asymptotes is given by

$$\tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right) = \tan^{-1}\left(\frac{2(4)(3)}{16-9}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right)$$

$$= \pi - 2\tan^{-1}\left(\frac{3}{4}\right)$$

2.

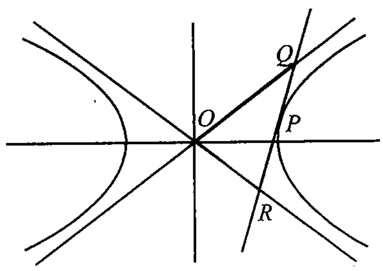


Fig. S–5.9

Consider point $P(a \sec \alpha, b \tan \alpha)$ on the hyperbola.

Tangent at P is

$$\frac{x}{a}\sec\alpha - \frac{y}{b}\tan\alpha = 1 \qquad \text{(i)}$$

Asymptotes are

$$y = (b/a)x \qquad \text{(ii)}$$

and

$$y = -(b/a)x \qquad \text{(iii)}$$

Solving (i) and (ii), we have

$$Q\left(\frac{a}{\sec\alpha - \tan\alpha}, \frac{b}{\sec\alpha - \tan\alpha}\right)$$

Solving (i) and (iii), we get

$$R\left(\frac{a}{\sec\alpha + \tan\alpha}, \frac{-b}{\sec\alpha + \tan\alpha}\right)$$

Then area of $\Delta OQR = \frac{1}{2}$

$$\begin{vmatrix} 0 & 0 & 1 \\ \dfrac{a}{\sec\alpha - \tan\alpha} & \dfrac{b}{\sec\alpha - \tan\alpha} & 1 \\ \dfrac{a}{\sec\alpha + \tan\alpha} & \dfrac{-b}{\sec\alpha + \tan\alpha} & 1 \end{vmatrix} = ab$$

3. Since equations of a hyperbola and its asymptotes differ in constant terms only, therefore the pair of asymptotes is given by

$$xy - 3y - 2x + \lambda = 0 \quad \text{(i)}$$

where λ is any constant such that it represents two straight lines. Hence,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 0 + 2 \times \left(-\frac{3}{2}\right) \times (-1) \times \left(\frac{1}{2}\right) - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow \lambda = 6$$

From (i), the asymptotes of the given hyperbola are given by

$$xy - 3y - 2x + 6 = 0$$

or

$$(y-2)(x-3) = 0$$

Therefore the asymptotes are $x - 3 = 0$ and $y - 2 = 0$.

4. Equation of the asymptotes of the given hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \quad b^2x^2 - a^2y^2 = 0$$

If θ is an angle between the asymptotes, then

$$\tan\theta = \pm\frac{\sqrt{a^2b^2}}{b^2 - a^2} = \pm\frac{ab}{a^2 - b^2}$$

$$\therefore \quad \tan\theta = \frac{ab}{a^2 - b^2}$$

$$\Rightarrow \quad \cos(\theta/2) = \sqrt{\frac{a^2}{a^2 + b^2}} = \frac{1}{e}$$